Spec and Proj

1 Introduction

In what follows, we always work over the field of complex numbers $\mathbb{C}$. For purely algebraic considerations, the only key property of $\mathbb{C}$ that we use is that $\mathbb{C}$ is algebraically closed, i.e. that every non-constant polynomial with complex coefficients has at least one complex root. Furthermore, using $\mathbb{C}$ gives us the possibility to use complex topology, which is useful for illustrative purposes and also to make contact with complex or differential geometry.

The main point is the idea of *duality* between spaces and rings, or algebras, of functions on spaces. This duality is non-trivial in the sense that it exchanges difficult notions on one side with easy notions on the other side. It is also a contravariant operation: all the natural maps on one side go to natural maps on the other side going in the opposite direction.

The idea of duality between spaces and functions is a vague general idea that can be made precise in specific contexts.

Example: Let V be a finite-dimensional vector space over $\mathbb{C}$. Natural functions on V are given by linear functions. The set of linear functions on V is a finite-dimensional vector space V^* called the *dual* of V. It is possible to recover the space V from its space of functions V^* in the following way. Every $x \in V$ defines a linear function

$$\begin{array}{rccc} ev_x \colon & V^* & \longrightarrow & \mathbb{C} \\ & f & \mapsto & f(x) \end{array}$$

called the *evaluation map at x*. By dimension considerations, one shows that every linear function on V^* is of this form, i.e. the natural linear map $V \to V^{**}$ given by the evaluation is an isomorphism.

Example: Gelfand-Naimark, classification of commutative C^*-algebras. Let X be a Hausdorff compact topological space. We consider the space $C(X)$ of continuous complex-valued functions $f \colon X \to \mathbb{C}$. It is a vector space over $\mathbb{C}$, in general infinite dimensional. It also has a natural algebra structure (with unit) given by the product of functions, and a norm defined by $||f|| = sup_{x\in X}|f(x)|$. It is easy to show that for every f and g in $C(X)$ we have $||f.g|| \leq ||f||.||g||$, and that $C(X)$ with the norm $||.||$ is a complete normed vector space. In other words, $C(X)$ has a natural structure of a commutative *Banach algebra*. Furthermore, we define a map $f \mapsto f^*$ from $C(X)$ to itself by $f^*(x) = \overline{f(x)}$ where $z \mapsto \overline{z}$ is the complex conjugation. The map $*$ is antilinear, it is a morphism of rings, and it satisfies obvious compatibilities with the norm, such that $||f.f^*|| = ||f||^2$. A Banach algebra with such an operation $*$ is called a *C^*-algebra* and we have just shown how to associate to every Hausdorff compact topological space X its C^*-algebra of continuous functions $C(X)$.

Every $x \in X$ defines a $*$-homomorphism, i.e. a continuous algebra homomorphism compatible with the operation $*$,

$$\begin{array}{rccc} ev_x: & C(X) & \longrightarrow & \mathbb{C} \\ & f & \mapsto & f(x) \end{array}$$

called the *evaluation map at* x. It is possible to show that any $*$-homomorphism is of this form and so that a compact topological space X can be recovered from the C^*-algebra $C(X)$. One can also show that any commutative C^*-algebra with unit is of the form $C(X)$ for some Hausdorff compact topological space X called the *spectrum* of the C^*-algebra. This means that the map $X \mapsto C(X)$ identifies a notion of "space", the notion of Hausdorff compact topological space, with a notion of "algebra of functions", the notion of commutative C^*-algebra with unit.

Exercise: Show that if one forgets the operation $*$ on $C(X)$, one gets something like a complexification of X. For example, let $X = S^1$ be a circle parametrized by some angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and let R be the algebra over $\mathbb{C}$ of finite Fourier series $f = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ on S^1. Show that the space of $*$-homomorphisms $R \to \mathbb{C}$ can be identified with S^1 whereas the space of algebra homomorphisms $R \to \mathbb{C}$ can be identified with $\mathbb{C}^*$. Hint: Write $f = \sum_{n\in\mathbb{Z}} a_n z^n$ so that $R = \mathbb{C}[z, z^{-1}]$.

2 Spec

In the introduction, we have explained the general idea of duality between spaces and rings of functions, and we have given two examples: one linear algebraic and the other topological. We present in what follows an algebro-geometric example. The functions considered will be *algebraic*. More precisely, the starting point will be to take as our ring of functions the ring of *polynomials* $\mathbb{C}[x_1, ..., x_n]$. *Algebraic geometry* is concerned with the geometry of spaces whose functions are of this kind. Using different starting rings, it is possible to develop in a parallel way different theories, such as *analytic geometry*, starting from the ring $\mathbb{C}\{x_1, \ldots, x_n\}$ of power series converging on an open neighborhood of the origin, or *formal geometry*, starting from the ring $\mathbb{C}[[x_1, \ldots, x_n]]$ of all formal power series.

An *affine (algebraic) variety* over $\mathbb{C}$ is the vanishing locus in $\mathbb{C}^n$ of finitely many polynomials $p_1, \ldots, p_k \in \mathbb{C}[x_1, \ldots, x_n]$:

$$\{x \in \mathbb{C}^n | p_1(x) = \ldots = p_k(x) = 0\}.$$

Affine varieties define a notion of "space" in algebraic geometry.

We take as our notion of "ring of functions" the *finitely generated unital commutative algebras* over $\mathbb{C}$. Let R be such an algebra. By hypothesis, there exists a finite number of generators $x_1, \ldots, x_n$ of R, i.e. R is the quotient of the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ by some ideal of relations. By the Hilbert basis theorem, i.e. the fact that the ring $\mathbb{C}[x_1, \ldots, x_n]$ is *Noetherian*, the ideal of relations is itself finitely generated. Let $p_1, \ldots, p_k \in \mathbb{C}[x_1, \ldots, x_n]$ be some generators of the relations. We denote by $(p_1, ..., p_k)$ the *ideal* generated by the p_i's, i.e. the set of all elements of the form $\sum_{i=1}^k f_i p_i$, $f_i \in \mathbb{C}[x_1, \ldots, x_n]$, i.e. the

$\mathbb{C}[x_1, ..., x_n]$-submodule of $\mathbb{C}[x_1, \ldots, x_n]$ generated by the p_i's. Then we have an isomorphism:

$$R = \mathbb{C}[x_1, \ldots, x_n]/(p_1, \ldots, p_k).$$

We associate to such a ring R the affine variety:

$$X = \{x \in \mathbb{C}^n \mid p_1(x) = \ldots = p_k(x) = 0\}.$$

For example, we associate $\mathbb{C}^n$ to the ring $\mathbb{C}[x_1, \ldots, x_n]$. Each point $x \in X$ defines a ring homomorphism:

$$\begin{array}{cccc} ev_x\colon & R & \longrightarrow & \mathbb{C} \\ & f & \mapsto & f(x) \end{array}$$

called the *evaluation map at* x. The *spectrum* of the ring R, denoted by Spec R, is the set of all ring homomorphisms $R \to \mathbb{C}$ (since all our rings are in fact $\mathbb{C}$-algebras, "ring homomorphism" will always mean "homomorphism of algebras"). The evaluation map induces a map $X \longrightarrow$ Spec R defined by $x \mapsto ev_x$. It is possible to show that this map is a bijection, i.e. every ring homorphism $R \longrightarrow \mathbb{C}$ is the evaluation ev_x for some $x \in X$. In other words, we have

$$\{p_1 = \ldots = p_k = 0 \text{ in } \mathbb{C}^n\} = \text{Spec } \mathbb{C}[x_1, \ldots, x_n]/(p_1, \ldots, p_k).$$

Exercise: Check this is true by assuming that this is known for $R = \mathbb{C}[x_1, \ldots, x_n]$, i.e. that Spec $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}^n$. The space $\mathbb{C}^n$ seen as an affine variety is called the *affine space of dimension n*.

Conversely, starting from an affine variety X, it is possible to construct a ring of functions on X by quotienting $\mathbb{C}[x_1, \ldots, x_n]$ by the ideal of functions vanishing on X. The resulting finitely generated unital algebra is called the *coordinate ring of* X and is denoted by $\mathcal{O}_X$.

Example: We have an isomorphism $\mathbb{C}[x, y]/(y) = \mathbb{C}[x]$. The ring $\mathbb{C}[x, y]/(y)$ is the coordinate ring of the x-axis in $\mathbb{C}^2$. The isomorphism with $\mathbb{C}[x]$ corresponds to the obvious geometric fact that the x-axis in $\mathbb{C}^2$ is a copy of $\mathbb{C}$.

Example: We have an isomorphism $\mathbb{C}[x, y]/(y-x^2) = \mathbb{C}[x]$ given by $y = x^2$. The ring $\mathbb{C}[x, y]/(y-x^2)$ is the coordinate ring of the parabola $y = x^2$ in $\mathbb{C}^2$. The isomorphism with $\mathbb{C}[x]$ corresponds to the geometric fact that the projection of the parabola on the x-axis is an isomorphism of affine varieties.

Example: Let $f \in \mathbb{C}[x]$ and $R = \mathbb{C}[x, y]/(1 - yf(x))$. The ring R is the coordinate ring of the graph in $\mathbb{C}^2$ of $y = 1/f(x)$ for x such that $f(x) \neq 0$. The projection on the x-axis gives an isomorphism between this affine variety and the x-axis minus the points where $f = 0$. In other words, we have isomorphisms Spec $R = \mathbb{C}_x \setminus \{f = 0\}$ and $R = \mathbb{C}[x][f(x)^{-1}]$. The ring $R = \mathbb{C}[x][f(x)^{-1}]$ is generally denoted by $\mathbb{C}[x]_{(f)}$ and is called the *localization* of $\mathbb{C}[x]$ along the multiplicative set $\{1, f, f^2, \ldots\}$ consisting of powers of f.

We have explained how to associate an affine variety to a finitely generated unital commutative algebra R, by taking its spectrum Spec R, and how to associate a finitely generated unital commutative algebra to an affine variety X, by taking its ring of functions $\mathcal{O}_X$. But the constructions $R \mapsto$ Spec R and $X \mapsto \mathcal{O}_X$ are not in general the inverse of each other.

Example: Let $R_1 = \mathbb{C}[x]/(x) = \mathbb{C}$ and $R_2 = \mathbb{C}[x]/(x^2) = \mathbb{C} \oplus \mathbb{C}x$. The equations $x = 0$ and $x^2 = 0$ define the same subset X of $\mathbb{C}$, so we have $X =$ Spec $R_1 =$ Spec R_2, but $\mathcal{O}_X = \mathbb{C} = R_1 \neq R_2$.

To solve this difficulty and have a better correspondence between spaces and rings of functions, there are two possible approaches:

1. Restrict the notion of ring. Rather than to consider rings of the form $\mathbb{C}[x_1, \ldots, x_n]/I$ for a general ideal I of relations, one can consider only rings of the form $\mathbb{C}[x_1, \ldots, x_n]/I$ for an ideal which is *radical*, i.e. such that $\sqrt{I} = I$ where $\sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^N \in I \text{ for some } N > 0\}$. With this notion of rings, one obtains a one-to-one correspondence bewteen affine varieties and rings. This results from the *Nullstellensatz* which states that $\mathcal{O}_{\text{Spec } \mathbb{C}[x_1,\ldots,x_n]/I} = \mathbb{C}[x_1, \ldots, x_n]/\sqrt{I}$.

2. Enlarge the notion of space. Rather than considering spaces as subsets of $\mathbb{C}^n$, one could remember more information. For example, one would like to say that the space associated to the ring $\mathbb{C}[x]/(x^2)$ is different from the one associated to the ring $\mathbb{C}[x]/(x)$, the former being a first order "infinitesimal thickening" of the latter. The notion of *scheme* enlarges the notion of space to take into account the possibility of "infinitesimal thickenings". There is a one-to-one correspondence between affine schemes and rings of the form $\mathbb{C}[x_1, \ldots, x_n]/I$.

We will describe the first approach in more detail. Given an ideal $I \lhd \mathbb{C}[x_1, \ldots, x_n]$ we have the associated affine variety $X = \mathbb{V}(I)$. To an affine variety X we have associated the ideal of polynomials which vanish on it $\mathbb{I}(X) = \{f \in C[x_1, \ldots, x_n] | f(x) = 0 \text{ for all } x \in\}$ and have defined its ring of functions to be $\mathcal{O}_X = \mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(X)$. It is straightforward to check that $X = \mathbb{V}(\mathbb{I}(X))$. But in general $\mathbb{I}(\mathbb{V}(I)) \neq I$. For example, $\mathbb{I}(\mathbb{V}(x^2)) = (x)$. One of the many forms of the Nullstellensatz states that for an affine variety $X = \mathbb{V}(I)$ we have $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$. So the ring of functions on $X = \text{Spec } \mathbb{C}[x_1, \ldots, x_n]/I$ is just $\mathbb{C}[x_1, \ldots, x_n]/\sqrt{I}$, that is $\mathcal{O}_{\text{Spec } \mathbb{C}[x_1,\ldots,x_n]/I} = \mathbb{C}[x_1, \ldots, x_n]/\sqrt{I}$. This gives a bijection between radical ideals and affine varieties which in turn sets up a contravariant equivalence between the category of affine varieties and the category of finitely-generated reduced algebras (i.e.Abstractwithout nilpotent elements, it is easy to see that $C[x_1, \ldots, x_n]/I$ is reduced if and only if I is radical).

For instance, given a morphism of varieties $F : X \to Y$ we have an induced homomorphism on the rings of functions $F^* : \mathbb{C}[y_1, \ldots, y_n]/J \to \mathbb{C}[x_1, \ldots, x_m]/I$ given by $y_i \mapsto y_i \circ F$.

Going the other way, given a homomorphism between finitely-generated reduced $\mathbb{C}$-algebras, $\phi : S \to R$, we can choose representations

$$R = \mathbb{C}[x_1, \ldots, x_n]/I, \;\; S = \mathbb{C}[y_1, \ldots, y_m]/J$$

and define an induced morphism between the associated affine varieties $X = \mathbb{V}(I)$ and $Y = \mathbb{V}(J)$ to be the restriction of the polynomial map $\phi_* : \mathbb{C}^n \to \mathbb{C}^m$, $x \mapsto (f_1(x), \ldots, f_n(x))$ where $f_i \in \mathbb{C}[x_1, \ldots, x_n]$ represents $\phi(y_i)$. This gives a well defined morphism when restricted to X. Then, if $x \in X$ and $p \in J$ we have $p(\phi_*(x)) = p(f_1(x), \ldots, f_n(x))$ but $p(f_1, \ldots, f_n) \equiv p(\phi(y_1), \ldots, \phi(y_m)) \equiv \phi(p) \equiv 0 \mod I$ so $p(\phi_*(x)) = 0$ and $\phi_*(x) \in Y$. It is straight forward to check that $(F \circ G)^* = G^* \circ F^*$ and $(\phi \circ \psi)_* = \psi_* \circ \phi_*$ so that isomorphisms in one category induce isomorphisms in the other and vice versa.

Example: Let $X = \mathbb{C}$, $Y = \mathbb{V}(y - x^2) \subset \mathbb{C}^2$. The morphism $F : X \to Y$, $t \mapsto (t, t^2)$ has inverse morphism $(x, y) \mapsto x$. This corresponds to the isomorphisms

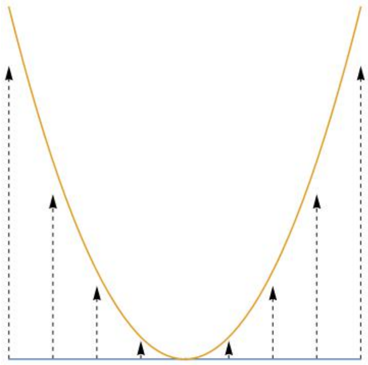

Figure 1: $y = x^2$.

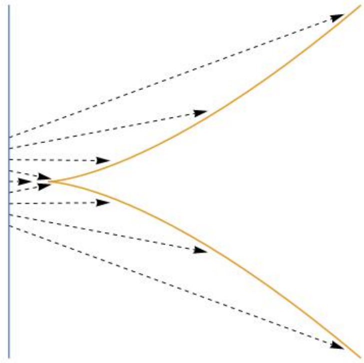

Figure 2: $y^2 = x^3$.

of rings of functions $\mathbb{C}[x, y]/(y - x^2) \to \mathbb{C}[t]$,

$$x \mapsto x \circ F(t) = t, y \mapsto y \circ F(t) = t^2$$

Example: Let $X = \mathbb{C}$, $Y = \mathbb{V}(y^2 - x^3) \subset \mathbb{C}^2$. The morphism $F : X \to Y$, $t \mapsto (t^2, t^3)$ induces the following homomorphism on the rings of functions $\mathbb{C}[x, y]/(y^2 - x^3) \to \mathbb{C}[t]$,

$$x \mapsto x \circ F(t) = t^2, \;\; y \mapsto y \circ F(t) = t^3.$$

This is clearly not onto and so this is not an isomorphism of varieties. (In fact, there is no surjective homomorphism $\mathbb{C}[x, y]/(y^2 - x^3) \to \mathbb{C}[t]$. Suppose $x \mapsto p(t)$ and $y \mapsto q(t)$ with $f(p(t), q(t)) = t$ for some $f \in \mathbb{C}[x, y]$. Then, differentiating, $f_x(p, q)p' + f_y(p, q)q' = 1$ so p' and q' are coprime and in particular p and q cannot have any multiple zeros in common. But we also have $p^3 = q^2$ so p and q must have the same roots but neither can have any single roots. This is a contradiction.)

There is a table of dualities:

$$
\begin{array}{rcl}
\mathbb{C}^n & \longleftrightarrow & \mathbb{C}[x_1,\ldots,x_n] \\
X = \{x \in \mathbb{C}^n | p_1(x) = \ldots = p_k(x) = 0\} & \longleftrightarrow & R = \mathbb{C}[x_1,\ldots,x_n]/(p_1,\ldots,p_k) \\
 & \longleftrightarrow & I = (p_1,\ldots,p_k) \\
X_1 \cap X_2 & \longleftrightarrow & I_1 + I_2 \\
X_1 \cup X_2 & \longleftrightarrow & I_1 \cap I_2 \\
X \text{ irreducible} & \longleftrightarrow & I \text{ prime ideal} \\
\text{Points of } X & \longleftrightarrow & \text{Maximal ideals of } R
\end{array}
$$

There is an alternative description of Spec R as the set of maximal ideals of R; we will now show that this is equivalent to our original definition. Firstly, if $\mathfrak{m}$ is a maximal ideal of R then we let $f_\mathfrak{m}$ be the quotient morphism $R \to R/\mathfrak{m}$. It is a basic fact (see the accompanying exercises) that $R/\mathfrak{m} \cong \mathbb{C}$, so we have a morphism $f_\mathfrak{m} : R \to \mathbb{C}$. In the reverse direction, given a ring homomorphism $f : R \to \mathbb{C}$, we let $\mathfrak{m}_\mathfrak{f} = \ker f$. Then m_f is an ideal of R, and it is maximal because $\operatorname{Im} f = \mathbb{C}$ (since f is an algebra morphism and $\mathbb{C} \hookrightarrow R$), and so $R/\mathfrak{m}_\mathfrak{f}$ is a field. It is straightforward to check that these constructions are mutually inverse.

Exercise: Show that Spec $\mathbb{C}[x_1,\ldots,x_n] = \mathbb{C}^n$, i.e. that for every maximal ideal I of $\mathbb{C}[x_1,\ldots,x_n]$ is of the form $(x_1 - a_1,\ldots,x_n - a_n)$ for some (unique) $a \in \mathbb{C}^n$.

Let $a \in \mathbb{C}^n$ and consider the evaluation map $ev_a : \mathbb{C}[x_1,\ldots,x_n] \to \mathbb{C}$ defined by $ev_a(f) = f(a)$. It is clear that the ideal $(x_1 - a_1,\ldots,x_n - a_n)$ is contained in the kernel of ev_a. Now, given $g \in \mathbb{C}[x_1,\ldots,x_n]$ we can write it in the form

$$
g(x_1,\ldots,x_n) = g(a_1,\ldots,a_n) + \sum c_i(x_i - a_i) + \sum c_{ij}(x_i - a_i)(x_j - a_j) + \cdots
$$

and see that if $g(a) = 0$ then $g \in \operatorname{Ker}(ev_a)$. Hence, $\operatorname{kernel}(ev_a) = (x_1,\ldots,x_n)$.

Conversely, given a maximal ideal $M \lhd \mathbb{C}[x_1,\ldots,x_n]$ we know by exercise 1 that the quotient is isomorphic to $\mathbb{C}$. If a_i is identified with $x_i + M$ under this isomorphism then $(x_1 - a_1,\ldots,x_n - a_n)$ is contained in the natural projection $\mathbb{C}[x_1,\ldots,x_n] \to \mathbb{C}$ since $x_i - a_i \mapsto a_i - a_i = 0$. By maximality, we conclude that it must be equal to M.

Exercise: Using that Spec $\mathbb{C}[x_1,\ldots,x_n] = \mathbb{C}^n$, prove the Nullstellensatz, i.e. for $I \lhd \mathbb{C}[x_1,\ldots,x_n]$ we have $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.

Checking that $\sqrt{I} \subset \mathbb{I}(\mathbb{V}(I))$ is straightforward.

Let $f \in \mathbb{I}(\mathbb{V}(I))$. Now introduce a new variable t and consider the ideal $I' = (I, ft - 1) \in \mathbb{C}[x_1,\ldots,x_n,t]$. The corresponding variety $\mathbb{V}(I')$ must be empty. For if not, there exists some $(a_1,\ldots,a_n,b) \in \mathbb{C}^{n+1}$ such that $f(a)b = 1$ and yet $f(a) = 0$ since $a \in \mathbb{V}(I)$ and $f \in \mathbb{I}(\mathbb{V}(I))$. Now it must be the case that $1 \in I'$. Again, supose not, then I' lies in some maximal ideal $M \lhd \mathbb{C}[x_1,\ldots,x_n]$ but then we would have $\emptyset \neq \mathbb{V}(M) \subset \mathbb{V}(I') = \emptyset$. So write

$$
1 = \sum_{i=1}^{n} c_i f_i + c_0(ft - 1)
$$

for some generators f_i of I and $c_i \in \mathbb{C}[x_1,\ldots,x_n,t]$. Formally substituting $t = 1/f$ and then multiplying through by a sufficiently higher power of f we get an expression of the form

$$f^N = \sum_{i=1}^{n} c'_i f_i$$

and conclude that $f \in \sqrt{I}$.

Remark: What we call spectrum is what is generally called the *maximal spectrum*, since it is the set of maximal ideals. What is generally called the spectrum is the set of prime ideals. If we only consider varieties over $\mathbb{C}$, the maximal spectrum is enough.

Exercise: Why the name "spectrum"? Given $A \in M_{n\times n}(\mathbb{C})$ consider the (commutative!) $\mathbb{C}$-algebra generated by A. What is its Spec?

3 Proj

For the complex topology, any connected component of an affine variety is either a single point or non-compact. Indeed, if X is a connected component of an affine variety, the functions x_i can't have a maximum on X unless they are constant so if X is compact then each x_i is equal to some constant a_i on X, i.e. X is the point $(a_1, \dots, a_n)$. This non-compactness of affine varieties is a motivation to introduce projective spaces and projective varieties.

Let $\mathbb{P}^n$ be the space of complex lines in $\mathbb{C}^{n+1}$. We denote $x = (x_0, \dots, x_n)$ the points of $\mathbb{C}^{n+1}$. As a line is determined by some direction vector, well-defined up to scaling by non-zero scalars, $\mathbb{P}^n$ can be identified with the quotient

$$(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*.$$

Making use of the natural Hermitian structue on $\mathbb{C}^{n+1}$, i.eȯf the norm $|x| = \sum_{i=0}^{n} |x_i|^2$, we see that a line is well-defined by a unitary vector up to scaling by some scalar of unit norm. This gives an identification of $\mathbb{P}^n$ with the quotient S^{2n+1}/S^1 where S^k denotes the k-dimensional sphere. In particular, $\mathbb{P}^n$ is compact for the complex topology. The space

$$\mathbb{P}^n = \{\text{lines in } \mathbb{C}^{n+1}\} = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* = S^{2n+1}/S^1$$

is called the *projective space of dimension n*.

In order to describe the projective space in terms of algebra, it is enough to work on $\mathbb{C}^{n+1}$ in a $\mathbb{C}^*$-equivariant way. More precisely, we consider the action of $\mathbb{C}^*$ on $\mathbb{C}[x_0, \dots, x_n]$ with $\lambda \in \mathbb{C}^*$ acting on a degree d homogeneous polynomial by multiplication by λ^d. Let us denote $R = \mathbb{C}[x_0, \dots, x_n]$ and R_d the finite dimensional vector space of degree d homogeneous polynomials. As any polynomial can be uniquely decomposed in homogeneous pieces, we have the direct sum decomposition

$$R = \bigoplus_{d\geq 0} R_d.$$

This decomposition is the eigenspace decomposition for the $\mathbb{C}^*$-action on R: R_d is the eigenspace of the action of $\lambda \in \mathbb{C}^*$ for the eigenvalue λ^d. The space R_d is called the *weight space of weight d* of the $\mathbb{C}^*$-action on R.

Exercise: We say that a finite dimensional representation W of $\mathbb{C}^*$ is *algebraic* if the corresponding group homomorphism $\mathbb{C}^* \longrightarrow GL(W)$ is algebraic.

Show that the data of a $\mathbb{Z}$-graded vector space $V = \bigoplus_{d\in\mathbb{Z}} V_d$ with finite-dimensional graded components V_d is equivalent to the data of a vector space V with a $\mathbb{C}^*$-action such that every $v \in V$ is contained in a finite-dimensional $\mathbb{C}^*$-invariant subspace on which the $\mathbb{C}^*$-action is algebraic.

Let f be a degree d homogeneous polynomial. Then, for every $x \in \mathbb{C}^{n+1}$, we have $f(x) = 0$ if and only if $f(\lambda x) = \lambda^d f(x) = 0$. So it makes sense to say that f vanishes at some point $[x] \in \mathbb{P}^n$. An ideal I of $R = \bigoplus_{d\geq 0} R_d$ is called *homogeneous* if $I = \bigoplus_{d\geq 0}(I \cap R_d)$ or equivalently if I is generated by homogeneous polynomials. It makes sense to say that all the functions f in a given homogeneous ideal I of R vanish at some point $[x] \in \mathbb{P}^n$ and so every homogeneous ideal I of R defines a subvariety of $\mathbb{P}^n$. In other words, the subvarieties of $\mathbb{C}^{n+1}$ defined by the homogeneous ideals of R are precisely the $\mathbb{C}^*$-invariant subvarieties of $\mathbb{C}^{n+1}$ and so define subvarieties of $\mathbb{P}^n$. The $\mathbb{C}^*$-invariant subvariety of $\mathbb{C}^{n+1}$ associated to a subvariety of $\mathbb{P}^n$ is called the *affine cone* of this subvariety. A subvariety of $\mathbb{P}^n$ is called a *projective variety.*

Let I be a homogeneous ideal of R. Then the quotient ring R/I is also graded:

$$R/I = \bigoplus_{d\geq 0} R_d/(I \cap R_d).$$

We define the *homogeneous spectrum* of the graded ring R/I, denoted Proj R/I, as being the set of homogeneous ideals of R/I which are maximal amongst homogeneous ideals, except the so called *irrelevant ideal* $(R/I)^+ = \bigoplus_{d\geq 1} R_d/(I\cap R_d)$, which is the ideal of the origin in the $\mathbb{C}^*$-invariant subvariety of $\mathbb{C}^{n+1}$ defined by I, and thus corresponds to the empty set after passing to projective space. It is possible to show that Proj R/I can be naturally identified with the set of points of the projective variety defined by I. In other words, if $p_1, \ldots, p_k$ are homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$, we have

$$\{p_1 = \ldots = p_k = 0 \text{ in } \mathbb{P}^n\} = \text{Proj } \mathbb{C}[x_0, \ldots, x_n]/(p_1, \ldots, p_k).$$

Example: Let C be the projective curve in $\mathbb{P}^2$ defined by the degree 3 homogeneous polynomial $zy^2 - x(x^2 - z^2)$. Its affine cone (with real coordinates) is a union of lines through the origin. Consider its intersection with the plane $\{z = 1\}$. We can identify $C \cap \{z = 1\}$ with the plane affine curve $Y^2 = X(X^2-1)$ via the map $[x : y : z] \mapsto (x/z, y/z)$. We think of the remaining points on C as "at infinity" with respect to this affine set.

Exercise: Rewrite the table of dualities by replacing Spec by Proj, ideals by homogeneous ideals, rings by graded rings, $\mathbb{C}^n$ by $\mathbb{P}^n$...

Exercise: Work out the details of $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$. Pass back and forth between homogeneous polynomials in $\mathbb{P}^n$ and polynomials in $\mathbb{C}^n$.

A degree d homogeneous polynomial is a function on $\mathbb{C}^{n+1}$ but is not a function on $\mathbb{P}^n$. A natural question is: what is the interpretation of the degree d homogeneous polynomials in terms of the projective space $\mathbb{P}^n$? For $[x] \in \mathbb{P}^n$, a degree d homogeneous polynomial defines a function on the line $\mathbb{C}x$ which is of degree d. For example, if $d = 1$, the coordinates x_i are linear functionals on the line $\mathbb{C}x$. We call $\mathcal{O}_x(-1)$ the line $\mathbb{C}x \subset \mathbb{C}^{n+1}$ and $\mathcal{O}(-1)$ the union of these lines, i.e.

$$\mathcal{O}(-1) = \{(v, [x]) \in \mathbb{C}^{n+1} \times \mathbb{P}^n \mid v \in [x]\}.$$

As the fibres of the natural map from $\mathcal{O}(-1)$ to $\mathbb{P}^n$, $(v, [x]) \mapsto [x]$, are copies of $\mathbb{C}$, i.e. lines, we see that $\mathcal{O}(-1)$ is a *line bundle* on $\mathbb{P}^n$. As the fibre over a

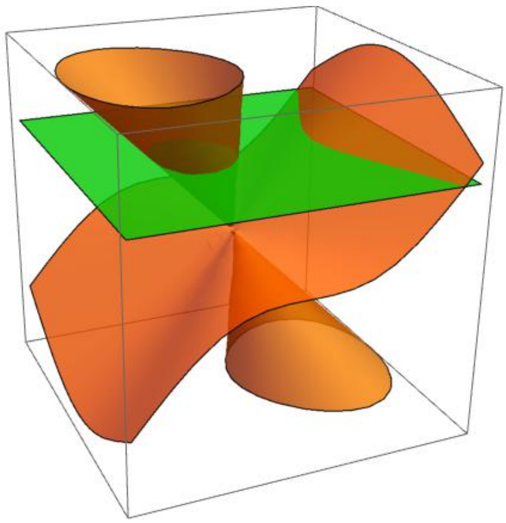

Figure 3: Affine cone of projective curve $zy^2 = x(x^2 - z^2)$.

point of $\mathbb{P}^n$, i.e. over a line in $\mathbb{C}^{n+1}$, is precisely this line, $\mathcal{O}(-1)$ is called the *tautological* line bundle on $\mathbb{P}^n$.

Let us consider the natural map from $\mathcal{O}(-1)$ to $\mathbb{C}^{n+1}$, $\pi\colon (v, [x]) \mapsto v$. The fibre of π over $v \in \mathbb{C}^{n+1}$ is the set of lines passing through v. If v is non-zero, there is a unique such line and so π is one-to-one above $\mathbb{C}^{n+1} \setminus \{0\}$. If $v = 0$, any line passes through zero and so the fibre $\pi^{-1}(0)$ is $\mathbb{P}^n$. This map $\pi\colon \mathcal{O}(-1) \to \mathbb{C}^{n+1}$ is called the *blow-up of the origin in* $\mathbb{C}^{n+1}$: one goes from $\mathbb{C}^{n+1}$ to $\mathcal{O}(-1)$ by separating the various lines intersecting each other at the origin and this effectively replaces the origin of $\mathbb{C}^{n+1}$ by the projective space $\mathbb{P}^n$.

As the coordinates x_i are linear functionals on the lines $\mathbb{C}x$, the natural interpretation of the x_i's is as sections of the dual line bundle of $\mathcal{O}(-1)$, called $\mathcal{O}(1)$. More generally, degree d homogeneous polynomials naturally define sections of the line bundle obtained by the d-th tensor product of the dual of $\mathcal{O}(-1)$, i.e. of $\mathcal{O}(1)$. This line bundle is called $\mathcal{O}(d)$.

Exercise: What is the interpretation of the quotient of an affine variety by the action of a finite group in terms of coordinate rings?

Exercise: Prove the degree-genus formula, a smooth degree d curve in $\mathbb{P}^2$ is of genus $g = \frac{1}{2}(d-1)(d-2)$, by degenerating the degree d curve to a union of d lines.

Exercise: Let X be a "reasonable" topological space. Show that the closed subsets of X are exactly the subsets of the form $\{f_1 = \ldots = f_k = 0\}$ where the f_i are continuous functions on X. Now invent the Zariski topology by replacing continuous functions by polynomials.