The Ordinary Double Point

(Notes written by S. Stark based on a lecture by R. Thomas.)

1. **Introduction.** Consider an analytic function

$$f : \mathbf{C}^{n+1} \to \mathbf{C}.$$

Assume that f and its partial derivatives vanish at the origin, so that the hypersurface X in $\mathbf{C}^{n+1}$ defined by f has a singularity at 0. Then the lowest nonvanishing term of the Taylor series expansion of f about the origin is the quadratic one. If this quadratic form is nondegenerate, then in a neighbourhood of 0 there is a system of analytic coordinates in which f takes the form

$$f(x) = x_1^2 + \cdots + x_{n+1}^2. \tag{1.1}$$

(This is essentially the Morse lemma.) In this case one says the singularity of X at 0 is an *ordinary double point* (odp for short). This is the simplest kind of hypersurface singularity; it is ubiquitous in geometry.

Remark 1. (i) Of course, this can be expressed purely in terms of the local ring $\mathscr{O}_{\mathrm{X},0}$: X has an ordinary double point at 0 if and only if the completion $\hat{\mathscr{O}}_{\mathrm{X},0}$ is isomorphic to $\mathbf{C}[[\mathrm{T}_1, \ldots, \mathrm{T}_{n+1}]]/(\mathrm{T}_1^2 + \cdots + \mathrm{T}_{n+1}^2)$.
(ii) Replacing (1.1) with $x_1^2 + \cdots + x_n^2 + x_{n+1}^{k+1}$ yields the family of A_k-*singularities* ($k \geqslant 1$). Odps are therefore also called A_1-singularities or *nodes*. Nodal curves occur for instance in the (Deligne-Mumford) compactification $\overline{\mathscr{M}}_g$ of the moduli space $\mathscr{M}_g$ of curves of genus g.

Exercise 1. Let $\mathscr{L}$ be a (globally generated) line bundle on a smooth projective variety X, $\mathrm{P} \subset \mathbf{P}(\mathrm{H}^0(\mathrm{X}, \mathscr{L}))$ a generic pencil of sections, with zero sets $(\mathrm{X}_t)_{t \in \mathbf{P}^1}$. Show that each X_t has at worst one odp. (For a special case see [**7**], section 2).

In the following we discuss two approaches to understanding the geometry of the ordinary double point; X will usually denote the affine hypersurface given by

$$x_1^2 + \cdots + x_{n+1}^2 = 0. \tag{1.2}$$

The first approach is called *smoothing*, while the second one is called *resolving*.

2. **Smoothing.** Consider the hypersurface $\mathfrak{X}$ in $\mathbf{C}^{n+1} \times \mathbf{C}$ given by

$$x_1^2 + \cdots + x_{n+1}^2 = t,$$

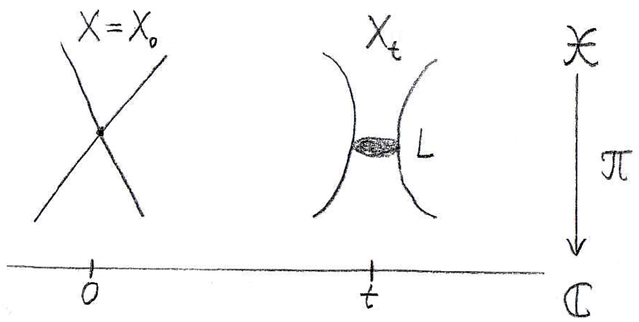
which we regard as fibered over $\mathbf{C}$ via the morphism

$$\pi : \mathfrak{X} \to \mathbf{C} \tag{2.1}$$

induced by the projection $\mathbf{C}^{n+1} \times \mathbf{C} \to \mathbf{C}$. The fibre X_t over a fixed point $t \neq 0$ in $\mathbf{C}$ is the smooth hypersurface in $\mathbf{C}^{n+1}$ defined by

$$x_1^2 + \cdots + x_{n+1}^2 = t,$$

while the fibre over 0 (the singular fibre) is $\mathrm{X}_0 = \mathrm{X}$. One calls X_t the *smoothing* or *Milnor fibre* of X. Consider first the case $n = 1$, which guides our intuition [(1)]. After a change of variables the equation $x_1^2 + x_2^2 = 0$ becomes $uv = 0$; thus X is a cone, while X_t is a quadric. The real slice $\mathrm{L} = \mathrm{L}_t$ of X_t which degenerates to the singularity of $\mathrm{X} = \mathrm{X}_0$ as $t \to 0$ is called the *vanishing cycle*.



We view X_t as a symplectic manifold with symplectic structure ω induced by the one of $\mathbf{C}^{n+1}$. It is a fundamental fact that the vanishing cycle L is a *Lagrangian submanifold* of X_t, i.e. $\dim \mathrm{L} = \frac{1}{2} \dim \mathrm{X}_t$ and $\omega|\mathrm{L} = 0$. We will outline three different descriptions of L and, correspondingly, three different proofs that L is Lagrangian; they all indicate that smoothings should be regarded within the framework of *symplectic geometry*. To begin with, L is the fixed locus of the anti-symplectic involution $\sigma : \mathrm{X}_t \to \mathrm{X}_t$ induced by complex conjugation.

Exercise 2. Let (X, ω) be a symplectic manifold and $\sigma : \mathrm{X} \to \mathrm{X}$ an anti-symplectic involution, i.e. $\sigma^2 = \mathrm{id}$ and $\sigma^* \omega = -\omega$. If the fixed locus is nonempty, then it is a Lagrangian submanifold.

In dimension $n = 1$ it is easy to see that X_t is diffeomorphic to $\mathrm{S}^1 \times \mathbf{R}$, with L corresponding to a circle S^1. There is a generalisation of this observation, holding

(1) For $n = 0$ the fibre over 0 is the singleton consisting of 0, while the general fibre is the set of square roots of t. (The scheme theoretic fibres are $\mathrm{Spec}(\mathbf{C}[\mathrm{T}]/(\mathrm{T}^2))$ and $\mathrm{Spec}(\mathbf{C} \times \mathbf{C})$, respectively.)

in any dimension: X_t is symplectomorphic to the cotangent bundle $\mathrm{T}^*\mathrm{S}^n$, with L corresponding to the zero section $\mathrm{S}^n \subset \mathrm{T}^*\mathrm{S}^n$.

Exercise 3. Let $t > 0$, and identify $\mathbf{C}^{n+1} \simeq \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ via $x_j = a_j + ib_j$. Then for $x \in \mathrm{X}_t$ we have $t = x_1^2 + \cdots + x_{n+1}^2 = |a|^2 + 2i\langle a, b\rangle - |b|^2$, in particular $\langle a, b\rangle = 0$ and $|a| \neq 0$. Identify

$$\mathrm{T}^*\mathrm{S}^n \simeq \mathrm{TS}^n = \left\{(a, b) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \mid |a| = 1,\ \langle a, b\rangle = 0\right\},$$

and show that the map

$$\mathrm{X}_t \to \mathrm{T}^*\mathrm{S}^n \quad \text{given by} \quad x \mapsto (a/|a|, |a|b) \tag{2.2}$$

is a symplectomorphism.

This gives the second proof that the vanishing cycle is a Lagrangian submanifold: it is the zero section of a cotangent bundle.

Remark 2. Locally this is typical for Lagrangian submanifolds: Weinstein's neighbourhood theorem says that for every compact Lagrangian L in a symplectic manifold X there is a neighbourhood of L in X symplectomorphic to a neighbourhood of the zero section of $\mathrm{T}^*\mathrm{L}$.

We come to the third and final description; it relies on the notion of *symplectic connection*, which gives rise to *monodromy*, the Riemannian analogoue of which is holonomy. Let us consider a family of projective varieties

$$\pi : \mathfrak{X} \to \mathrm{T},$$

whose smooth locus will be denoted by $\mathrm{T}^* \subset \mathrm{T}$. Regarded as a family of smooth manifolds over T^* this is a locally trivial fibration by a theorem of Ehresmann. In particular, the fibres are diffeomorphic, but the complex structure may vary. However, the symplectic structure does not vary: it is a symplectic fibre bundle. By taking the annihilator (with respect to the symplectic form) of $\mathrm{T}_{\mathfrak{X}/\mathrm{T}}$ we obtain a connection (horizontal subbundle of $\mathrm{T}_{\mathfrak{X}}$) on $\mathfrak{X}$ over T^*. The parallel transport map associated to a path in T^* is a symplectomorphism, and for loops we in particular obtain the *monodromy representation*

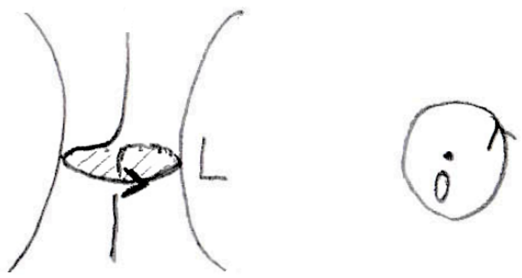
$$\pi_1(\mathrm{T}^*, t) \to \mathrm{Aut}(\mathrm{X}_t),$$

where $\mathrm{Aut}(\mathrm{X}_t)$ is the group of symplectomorphisms of X_t modulo Hamiltonian isotopies [(2)].

We return to the particular family (2.1), with $\mathrm{T}^* = \mathbf{C}^*$. Consider the straight line path from t to 0 in $\mathbf{C}$; the third and final description of the vanishing cycle L

(2) It turns out that the parallel transport maps corresponding to isotopic loops are distinct but Hamiltonian isotopic, see [**6**], section 2.

is then that it is the set of all $x \in \mathrm{X}_t$ which parallel transport to the singularity of X_0. This is a Lagrangian submanifold exactly because the connection preserves ω. The monodromy about a circle around the origin in $\mathbf{C}$ is particularly interesting; it is called the *Dehn twist* $\mathrm{T}_\mathrm{L} : \mathrm{X}_t \to \mathrm{X}_t$ about L, shown in the following picture.



If we identify $\mathrm{X}_t \simeq \mathrm{T}^*\mathrm{S}^n$ via (2.2), then the Dehn twist can be described as a Hamiltonian flow (with Hamiltonian essentially equal to $|b|$), or as a geodesic flow (the flow of the geodesic vector field on TS^n associated to the Levi-Cevita connection). Using this local model one then defines the Dehn twist more generally by appealing to Weinstein's neighbourhood theorem.

Remark 3. (i) One can generalise this dicussion to the A_k-singularity (see remark 1 (ii)) with smoothings X_p given by $x_1^2 + \cdots + x_n^2 = p(x_{n+1})$, where p is a monic polynomial of degree $k+1$ with distinct roots. The vanishing cycle is a chain of $k+1$ spheres, and the fundamental group of the smooth locus (the configuration space of $k+1$ unordered points in $\mathbf{C}$) is the *Braid group* B_{k+1}. Khovanov and Seidel proved that in this case the monodromy representation is faithful; for more on this we refer to [**6**], section 2.3.
(ii) For $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$, $\alpha_i \geqslant 2$, let $\mathrm{X}(\alpha) \subset \mathbf{C}^{n+1}$ be defined by

$$x_1^{\alpha_1} + \cdots + x_{n+1}^{\alpha_{n+1}} = 0.$$

Then $\mathrm{X}(\alpha)$ is a topological manifold if and only if the *link* $(\alpha) = \mathrm{X}(\alpha) \cap \mathrm{S}^{2n+1}$ (where S^{2n+1} is a small sphere centred at the origin) is topologically a sphere. For $\alpha_1 = \cdots = \alpha_{n+1} = 2$ it is easy to see that (α) is the Stiefel manifold $\mathrm{V}_2(\mathbf{R}^{n+1})$. Brieskorn [**2**] (see also [**4**]) shows that $\mathrm{X}(\alpha)$ is a topological manifold if and only if

$$\prod \left(1 - \prod_{k=1}^{n+1} \exp(2\pi/\alpha_k)^{i_k} \right) = 1, \tag{2.3}$$

where the product is taken over all $0 < i_k < \alpha_k$. The topological manifold (α) carries a natural differentiable structure, and Brieskorn proves that the (α) with $\alpha = (2,2,2,3,6k-1)$ $(1 \leqslant k \leqslant 28)$ give the 28 differentiable structures on S^7.
(iii) The fundamental class [L] generates $\mathrm{H}_n(\mathrm{X}_t;\mathbf{Z}) \simeq \mathbf{Z}$, and the map induced by

$\mathrm{T_L}$ on $\mathrm{H}_n(\mathrm{X}_t; \mathbf{Z})$ is given by the *Picard-Lefschetz formula*

$$(\mathrm{T_L})_*(a) = a + (-1)^{(n+1)(n+2)/2}(a.[\mathrm{L}])[\mathrm{L}],$$

where $(a.[\mathrm{L}])$ denotes the intersection number (see for instance [**1**], chapter 2).

3. **Resolution.** In algebraic geometry one deals with singularities by resolving them. A *resolution of singularities* of a variety X is a smooth variety $\tilde{\mathrm{X}}$ with a proper birational map $\pi : \tilde{\mathrm{X}} \to \mathrm{X}$ which is an isomorphism over the smooth locus of X. The existence theorem for resolutions of singularities is due to Hironaka; he proved that one can obtain a resolution of singularities by repeated blow ups (in particular, π can be chosen to be projective). In our case (1.2), the mild nature of the singularity allows us to resolve X by a single blow up

$$\pi : \tilde{\mathrm{X}} = \mathrm{Bl_0 X} \to \mathrm{X}.$$

Exercise 4. Show that the exceptional divisor of this blow up is a smooth projective n-dimensional quadric, and that π is a resolution of singularities (3).

An interesting phenomenon occurs in dimension $n = 3$. By a change of variables we can rewrite the equation (1.2) as $uv - wz = 0$. Apart from blowing up 0, one can also resolve X by blowing up the 2-planes D^+ and D^- given by $u = w = 0$ and $u = z = 0$, respectively. So we consider the blow ups

$$\pi^+ : \mathrm{X}^+ = \mathrm{Bl}_{\mathrm{D}^+}(\mathrm{X}) \to \mathrm{X} \quad \text{and} \quad \pi^- : \mathrm{X}^- = \mathrm{Bl}_{\mathrm{D}^-}(\mathrm{X}) \to \mathrm{X}$$

of X along D^+ and D^-, respectively.

Exercise 5. Show that the divisors D^+ and D^- are not Cartier, but that they are Cartier away from the origin. Prove that X^+ can be regarded as the closure of the graph of the rational function $u/w = z/v$ on X. Find a similar description for X^-. Check that π^+ and π^- are resolutions of singularities of X.

The inverse images of D^+ and D^- under the blow up $\tilde{\mathrm{X}} \to \mathrm{X}$ are effective Cartier divisors on $\tilde{\mathrm{X}}$ (they are of pure codimension 1). The universal property of the blow up thus gives rise to a commutative diagram

$$\begin{array}{ccc} \tilde{\mathrm{X}} & \longrightarrow & \mathrm{X}^- \\ \downarrow & & \downarrow \\ \mathrm{X}^+ & \longrightarrow & \mathrm{X}. \end{array}$$

The fibres of π^+ and π^- over $0 \in \mathrm{X}$ are rational curves C^+ and C^-, and so π^+ and π^- are *small resolutions* of X in the sense that the exceptional sets are of

(3) The exceptional divisor is the Proj of the associated graded ring of the local ring of X at 0, so it suffices so show that the latter ring is isomorphic to $\mathbf{C}[\mathrm{T}_1, \ldots, \mathrm{T}_{n+1}]/(\mathrm{T}_1^2 + \cdots + \mathrm{T}_{n+1}^2)$.

codimension 2. The maps $\tilde{X} \to X^+$ and $\tilde{X} \to X^-$ can be regarded as blow ups along these curves; each of them contracts one of the rulings of the exceptional quadric $E \subset \tilde{X}$. The above diagram is actually a Cartesian diagram, $\tilde{X} \simeq X^+ \times_X X^-$. It is easy to see that the schemes X^+ and X^- are isomorphic as schemes over $\mathbf{C}$ (there is an automorphism of X exchanging D^+ and D^-), but not as schemes over X [(4)]. However, π^+ and π^- induce an isomorphism

$$X^+ - C^+ \xrightarrow{\sim} X^- - C^-$$

and in particular a birational map

$$X^+ \dashrightarrow X^-,$$

the so-called *Atiyah flop*. It can be thought of as an 'algebraic surgery': it exchanges one rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ by another.

Remark 4. (i) Small resolutions are crepant, and X is therefore an example of a variety with two distinct crepant resolutions. (For surfaces these are unique.)
(ii) Let V be vector space of dimension 2, and view $X \subset \mathrm{End}(V)$ as the locus of endomorphisms u with $\mathrm{rank}(u) \leqslant 1$. Then X^+ (resp. X^-) can be viewed as the subvariety of $\mathrm{End}(V) \times \mathrm{Gr}_1(V)$ given by pairs (u, L) with $L \subset \mathrm{Ker}(u)$ (resp. $\mathrm{Im}(u) \subset L$), and the Atiyah flop takes $(u, \mathrm{Ker}(u))$ to $(u, \mathrm{Im}(u))$.
(iii) The varieties X, X^+, X^- and $\tilde{X}$ are toric, and π^+, π^-, and π are toric resolutions of singularities (see [**3**], example 1.13, for a toric description of these maps).

As pointed out in the above remark, there cannot be an analogue of the Atiyah flop in dimension two; this is only one of the many differences between dimension two and three. For instance, the surface odp can be regarded as a quotient of the affine plane $\mathbf{C}^2$ by the cyclic group $\mathbf{Z}/2\mathbf{Z}$ generated by the involution $x \mapsto -x$. It is not possible to express the 3-fold odp as a quotient of an affine space by a finite group [(5)], essentially because the divisor class group of the 3-fold is $\mathbf{Z}$ (with generator $[D^+] = [D^-]$), which is not torsion. (In contrast, the class group of the surface odp is $\mathbf{Z}/2\mathbf{Z}$.)

Maybe more interesting is the observation that the smoothing and resolution of the surface odp are diffeomorphic; we leave this as an exercise.

Exercise 6. The small resolution of the 3-fold odp X has a natural map to $\mathbf{C}$ induced by the fourth projection of $\mathbf{C}^4$. Notice that the fibre over 0 is the resolution

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(4) In fact there is no morphism $f : X^+ \to X^-$ of schemes over X. If there were such a morphism (necessarily dominant), then the pullback of D^- to X^- would be a Cartier divisor whose pullback under f would be a Cartier divisor in X^+. But this divisor would be equal the pullback of D^- to X^+, which is not a Cartier divisor.

(5) However, one can regard the 3-fold odp as a quotient of $\mathbf{C}^4$ by the multiplicative group $\mathbf{G}_m$ acting with weight $(1, 1, -1, -1)$. By varying the linearisation of the trivial bundle on $\mathbf{C}^4$, one obtains X^+ and X^- (this is sometimes called 'variation of GIT'). See [**5**], example 1.16, for details.

of the surface odp, while the fibre over $t \neq 0$ is the smoothing. (Another approach would be to view the surface odp as $\mathbf{C}^2/\{\pm 1\}$ as indicated above, and to use that $\mathrm{Bl}_0(\mathbf{C}^2/\{\pm 1\}) \simeq \mathrm{Bl}_0(\mathbf{C}^2)/\{\pm 1\}$.)

However, in three dimensions resolution and smoothing are no longer diffeomorphic, in fact not even homeomorphic, essentially because smoothing replaces the singularity by a S^3, while resolution replaces it by a $\mathbf{P}^1 \simeq \mathrm{S}^2$ (6). There is another difference between dimension two and dimension three. If X is a projective surface and $\pi : \tilde{\mathrm{X}} \to \mathrm{X}$ a resolution of singularities, then $\tilde{\mathrm{X}}$ is a smooth proper surface, in particular projective by a theorem of Zariski. The following construction of Hironaka shows that in dimension three this no longer holds; it also gives an example of a smooth variety which is not quasi-projective.

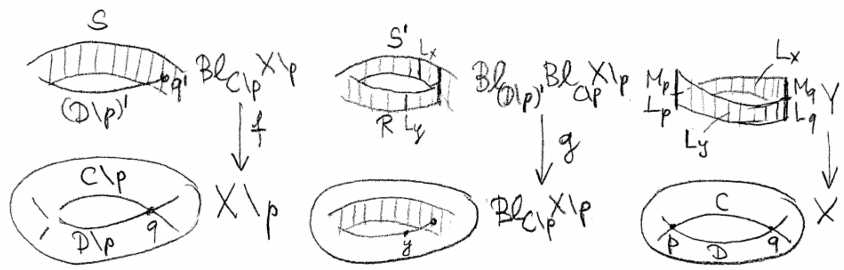
Exercise 7 (Hironaka). (Fill in the details.) Let X be a smooth projective 3-fold, and $\mathrm{C}, \mathrm{D} \subset \mathrm{X}$ smooth (rational, say, for simplicity) curves intersecting transversally in two points p, q. Consider the composites

$$\mathrm{Bl}_{(\mathrm{D}-p)'}\left(\mathrm{Bl}_{\mathrm{C}-p}\left(\mathrm{X}-p\right)\right) \xrightarrow{g} \mathrm{Bl}_{\mathrm{C}-p}\left(\mathrm{X}-p\right) \xrightarrow{f} \mathrm{X}-p, \tag{3.1}$$

$$\mathrm{Bl}_{(\mathrm{C}-q)'}\left(\mathrm{Bl}_{\mathrm{D}-q}\left(\mathrm{X}-q\right)\right) \to \mathrm{Bl}_{\mathrm{D}-q}\left(\mathrm{X}-q\right) \to \mathrm{X}-q, \tag{3.2}$$

where $(\mathrm{D}-p)'$ is the proper transform of $\mathrm{D}-p$. The inverse images of $\mathrm{X}-\{p,q\}$ under these morphisms are canonically isomorphic, and the morphisms coincide over $\mathrm{X}-\{p,q\}$. This allows one to glue, and we obtain a proper 3-fold Y with a morphism $\mathrm{Y} \to \mathrm{X}$. Consider now (3.1). The exceptional surface of the blow-up f is a $\mathbf{P}^1$-bundle $\mathrm{S} \to \mathrm{C}-p$ whose fibres are all linearly equivalent. We have $f^{-1}(\mathrm{D}-p) = f^{-1}(q) \cup (\mathrm{D}-p)'$, and $\mathrm{S} \cap (\mathrm{D}-p)'$ consists of a single point q' which is taken to q by f. Now $g^{-1}(\mathrm{S}) = g^{-1}(q') \cup \mathrm{S}'$ with $\mathrm{S}' = \mathrm{Bl}_{q'}\mathrm{S}$. As $g^{-1}(\mathrm{S})$ is irreducible, we have $g^{-1}(q') \subset \mathrm{S}'$. The fibre $g^{-1}(f^{-1}(q))$ has two components L_q and M_q, where $\mathrm{L}_q = g^{-1}(q')$ and g induces an isomorphism $\mathrm{M}_q \xrightarrow{\sim} f^{-1}(q)$. For $x \neq q$ in $\mathrm{C}-p$ the fibre $g^{-1}(f^{-1}(x))$ is irreducible and we denote it by L_x. Then $\mathrm{L}_x \approx \mathrm{L}_q + \mathrm{M}_q$, where $\approx$ denotes numerical equivalence. The exceptional surface $g : \mathrm{R} \to (\mathrm{D}-p)'$ is a $\mathbf{P}^1$-bundle with fibres L_y and $\mathrm{L}_{q'} = \mathrm{L}_q$. The surfaces R and S' meet along L, and we have $\mathrm{L}_x \approx \mathrm{L}_q + \mathrm{M}_q \approx \mathrm{L}_y + \mathrm{M}_q$. By doing analogous considerations with (3.2) we arrive at $\mathrm{M}_p + \mathrm{M}_q \approx 0$ which implies that Y cannot be projective.

(6) This is closely related to a phenomenon which in the physics literature is called ‘conifold transition’, usually illustrated by an enlightening picture showing X as a cone over $\mathrm{S}^2 \times \mathrm{S}^3$. From there also stems the view that, in a sense, smoothing and resolution are ‘mirror’ to each other.



The universal property of the blow up $\mathrm{Bl}_{\mathrm{C}\cup\mathrm{D}}\,\mathrm{X} \to \mathrm{X}$ allows us to factor the morphism $\mathrm{Y} \to \mathrm{X}$ as

$$\mathrm{Y} \to \mathrm{Bl}_{\mathrm{C}\cup\mathrm{D}}\,\mathrm{X} \to \mathrm{X}.$$

The morphism $\mathrm{Y} \to \mathrm{Bl}_{\mathrm{C}\cup\mathrm{D}}\,\mathrm{X}$ is a small resolution of the nodal threefold $\mathrm{Bl}_{\mathrm{C}\cup\mathrm{D}}\,\mathrm{X}$.

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