Symplectic reduction

1 Basic Symplectic Geometry

Definition 1.1. Let M be a smooth manifold. A 2-form $\omega \in \Omega^2(M)$ is called a *symplectic form* if $d\omega = 0$ and ω is nondegenerate. In this case, we call the pair (M, ω) a *symplectic manifold.*

One can think of the closed condition as follows. By Stokes' Theorem, for any 3-dimensional submanifold with boundary $V \subseteq M$, we have

$$\int_{\partial V} \omega = \int_V d\omega = 0.$$

In particular, if Σ_1 and Σ_2 are surfaces inside M such that $\Sigma_1 \cup \Sigma_2 = \partial V$ for some V, then (provided we orient things correctly),

$$\int_{\Sigma_1} \omega = \int_{\Sigma_2} \omega.$$

What about non-degeneracy? Since ω is a 2-form, for each $p \in M$, we get an alternating map

$$\omega : T_pM \times T_pM \to \mathbb{R}.$$

This defines a map

$$\begin{aligned} T_pM &\to T_p^*M \\ v &\mapsto \iota_v\omega, \end{aligned}$$

where $\iota_v\omega$ is defined by

$$\langle \iota_v\omega, w\rangle = \omega(v, w) \quad \text{for all} \quad w \in T_pM.$$

We say that ω is *nondegenerate* if the map $T_pM \to T_p^*M$ is an isomorphism for all $p \in M$.

Exercise 1.2. Let V be a vector space with $\dim_{\mathbb{R}} V = 2n$. Show that $\omega \in \Lambda^2 V^*$ satisfies $\omega^n \neq 0$ if and only if the induced map $V \to V^*$ as described above is an isomorphism.

Suppose we also have an inner product $\langle\, ,\, \rangle$, and so an isomorphism $V \to V^*$. Show that, if the composition $J : V \xrightarrow{\omega} V^* \xrightarrow{\langle,\rangle} V$ is orthogonal with respect to $\langle\, ,\, \rangle$, then $J^2 = -1$, so J defines a complex structure on V.

Conversely, suppose we have a complex structure J and inner product $\langle\, ,\, \rangle$ on V, which is compatible with J in the sense that $\langle Jv, Jw\rangle = \langle v, w\rangle$. Prove that $\omega(u, v) := \langle u, Jv\rangle$ is a skew non-degenerate 2-form.

Example 1.3. The following are examples of symplectic manifolds.

1. (Almost) Kähler manifolds. If X is an almost Hermitian manifold with Hermitian form ω, then ω is a nondegenerate 2-form by Exercise 1.2. So ω is a symplectic form if and only if $d\omega = 0$, i.e., if and only if X is an almost Kähler manifold.

2. Cotangent bundles. Consider $\pi : T^*N \to N$. This has a canonical 1-form on it, defined by $\theta_{(n,\xi)} = \pi^*\xi$ for $n \in N$ and $\xi \in T_n^*N$. Then (claim) the 2-form $\omega = d\theta$ is actually a symplectic 2-form. So (T^*N, ω) is naturally a symplectic manifold (and is a good model for the phase space of classical physics—think about the case $N = \mathbb{R}^n$)

Exercise 1.4. Take local coordinates x_i on $U \cong \mathbb{R}^n \subset N$. Let y_i be coordinates on the fibres of $T^*U \cong \mathbb{R}^n \times (\mathbb{R}^n)^* \subset T^*N$.

Check that in these coordinates $\theta = \Sigma_{i=1}^n y_i dx_i$ and $\omega := d\theta = \Sigma_{i=1}^n dy_i \wedge dx_i$ and that this is indeed non-degenerate.

The construction of the isomorphism $T_pM \to T_p^*M$ above globalises to give an isomorphism between vector fields and 1-forms: given a vector field v on a symplectic manifold (M, ω), we get a 1-form $\iota_v\omega$ satisfying

$$\langle \iota_v\omega, w\rangle = \omega(v, w)$$

for all vector fields w on M.

Definition 1.5. We say that a vector field v on (M, ω) is *symplectic* if the 1-form $\iota_v\omega$ is closed, i.e., $d\iota_v\omega = 0$. We say that v is *Hamiltonian* if $\iota_v\omega$ is exact, i.e., $\iota_v\omega = -dH$ for some smooth function H on M. In this case, we call the function H a *Hamiltonian* for the vector field v. We write $\mathrm{symp}(M, \omega)$ and $\mathrm{ham}(M, \omega)$ respectively for the spaces of symplectic and Hamiltonian vector fields on M.

Remark 1.6. Note that the Hamiltonian H is only defined up to constant translation. Indeed $C^\infty(M, \mathbb{R}) \to \mathrm{ham}(M, \omega)$ given by $H \mapsto v_H$ (where v_H is the unique vector field satisfying $\iota_{v_H}\omega = -dH$) has kernel precisely the constants $\mathbb{R}$.

Example 1.7. Consider $M = \mathbb{R}^2$ with coordinates (p, q), and the symplectic form $\omega = dp \wedge dq$. Consider the Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$. We have

$$dH = pdp + qdq$$

For any vector field v

$$\text{if} \quad v = a\frac{\partial}{\partial p} + b\frac{\partial}{\partial q} \quad \text{then} \quad \iota_v\omega = adq - bdp.$$

So taking $b = p$ and $a = -q$, we get

$$-dH = \iota_v\omega \quad \text{for} \quad v = -q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q}.$$

So v above is a Hamiltonian vector field.

Remark 1.8. On $\mathbb{R}^2$, every symplectic vector field is Hamiltonian, since every closed 1-form is exact.

Example 1.9. Consider the the torus $M = \mathbb{R}^2/\mathbb{Z}^2$. If (θ_1, θ_2) are coordinates on $\mathbb{R}^2$, then the symplectic form $d\theta_1 \wedge d\theta_2$ descends to a symplectic form ω on M. The vector field $v = \frac{\partial}{\partial \theta_2}$ has $\iota_v \omega = -d\theta_1$, which is closed but not exact (as θ_1 is not a function on M). Hence v is symplectic but not Hamiltonian.

Remark 1.10. Let (X, ω) be a symplectic manifold and assume that $\phi_t : X \to X$ is a smoothly varying family of diffeomorphisms. Then we have

$$\frac{d}{dt}\phi_t^* \omega = d\iota_{v_t}\omega + \iota_{v_t} d\omega,$$

where $v_t = \frac{d\phi_t}{dt}$ is the vector field defined by ϕ_t and $\iota_{v_t} d\omega$ is the 2-form given by $(\iota_{v_t} d\omega)(u, w) = (d\omega)(v_t, u, w)$. (The left hand side of this equation is called the *Lie derivative* of ω in the direction v_t.) In particular, since ω is symplectic, $\frac{d}{dt}\phi_t^*\omega = d\iota_{v_t}\omega$ is 0 if and only if v_t is a symplectic vector field. In particular, specialising to the case where $v_t = v$ is constant in t, we see that v is a symplectic vector field if and only if flowing along v preserves the symplectic form ω.

Example 1.11. Consider $M = \mathbb{R}^2$ with coordinates (p, q) and symplectic form $dp \wedge dq$. Consider the Hamiltonian

$$H = \frac{p^2}{2m} + V(q),$$

where V is some smooth real-valued function. The associated Hamiltonian vector field is

$$v_H = -V'(q)\frac{\partial}{\partial p} + \frac{p}{m}\frac{\partial}{\partial q}.$$

The equations for flow along v_H are therefore

$$\begin{aligned} \dot{p} &= -V'(q) \\ \dot{q} &= \frac{p}{m}. \end{aligned}$$

Interpreting q as position, p as momentum, m as mass and V as potential enegery, these are precisely the equations of motion in classical mechanics.

Remark 1.12. The vector field $X_H = -v_H$ is called the "symplectic gradient of H". It is the vector field dual to dH under $\omega : TM \to T^*M$ and, once we have picked a compatible almost complex structure and metric, is precisely $-J\nabla H$. Since this is orthogonal to ∇H, flowing under X_H preserves H. Formally this follows from $X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0$.

Exercise 1.13. Let S^1 act on $\mathbb{C}^n$ diagonally in the usual way. Show, by differentiating, that the infinitesimal vector field defined by $X_\xi(z) = \frac{d}{dt}|_{t=0}(\exp(t\xi).z)$, is $X_\xi(z_1, \ldots, z_n) = i\xi(z_1, \ldots, z_n)$ for $\xi \in \mathrm{Lie}(S^1) \cong \mathbb{R}$. Putting $\xi = 1$, show that this is really $\Sigma_{i=1}^n r_i \frac{\partial}{\partial \theta_i}$ where $z_j = r_j e^{i\theta_j}$. Furthermore, verify that this is a Hamiltonian vector field with Hamiltonian $\frac{-1}{2}\Sigma_{i=1}^n r_i^2 + c$ for any constant c.

Exercise 1.14. Let X be a symplectic vector field on (M, ω) and γ a path in M. Denote by T_ϵ the surface swept out after time ϵ by the flow along the vector field X starting at γ. Then define $\mathrm{Flux}_\gamma(X) := \lim_{\epsilon \to 0} \frac{\int_{T_\epsilon} \omega}{\epsilon}$. Prove that X is Hamiltonian $\Leftrightarrow$ $\mathrm{Flux}_\gamma(X) = 0$ for every $\gamma \in H_1(M)$.

2 Hamiltonian group actions and moment maps

Definition 2.1. If (X, ω) is a symplectic manifold, the *group of symplectomorphisms of* X is the infinite-dimensional Lie group

$$\mathrm{Symp}(X, \omega) = \{\phi : X \to X \mid \phi^*\omega = \omega\}.$$

By Remark 1.10, the Lie algebra of $\mathrm{Symp}(X, \omega)$ is

$$\mathrm{Lie}(\mathrm{Symp}(X, \omega)) = \mathrm{symp}(X, \omega),$$

the space of symplectic vector fields on X. (This sits inside the space of all smooth vector fields, which is the Lie algebra of the group of diffeomorphisms of X.)

Exercise 2.2. If F and G are smooth functions on X, define the *Poisson bracket* of F and G by

$$\{F, G\} = \omega(v_F, v_G).$$

Show that $\{\ ,\ \}$ is a Lie bracket on $C^\infty(X, \mathbb{R})$, and that the map

$$\begin{aligned} C^\infty(X, \mathbb{R}) &\to \mathrm{symp}(X) \\ F &\mapsto v_F \end{aligned}$$

is a Lie algebra homomorphism with respect to the usual Lie bracket of vector fields. Deduce that the space of Hamiltonian vector fields $\mathrm{ham}(X) \subseteq \mathrm{symp}(X)$ is a Lie subalgebra.

Definition 2.3. We denote by $\mathrm{Ham}(X, \omega)$ the unique (connected) subgroup of $\mathrm{Symp}(X, \omega)$ with

$$\mathrm{Lie}(\mathrm{Ham}(X, \omega)) = \mathrm{ham}(X, \omega).$$

Definition 2.4. Let G be a connected Lie group acting on a symplectic manifold (X, ω). We say that the action is *Hamiltonian* if the action of G factors through

$$G \to \mathrm{Ham}(X, \omega) \to \mathrm{Diff}(X).$$

Equivalently, the action is Hamiltonian if for every $\xi \in \mathrm{Lie}(G)$, associated vector field v_ξ given by

$$(v_\xi)_x = \frac{d}{dt} \exp(t\xi) \cdot x|_{t=0}$$

is Hamiltonian.

Example 2.5. Let $G = T^n = U(1)^n$ be a torus. Then $\mathrm{Lie}(G) = \mathbb{R}^n$ with Lie bracket $[\ ,\ ] = 0$. If we have a Hamiltonian action of G on (X, ω), then if $\xi_1, \ldots, \xi_n$ is a basis for $\mathrm{Lie}(G)$, we get a collection $H_1, \ldots, H_n$ of Hamiltonians on X. Since $[\xi_i, \xi_j] = 0$, passing to the corresponding vector fields v_{H_i}, we have

$$v_{\{H_i, H_j\}} = [v_{H_i}, v_{H_j}] = 0$$

and hence $\{H_i, H_j\}$ is constant for all i, j. With a little more work, we can actually show that

$$\{H_i, H_j\} = 0 \quad \text{for all } i, j.$$

As a slight aside, note that since the v_{H_i} span the tangent space to each orbit, this implies that $\omega = 0$ when restricted to a T^n-orbit. Submanifolds of this type appear sufficiently often in symplectic topology that they have a special name.

Definition 2.6. A submanifold $L \subseteq X$ is called *isotropic* if $\omega|_L = 0$, and *Lagrangian* if in addition $\dim L = \frac{1}{2} \dim X$.

Remark 2.7. Lagrangian submanifolds appear in quantum mechanics as the smallest submanifolds on which a which a wavefunction can be localised. (cf. Heisenberg uncertainty principle)

Example 2.8. Consider T^n acting on $\mathbb{C}^n$ via

$$(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \cdot (z_1, z_2, \ldots, z_n) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_n} z_n).$$

The corresponding Hamiltonians are $H_i = \frac{1}{2}|z_i|^2$ generating the θ_i rotation.

Example 2.9. Consider $S^2 \subseteq \mathbb{R}^3$ with symplectic form given by the area form. (If we identify S^2 with $\mathbb{CP}^1$, then this is also the natural symplectic structure coming from the Fubini-Study form.) Then the $U(1)$ action on S^2 given by rotation about the z-axis is Hamiltonian, and the Hamiltonian H is just the height function (projection to z-coordinate).

Definition 2.10. Let X be a symplectic manifold with a Hamiltonian action of T^n and associated Hamiltonians $H_1, \ldots, H_n$. The *moment map* of the action is $\mu : X \to \mathbb{R}^n$ given by

$$\mu(x) = (H_1(x), \ldots, H_n(x)).$$

Remark 2.11. The moment map for a torus action is well-defined up to translation in $\mathbb{R}^n$.

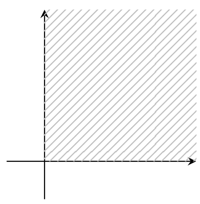
Example 2.12. For T^n acting on $\mathbb{C}^n$ as in Example 2.8, we have

$$\mu(z_1, \ldots, z_n) = (\frac{1}{2}|z_1|^2, \ldots, \frac{1}{2}|z_n|^2).$$

The moment image is

$$\mu(\mathbb{C}^n) = \{(x_1, \ldots, x_n) | x_i \geq 0\}.$$

For example, for $n = 2$, the moment image is the quadrant in $\mathbb{R}^2$ below.



For every point $c = (\frac{1}{2}r_1^2, \frac{1}{2}r_2^2)$ on the interior, we have

$$\mu^{-1}(c) = \{(e^{i\theta_1} r_1, e^{i\theta_2} r_2) \in \mathbb{C}^2\}$$

is a free T^2-orbit.

In general, we have the following cool fact.

Theorem 2.13. *If $\mu : X \to \mathbb{R}^n$ is the moment map for a Hamiltonian torus action, then $\mu(X) \subseteq \mathbb{R}^n$ is a convex polytope.*

We can define moment maps for more general group actions as follows.

Definition 2.14. Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$ and a Hamiltonian action on a symplectic manifold (X, ω). A map $\mu : X \to \mathfrak{g}^*$ is called an *equivariant moment map* if

- $\mu(gx) = \mathrm{Ad}(g)^* \mu(x)$ for all $g \in G$ and $x \in X$, and
- $\iota_{v_\xi}\omega = -d\langle \mu, \xi \rangle$ for $\xi \in \mathfrak{g}$.

Here $\mathrm{Ad}(g)^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is (the inverse of) the dual to the adjoint action $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$, which is the map induced on the tangent space at the identity of G by

$$\begin{aligned} G &\longrightarrow G \\ h &\longmapsto ghg^{-1}. \end{aligned}$$

Exercise 2.15. In the setup of Definition 2.14, assume that $\mu : X \to \mathfrak{g}^*$ satisfies

$$\iota_{v_\xi}\omega = -d\langle \mu, \xi \rangle.$$

Show that the following are equivalent.

(a) $\mu(gx) = \mathrm{Ad}(g)^* \mu(x)$ for all $g \in G$ and $x \in X$.

(b) The map

$$\tilde{\mu} : \mathfrak{g} \to C^\infty(X, \mathbb{R}) \quad \text{given by} \quad \tilde{\mu}(\xi)(x) = \langle \mu(x), \xi \rangle$$

is a homomorphism of (left) G-modules, where G acts on $\mathfrak{g}$ via the adjoint action, and on $C^\infty(X, \mathbb{R})$ by $(gf)(x) = f(g^{-1}x)$.

(c) The map $\tilde{\mu} : \mathfrak{g} \to C^\infty(X, \mathbb{R})$ is a homomorphism of $\mathfrak{g}$-modules.

(d) The map $\tilde{\mu} : \mathfrak{g} \to C^\infty(X, \mathbb{R})$ is a homomorphism of Lie algebras.

Lemma 2.16. *Let G be a compact Lie group with a Hamiltonian action on a symplectic manifold X. Then there exists an equivariant moment map $\mu : X \to \mathfrak{g}^*$.*

Proof. We have a commutative diagram of G-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(X, \mathbb{R}) & \longrightarrow & \mathrm{ham}(X) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & \nwarrow^{\tilde{\mu}} & \uparrow & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & V & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

with exact rows, where

$$V = \{(F, \xi) \in C^\infty(X, \mathbb{R}) \oplus \mathfrak{g} \mid v_F = v_\xi\}.$$

Since the bottom row is an exact sequence of finite-dimensional G-modules with G compact, it has a G-equivariant splitting. In particular, we can construct a map of G-modules $\tilde{\mu} : \mathfrak{g} \to C^\infty(X, \mathbb{R})$ covering $\mathfrak{g} \to \mathrm{ham}(X)$ as shown, which induces an equivariant moment map $\mu : X \to \mathfrak{g}^*$. $\square$

3 Symplectic reduction

The aim of symplectic reduction is to find a way of taking quotients of symplectic manifolds under group actions. For example, suppose we have a free action of S^1 on a symplectic manifold X. Naïvely, we might hope to find a symplectic structure on the topological quotient X/S^1. However, this cannot possibly work, since

$$\dim\left(X/S^1\right) = \dim X - 1$$

is odd, and symplectic manifolds always have even dimension.

Instead, we use the following trick: if the action of S^1 is Hamiltonian, then we can cut down the dimension by 1 by restricting the action to a level set of the moment map. Taking the quotient of this new manifold, we at least get something even-dimensional. The following proposition ensures that we get a natural symplectic structure.

Proposition 3.1. *Let (X, ω) be a symplectic manifold with a Hamiltonian action of a compact Lie group G and associated equivariant moment map $\mu : X \to \mathfrak{g}^* = \mathrm{Lie}(G)^*$. If $0 \in \mathfrak{g}^*$ is a regular value of μ such that G acts freely on $M = \mu^{-1}(0)$, then M/G is a symplectic manifold with symplectic structure induced by ω.*

Definition 3.2. The symplectic manifold M/G of Proposition 3.1 is called the *symplectic reduction of* X, and is denoted by $X//_0G$ if the choice moment map is understood.

Remark 3.3. If $\mu : X \to \mathfrak{g}^*$ is any equivariant moment map and $c \in \mathfrak{g}^*$ is invariant under the coadjoint action of G, then $\mu'(x) = \mu(x) - c$ defines another moment map. (In particular if G is abelian, then we can do this for any $c \in \mathfrak{g}^*$.) If c is a regular value for μ such that the G-action on $\mu^{-1}(c)$ is free, then we can form another symplectic reduction

$$(\mu')^{-1}(0)/G = \mu^{-1}(c)/G.$$

In general, different values of c will give different symplectic reductions. We often write

$$X//_cG = \mu^{-1}(c)/G,$$

where the choice of moment map is implicit.

Sketch of proof of Proposition 3.1. Write $B = M/G$. Since the G-action on M is free, B is a manifold with tangent space

$$\begin{aligned} T_pB &= T_{\tilde{p}}M/\mathfrak{g} \\ &= T_{\tilde{p}}M/(\mathbb{R}v_{H_1} \oplus \cdots \oplus \mathbb{R}v_{H_n}), \end{aligned}$$

where $\tilde{p} \in M$ is any preimage of $p \in B$, and $H_1, \ldots, H_n$ are the components for the moment map on X corresponding to some basis for $\mathfrak{g}^*$. Define the symplectic form $\omega_B \in \Omega^2(B)$ by

$$(\omega_B)_p(v, w) = \omega_{\tilde{p}}(\tilde{v}, \tilde{w}),$$

where $\tilde{p} \in M$ is a preimage of $p \in B$, and $\tilde{v}, \tilde{w} \in T_{\tilde{p}}M$ are preimages of $v, w \in T_pB$. The form ω_B is well-defined since ω is invariant under G and the H_i are constant restricted to M (so that $\omega(v_{H_i}, \tilde{w}) = 0$ for all $w \in T_pM$). One can check that ω_B is indeed a symplectic form. □

Example 3.4 ($\mathbb{CP}^n$)**.** Consider S^1 acting on $\mathbb{C}^{n+1}$ diagonally by

$$e^{i\theta}(z_0, \ldots, z_n) = (e^{i\theta} z_0, \ldots, e^{i\theta} z_n).$$

The moment map (which is just a Hamiltonian in this case) is

$$\mu(z_0, \ldots, z_n) = \frac{1}{2}|z_0|^2 + \cdots + \frac{1}{2}|z_n|^2.$$

So for every $r > 0$, we get a symplectic structure on

$$\mu^{-1}(r^2/2)/S^1 = S^{2n+1}/S^1 = \mathbb{CP}^n.$$

The associated symplectic form is the unique form ω_{FS} such that

$$\pi^* \omega_{FS} = \omega_{\mathbb{C}^{n+1}}|_{S^{2n+1}},$$

where $\pi : S^{2n+1} \to \mathbb{CP}^n$ is the quotient map.

Exercise 3.5. Show that for an appropriate choice of r, ω_{FS} agrees with the explicit expression for the Fubini-Study form in the lecture on Kähler geometry.

Example 3.6. The S^1-action of Example 3.4 factors through the T^{n+1}-action

$$(e^{i\theta_0}, \ldots, e^{i\theta_n})(z_0, \ldots, z_n) = (e^{i\theta_0} z_0, \ldots, e^{i\theta_n} z_n),$$

via the diagonal map

$$\begin{aligned} S^1 &\longrightarrow T^{n+1} \\ e^{i\theta} &\longmapsto (e^{i\theta}, \cdots, e^{i\theta}). \end{aligned}$$

So we get a residual action of $T = T^{n+1}/S^1$ on the symplectic reduction $\mathbb{CP}^n = \mathbb{C}^{n+1}//_{r^2/2} S^1$ with moment map

$$\mu' : \mathbb{CP}^n = H^{-1}(r^2/2)/S^1 \longrightarrow \mathbb{R}^{n+1},$$

induced by the moment map

$$\begin{aligned} \mu : \mathbb{C}^{n+1} &\longrightarrow \mathbb{R}^{n+1} \\ (z_0, \ldots, z_n) &\longmapsto \left(\frac{1}{2}|z_0|^2, \ldots, \frac{1}{2}|z_n|^2\right) \end{aligned}$$

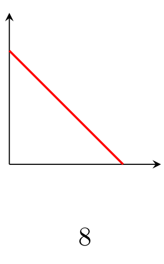
for the T^{n+1}-action. Note that the image of μ' is contained in

$$\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = r^2/2\},$$

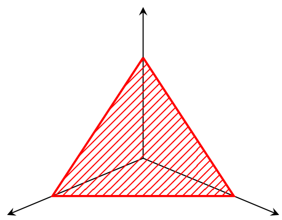
which, up to translation, is the same as

$$\mathrm{Lie}(T)^* \subseteq \mathrm{Lie}(T^{n+1})^* = \mathbb{R}^{m+1}.$$

So μ' does make sense as a moment map. For example, the moment image of $\mathbb{CP}^1$ is the interval shown below.



The moment image of $\mathbb{CP}^2$ is the triangle below.



In general, the moment image of $\mathbb{CP}^n$ is the n-simplex

$$\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0,\, x_0 + x_1 + \cdots + x_n = 1\}.$$