Symplectic toric varieties

1 Introduction

We are going to consider toric varieties from the point of view of symplectic geometry. Please be aware that this is work in progress. Comments and suggestions are welcome. For references other than Professor Thomas' lecture, I looked at http://www.math.ist.utl.pt/ acannas/Books/toric.pdf.

Let us first recall some basic notions. A *symplectic manifold* is a manifold M together with a symplectic form ω, i.e. a closed, everywhere non-degenerate 2-form on M. The dimension of M is necessarily even.
Furthermore, let

$$\mathbb{T}^n := (\mathbb{S}^1)^n$$

be the *n-torus.* Clearly, $\mathbb{T}^n$ is a compact Lie group.

Now let M be a symplectic manifold with symplectic form ω. Suppose that $\mathbb{T}^n$ acts on M smoothly and by symplectomorphisms. Suppose there exists a *moment map*

$$\mu\colon M \to (\mathbb{R}^n)^*,$$

i.e. a smooth map which is equivariant (with resprect to the coadjoint action on $(\mathbb{R}^n)^* \cong \mathrm{Lie}(\mathbb{T}^n)^*$) and such that for each $v \in \mathrm{Lie}(\mathbb{T}^n)$ we have

$$\omega(X_v, \cdot) \cong \mathrm{d}\mu_v,$$

where $\mu_v\colon M \to \mathbb{R}$ is defined by

$$\mu_v(x) := \mu(x)(v).$$

If such a moment map exists, the action is said to be *hamiltonian.*

Definition 1.1. A *symplectic toric manifold* is a compact symplectic manifold (M, ω) together with an effective action of $\mathbb{T}^n$ (i.e. $\mathbb{T}^n \to \mathrm{Sympl}(M, \omega)$ is injective) such that $\dim M = 2n$ and the choice of a moment map μ.

This definition resembles that of toric varieties in algebraic geometry. Indeed, $\mathbb{C}^*$ is the complexification of $\mathbb{S}^1$.

We have seen that toric varieties can be classified in terms of fans. Something similar is true for symplectic toric manifolds.
The fibres of the moment map are generically $\mathbb{T}^n$-orbits. Furthermore, we have

$$\mathrm{Lie}(\mathbb{T}^n) \cong \mathbb{R}^n \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$$

for some free $\mathbb{Z}$-module Λ of rank n. The lattice Λ is the kernel of the exponential map. The local model about a smooth fibre of $\mu\colon M \to (\mathbb{R}^n)^*$ is the projection map

$$\mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}^n \cong \Lambda^* \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let us now consider an

Example 1.2. Let $\mathbb{S}^1$ act by rotations on $\mathbb{S}^2$ with the symplectic form $\omega = \mathrm{d}h \wedge \mathrm{d}\theta$. The infinitesimal action is given by $\partial\theta$, and contracting ω with this vector field yields $\mathrm{d}h$, so the moment map is given by the height function h.

We observe that the image of h is a convex polytope with the fixed points of the torus action being mapped to the vertices of the polytope.

We have

Theorem 1.3. (Atiyah-Guillemin-Sternberg) The image of the moment map is a convex polytope. In fact, it is the convex hull of the images of the fixed points of the torus action.

Definition 1.4. A *Delzant polytope* in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is a convex polytope such that
(i) the vertices are in Λ,
(ii) the edges are generated by elements of Λ,
(iii) generators out of edges form a $\mathbb{Z}$-basis of Λ.

Clearly, the unit interval (Example 1.2) is a Delzant polytope. One can classify toric symplectic varieties in terms of Delzant polytopes. In what follows, we will sketch a proof of the (easier) existence part, i.e. we will show that for each Delzant polytope Δ there is a symplectic toric variety with moment polytope (image of the moment map) equal to Δ.
In order to prove this result, we first need the following result, due to Archimedes.

Lemma 1.5. *Imagine a sphere of radius a sitting inside a cylinder of height $2a$ and radius a. Then the projection from the sphere to the surface of the cylinder is area-preserving.*

Proof. Consider a plane through this figure along the middele axis of the cylinder. The intersection is a circle of radius a sitting inside a square with edges of length a. Now draw a rectangular triangle with one vertex at the centre of the circle, one vertex on the circle, such that the hypotenuse is the radius a and the other two edges are parallel to those of the square. Call one of them h. Let ϕ be the angle at the centre of the circle. Draw another rectangular triangle with one vertex at the same point at which the other triangle meets the circle such that the hypotenuse is tangent to the circle and the two other sides are parallel to those of the square. Call the hypotenuse s. Then the angle at this point (in the second triangle) is also equal to ϕ. Infinitesimally, we have

$$\frac{\delta s}{\delta h} = \frac{\sqrt{a^2 - h^2}}{a},$$

which implies that

$$2\pi\sqrt{a^2 - h^2}\delta s = 2\pi a \delta h$$

Now, the left hand side is the area of the ring on the sphere and the right hand side is that of the projection. The result follows by integrating. $\square$

Now suppose we are given a Delzant polytope Δ. To coinstruct a symplectic toric variety with moment polytope equal to Δ, we first consider the cotangent bundle

$$T^*\Delta \cong \Delta \times V^*.$$

Being a cotangent bundle, this has a canonical symplectic structure. We have $\Lambda^* \subseteq V^*$, so Λ^* acts on V^*. The symplectic form descends to a closed 2-form on $\Delta \times \mathbb{T}^n$ which is, however, not necessarily non-degenerate. In order to remedy that, we have to collapse the corresponding orbits.

Example 1.6. Consider again the manifold $\mathbb{S}^2$ with the symplectic form $\omega = \mathrm{d}\theta \wedge \mathrm{d}h$ and moment map equal to h. The corresponding moment polytope is $[-1, 1]$. The product $[-1, 1] \times \mathbb{S}^1$ is isomorphic to a cylinder. By collapsing the two components of the boundary, we recover the sphere.

Exercise 1.7. Do the same for $\mathbb{P}^2$ with the torus action

$$(\theta_1, \theta_2) \cdot [x : y : x] \colon = [x : e^{i\theta_1} y : e^{i\theta_2} z].$$

2 Symplectic toric manifolds from polytopes

We are now going to look at the proof of the existence part a bit more closely. Suppose $\Delta \subseteq \mathbb{R}^n$ is a Delzant polytope. Then there is $d \geq n$ such that we can write

$$\Delta = \{x \in \mathbb{R}^n \colon \langle x, v_i \rangle \leq \lambda_i \;\text{ for }\; i = 1, ..., d\}$$

for some $v_i \in \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$. One can check that the map $\pi \colon \mathbb{R}^d \to \mathbb{R}^n$ defined by $e_i \mapsto v_i$ (e_j being the standard basis) maps $\mathbb{Z}^d$ to $\mathbb{Z}^n$. Consider the symplectic manifold $\mathbb{C}^d$ with its canonical $\mathbb{T}^d$-action. The moment map is only defined up to a constant, so we can choose it to be

$$\varphi \colon (z_1, ..., z_d) \mapsto -\frac{1}{2}(|z_1|^2, ..., |z_d|^2) + (\lambda_1, ..., \lambda_d).$$

The map $\mathbb{R}^d \to \mathbb{R}^n$ from above descends to a surjective map $\pi \colon \mathbb{T}^d \to \mathbb{T}^n$. Call the kernel N. This is isomorphic to $\mathbb{T}^{d-n}$, and it acts in a hamiltonian way on $\mathbb{C}^d$. Let $i \colon N \to \mathbb{T}^d$ be the inclusion. We obtain a surjective map

$$i^* \colon (\mathbb{R}^d)^* \to \mathfrak{n}^*$$

of vector spaces, which is dual to the induced map on Lie algebras. The moment map of N acting on $\mathbb{C}^d$ is

$$\mu := i^* \circ \varphi : \mathbb{C}^d \to (\mathbb{R}^d)^* \to \mathfrak{n}^*.$$

One can show that that the set $\mu^{-1}(0)$ is compact and that N acts freely on it. Hence, by symplectic reduction, the quotient

$$M_\Delta := \mu^{-1}(0)/N$$

is a manifold of dimension $2n$, and the restriction of the symplectic form to $\mu^{-1}(0)$ descends to a symplectic form on M_Δ.

$$\begin{tikzcd}
0 \arrow[r] & (\mathbb{R}^n)^* \arrow[r, "\pi^*"] & (\mathbb{R}^d)^* \arrow[r, "i^*"] & (\mathbb{R}^{d-n})^* \arrow[r] & 0 \\
& M_\Delta = \mu^{-1}(0)/N \arrow[u, "\mu_\Delta"] & \mathbb{C}^d \arrow[u, "\varphi"] \arrow[ur, "\mu"] & & \\
& & \mu^{-1}(0) \arrow[u] \arrow[ul, "p"] & &
\end{tikzcd}$$

What remains to be proven is that M_Δ carries a hamiltonian torus action with the required moment polytope.

One can show that the exact sequence

$$0 \to N \to \mathbb{T}^d \to \mathbb{T}^n \to 0$$

splits, so the action of $\mathbb{T}^n$ descends to M_Δ. Now consider the sequence of maps

$$\mu^{-1}(0) \to \mathbb{C}^d \to (\mathbb{R}^d)^* \cong \mathfrak{n}^* \oplus (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$$

with the first map being the inclusion and the second the moment map. It follows from the N-equivariance that the composition is constant along N-orbits. Hence we obtain a map

$$\mu_\Delta \colon M_\Delta \to (\mathbb{R}^n)^*$$

which satisfies

$$\kappa = \mu_\Delta \circ p,$$

where $p\colon (i^* \circ \varphi)^{-1}(0) \to M_\Delta$ is the canonical projection and κ is the composition of the above maps. In particular, μ_Δ is the moment map of the action of $\mathbb{T}^n$ on M_Δ. We find

$$\mu_\Delta(M_\Delta) = \kappa(\mu^{-1}(0)) = \Delta.$$

using that $\varphi(\mu^{-1}(0)) = \varphi(\varphi^{-1}((i^*)^{-1}(0))) = \pi^*(\Delta)$.

3 Another way to get toric symplectic manifolds

The above construction relies on the short exact sequence of tori

$$0 \to N = \mathbb{T}^{d-n} \to \mathbb{T}^d \to \mathbb{T}^n \to 0$$

which via the splitting defines an action of $\mathbb{T}^n$ on the symplectic quotient $\mu^{-1}(0)/N$ where μ is the moment map.

Even without a polytope, we can construct symplectic toric varieties in a similar fashion by starting with a torus homomorphism as above.

Similarly to before, consider a torus $(S^1)^m$ with a Hamiltonian action on $\mathbb{C}^m$. Now let $r < m$, let $T = (S^1)^r$ be a torus of lower dimension and consider an injective group morphism

$$T \hookrightarrow (S^1)^m.$$

Since we have a canonical identification

$$\mathbb{Z}^m \to \mathrm{Hom}(S^1, (S^1)^m), \quad u \mapsto \lambda^u \text{ defined by } \lambda^u(t) = (t^{u_1}, \dots, t^{u_m}),$$

we conclude that the above injection can be represented by an $m \times r$ matrix M with integer entries, corresponding to the r maps $S^1 \to (S^1)^m$. The injection requirement says that M has full rank r.

Via $T \hookrightarrow (S^1)^m \curvearrowright \mathbb{C}^d$, this defines a Hamiltonian action of T on $\mathbb{C}^d$. The moment map of this action decomposes as

$$\mu_T : X \xrightarrow{\mu} \mathfrak{s}^* \xrightarrow{M^T} \mathfrak{t}^*$$

where $\mathfrak{s} := Lie((S^1)^m) \cong (\mathbb{R}^m)^*$ and $\mathfrak{t} := Lie(T) \cong (\mathbb{R}^r)^*$ are the corresponding Lie-algebras, μ is the moment map of the action of $(S^1)^m$, and M^T denotes the transpose of M, which represents the projection that is dual to $M : \mathbb{R}^r \hookrightarrow \mathbb{R}^d$.

If now $w \in t^*$ is a regular value of the moment map, we can form the symplectic quotient $X = \mu_T^{-1}(w)/T$ of real dimension $2(m-r)$. This is in fact a toric symplectic manifold: Analoguous to the construction in section 2, we have an exact sequence of tori

$$0 \to T \to (S^1)^m \to (S^1)^{m-r} \to 0$$

Since the $(S^1)^m$-action on X is trivial on the image of T, we get a torus action of $(S^1)^{m-r}$.

3.1 Where is the fan?

In the above setting, there is a nice construction of the corresponding fan.

Recall that we have $T \hookrightarrow (S^1)^m$ via an $m \times r$ matrix M. The moment map of the action of T on $\mathbb{C}^m$ was $\mu : \mathbb{C}^m \to \mathbb{R}^m \xrightarrow{M^T} \mathbb{R}^r$. Let $D_1, \ldots D_m \in \mathbb{R}^r$ be the column vectors of M^T. Recall that inside $\mathfrak{t}^* \cong \mathbb{R}^r$ there is a lattice of rank r corresponding to the character lattice $\mathrm{Hom}(T, S^1) \cong \mathbb{Z}^r$. Since each D_i defines a map $T \to S^1$, we can canonically identify each D_i with a point on the character lattice.
Now fix $w \in \mathfrak{t}^*$, a value of the moment map. For any $I \subseteq \{1, \ldots, m\}$ consider the cone σ_I that is spanned in $M \otimes \mathbb{R} = \mathbb{R}^r$ by the D_i with $i \in I$. This is called an anticone. We consider those anticones σ_I which contain w. Let

$$A_w = \{I \subseteq \{1, \ldots, m\} \mid \sigma_I \text{ is anticone containing } w\}.$$

Then different w give different maximal anticones in $\mathfrak{t}^*$. Note that all this happens in the Lie-algebra of T, while (as in section 2) we expect our fan to appear in a lattice corresponding to the quotient of $(S^1)^m$ by T. Where does this lattice come from?
The short exact sequence of tori $0 \to T \to (S^1)^m \to (S^1)^{m-r} \to 0$ transforms by taking cocharacters $\mathrm{Hom}(S^1, -)$ to a short exact sequence of lattices

$$0 \to \mathbb{Z}^r \to \mathbb{Z}^m \to N \to 0.$$

As we will see, N is the lattice we are looking for. Denote the images of the standard basis of $\mathbb{Z}^m$ in N by $v_1, \ldots, v_m$. Now for each $I \in A_w$ consider $\overline{I} = \{1, \ldots, m\} \setminus I$ and form

$$\mathrm{Cone}(v_i \mid i \in \overline{I}) \subseteq N \otimes \mathbb{R} = \mathbb{R}^{m-r}.$$

The collection of these cones forms a fan: For example, for $J \subseteq I$ we have $\overline{I} \subseteq \overline{J} \in A_w$ whenever $\overline{I} \subseteq A_w$. A similar statement about $I_1 \cap I_2$ gives the second fan condition.

This fan inside $N \otimes \mathbb{R}$ is the fan associated with X. To wrap things up:

$$w \in \mathfrak{t}^* \xrightarrow{\text{form cones in } \mathfrak{t}^*} \text{colection of anticones} \xrightarrow{\text{opposites in } N} \text{fan}$$

Note that starting with different values w of the moment map, we may get different fans. This can be made more precisely: As long as w_1, w_2 are in the interrior of the same maximal anticone, their symplectic quotients $\mu_T^{-1}(w_{1,2})/T$ are the same. But if we move w_2 into a different anticone, the symplectic quotients need not coincide. So we may get a different toric manifold and hence a different fan.

4 Toric blow-up

In this section, we will consider the blow-up of toric varieties, and in particular how the combinatorial description (polytope or fan) changes under this operation. We will particularly consider the example of $\mathbb{P}^2$ with the usual action of the torus, looking both at the symplectic and the algebro-geometric side of the theory.
Let us first recall briefly the construction of the fan associated to a toric variety. Suppose X is an affine toric variety over $\mathbb{C}$, of dimension n. By construction, X has an open orbit isomorphic to $(\mathbb{C}^*)^n$. Further, the $\mathbb{Z}$-module $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ of homomorphisms of algebraic groups is free of rank n. We say that an element $\psi \in \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ *converges* in X if there is a map $\bar{\psi}\colon \mathbb{C} \to X \supseteq (\mathbb{C}^*)^n$ such that

$$\bar{\psi}\mid_{\mathbb{C}^*} = \psi.$$

Define the *polyhedral cone* associated to X to be the cone

$$\sigma \subseteq \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) \otimes_{\mathbb{Z}} \mathbb{Q}$$

generated by the convergent morphisms. If X is not necessarily affine, we can cover X with finitely many open affine toric subvarieties, and we can patch the polyhedral cones together to obtain a *fan* . As it turns out, this combinatorial object determines the toric variety up to isomorphism.

Exercise 4.1. Find the polyhedral cone associated to $\mathbb{C}^n$ and $\mathbb{P}^n$ with the usual torus actions.

Now let us consider the variety

$$\mathrm{Bl}_0\,\mathbb{C}^2 := \{((x,y),[a:b]) \in \mathbb{C}^2 \times_{\mathbb{C}} \mathbb{P}^1 : ax - by = 0\}$$

with the torus action

$$(t_1,t_2)\cdot((x,y),[a:b]) := ((t_1x, t_2y),[t_1a : t_2b]).$$

This variety can be covered by open affine toric subvarieties given by $a \neq 0$ and $b \neq 0$. We know that any homomorphism of algebraic groups

$$\mathbb{C}^* \to (\mathbb{C}^*)^2$$

is given by

$$t \mapsto (t^{k_1}, t^{k_2})$$

with $k_j \in \mathbb{Z}$ for $j = 1,2$. Therefore, in order for such a morphism to converge, we must have $k_i \geq 0$ for $i = 1,2$. Furthermore, we have

$$[t^{k_1} : t^{k_2}] = [1 : t^{k_2-k_1}],$$

so (as "$[1:\infty] = [0:1]$"), we also obtain the condition that $k_2 \geq k_1$. Putting things together, we find that the fan associated to $\mathrm{Bl}_0\,\mathbb{C}^2$ consists of the cones generated by $(1,0),(1,1)$ and $(1,1),(0,1)$.
Using a very similar argument, we can show that the fan associated to $\mathrm{Bl}_{[0:1:1]}\,\mathbb{P}^2$ consists of the cones generated by $(1,0),(1,1)$, $(1,1),(0,1)$, $(0,1),(-1,-1)$ and $(-1,-1),(1,0)$, so it arises from the fan of $\mathbb{P}^2$ by *adding an extra ray.* This corresponds to the fact that the polytope of a blow-up (in a fixed point) corresponds to *chopping off* the vertex coming from this fixed point.