(Co)homology and Poincare Duality
(Enriching)

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1 Simplicial Homology

Definition 1.1. *A standard n-simplex is given by*

$$\Delta^n = \{(t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} : \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

For each $0 \leq j \leq n$, we define the j-th opposite face is give by

$$\{(t_0, t_1, ..., t_n) \in \Delta^n : t_j = 0\}.$$

and the inclusion map $\partial_j : \Delta^{n-1} \hookrightarrow \Delta^n$ given by

$$(t_0, t_1, ..., t_{n-1}) \mapsto (t_0, t_1, ..., t_{j-1}, 0, t_j, ..., t_{n-1}).$$

Definition 1.2. *A topological space X is a Δ-complex if*

- *there is a collection of maps $\{\sigma_\alpha : \Delta^n \to X\}$ (here, n depends of α) which is injective under restriction to $(\Delta^n)^o$, the interior of Δ^n.*
- *for all $x \in X$, there exists a unique $\sigma_\alpha|_{(\Delta^n)^o}$ such that its image contains x.*
- *a subset U in X is open if and only if $\sigma_\alpha^{-1}(U)$ is open in Δ^n for all $\alpha \in A$.*
- *For all $\alpha \in A$ and $0 \leq j \leq n$, we have $\sigma_\alpha \circ \partial_j = \sigma_\beta$ for some β.*

We shall abuse the notation a bit and call a map $\sigma_\alpha : \Delta^n \to X$ a n-simplex.

Roughly speaking, the n-th simplicial homology group describes the n dimensional holes of a topological space. Let X be a Δ-complex. Define the d-chain of X as a free abelian group generated by the d-simplex of X. In other words, we have

$$C_d = \bigoplus_{\text{d-simplex}} \mathbb{Z}\sigma.$$

An element of C_d can be written as a finite sum $\sum_{\alpha\in A} a_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^d \to X$ and $a_\alpha \in \mathbb{Z}$.

Definition 1.3. *The d-th boundary map $\partial : C_d \to C_{d-1}$ is given by*

$$\partial(\sum_{\alpha \in A} a_\alpha \sigma_\alpha) = \sum_{\alpha \in A} \sum_j (-1)^j a_\alpha (\sigma_\alpha \partial_j).$$

Example 1.4. *Consider the 2-simplex $[v_0, v_1, v_2]$*

Theorem 1.5. *Let X be a chain complex. For all $d \geq 2$, we have $\partial^2 : C_d \to C_{d-2}$.*

Proof. See Exercise 2. □

Now we can form a chain complex, namely

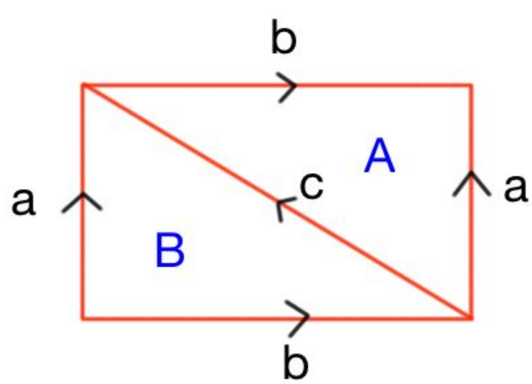
$$\ldots \longrightarrow C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} C_{d-2} \to \ldots$$

Since $\partial^2 = 0$, we have that $\ker(\partial : C_d \to C_{d-1}) \supseteq \operatorname{im}(\partial : C_{d+1} \to C_d)$. The d-th homology group measures the exactness of this inclusion.

Definition 1.6. *Let X be a Δ-complex. The d-th homology group of X is given by*

$$H_d(X) = \frac{\ker(\partial : C_d \to C_{d-1})}{im(\partial : C_{d+1} \to C_d)}.$$

Example 1.7. *Consider that torus T^2.*



We have $\partial A = \partial B = c - a + b$ and $\partial a = \partial b = \partial c = 0$. Therefore, the chain complex is given by

$$\mathbb{Z}_A \oplus \mathbb{Z}_B \xrightarrow{\partial_2} \mathbb{Z}_a \oplus \mathbb{Z}_b \oplus \mathbb{Z}_c \xrightarrow{\partial_1} \mathbb{Z}_{point}$$

where ∂_2 is given by the linear map $\begin{pmatrix} -1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ *and ∂_1 is the zero map. Therefore, we can compute its homology groups as*

$$H_2(T^2) \cong \mathbb{Z}_{(A-B)},$$
$$H_1(T^2) \cong \mathbb{Z}_a \oplus \mathbb{Z}_b,$$
$$H_0(T^2) \cong \mathbb{Z}_{point}.$$

It is worth mentioning that in the above example $H_1(T^2) \cong \pi_1(T^2)$, the fundamental group. However, this is far from true in general. See Exercise ??. Nevertheless, it is true that H_1 is always isomorphic to the abelianisation of the fundamental group π_1. In other words, we have $H_1 = \dfrac{\pi_1}{[\pi_1, \pi_1]}$ where $[\cdot, \cdot]$ is the commutator.

2 Singular Homology

We are introducing the singular homology in this section, which is more ambitious than the simplicial homology: besides the injective simplexes, we also take all the continuous maps into consideration. In other words, for any topological space, we define

$$C_d(X) := \bigoplus_{f:\Delta^d \to X} \mathbb{Z}_f$$

where the sum is over all continuous maps f. To achieve a chain, we shall define a boundary map $\partial : C_d(X) \to C_{d-1}(X)$ by

$$\partial f := \sum_j (-1)^j (f \circ \partial_j)$$

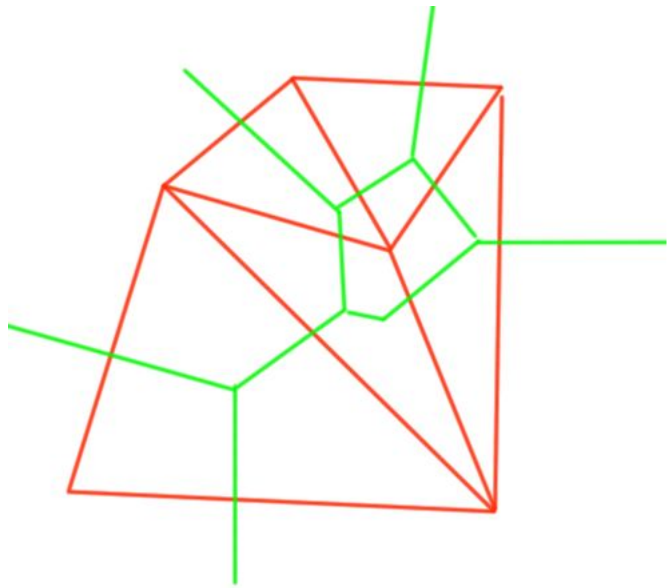
and extend the map by linearity. One can show that $\partial^2 = 0$ (See Exercise ??).

Despite being ambitious, the singular and simplicial homology agree for "reasonably" good spaces X.

1. They agree for a cell Δ^n.
2. They both satisfy the Mayer-Vietoris.

3 Dual Triangulation and Poincare Duality

Let X^n be a compact manifold, Consider a simplicial complex which is triangulated by 2-simplexes. We connect the centroid of each triangle with a green line and obtain the dual triangulation (Tragically, it is not entirely consisted of triangles).



One can show that each k-simplex, it induces a $(n-k)$-cell in the dual triangulation. In particular, this gives rise to a map of chain complexes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_d(X) & \xrightarrow[\partial]{} & C_{d-1}(X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longleftarrow & \mathrm{Hom}(\widetilde{C_{n-d}}(X),\mathbb{Z}) & \xleftarrow{\delta} & \mathrm{Hom}(\widetilde{C_{n-d+1}}(X),\mathbb{Z}) & \longleftarrow & \cdots \end{array}$$

The map $\delta : \mathrm{Hom}(\widetilde{C_{n-d+1}}(X),\mathbb{Z}) \to \mathrm{Hom}(\widetilde{C_{n-d}}(X),\mathbb{Z})$ is simply given by $f \mapsto f \circ \partial$, where $\widetilde{C_\star}$ denotes the cell in the dual chain complex. One can show that $\delta^2 = 0$. In particular, $\ker(\delta : \mathrm{Hom}(\widetilde{C_d}(X),\mathbb{Z}) \to \mathrm{Hom}(\widetilde{C_{d-1}}(X),\mathbb{Z})) \supseteq \mathrm{im}(\delta : \mathrm{Hom}(\widetilde{C_{d-1}}(X),\mathbb{Z}) \to \mathrm{Hom}(\widetilde{C_{d-2}}(X),\mathbb{Z}))$. Therefore, analogously we can define the cohomology groups.

Definition 3.1. *Let X be a simplicial complex. The d-th cohomology group is given by*

$$H^d(X) := \frac{\ker(\delta : Hom(\widetilde{C_d}(X),\mathbb{Z}) \to Hom(\widetilde{C_{d-1}}(X),\mathbb{Z}))}{im(\delta : Hom(\widetilde{C_{d-1}}(X),\mathbb{Z}) \to Hom(\widetilde{C_{d-2}}(X),\mathbb{Z}))}.$$

We should remark that if the coefficients are chosen over a field, then the cohomology group H^d is indeed the dual of the homology group H_d. See Exercise ??. However, this is not true if the coefficients are not chosen over a field. A simple example is to consider the multiplication-by-2 map $m : \mathbb{Z} \to \mathbb{Z}$, $x \mapsto 2x$. Dualizing the map gives the same map $m^* : \mathbb{Z} \to \mathbb{Z}$, $x \mapsto 2x$.

Therefore, the kernel is 0 whereas the cokernel is $\mathbb{Z}/2\mathbb{Z}$.

Now we can formulate the Poincare Duality, which gives a magical relation between homology and cohomology groups over a specific class of manifolds.

Theorem 3.2 (Poincare Duality). *If X^n is a manifold which is compact, closed and orientable, then we have*

$$H_k(X) \cong H^{n-k}(X)$$

for any $0 \leq k \leq n$.

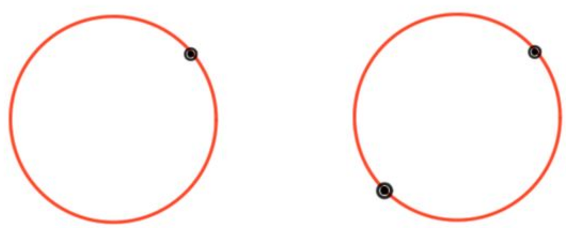
Roughly speaking (at least true over a field), the group $H_k(X)$ is dual to the group $H_{n-k}(X)$. The induced pairing from the Poincare Duality is called the intersection pairing.

4 Exercise

Exercise 1. What is the minimum number of 2-simplex on a torus T^2?

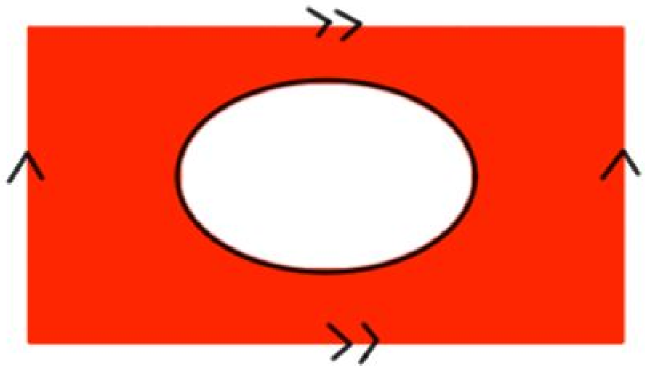
Exercise 2. Let ∂ be the boundary map with respect to the simplicial homology. For all $d \geq 2$, show that $\partial^2 : C_d \to C_{d-2}$.

Exercise 3. Calculate the homology groups of the circles.



Exercise 4. Let ∂ be the boundary map with respect to the singular homology. For all $d \geq 2$, show that $\partial^2 : C_d(X) \to C_{d-2}(X)$.

Exercise 5. Let X be the following Δ-complex.



Show that $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_1 = \langle a, b \rangle$, a free group generated by two elements.

Exercise 6. Show that $\pi_2(T^2) = 0$.

Exercise 7.