Lefschetz pencils in algebraic and symplectic geometry

1 Lefschetz hyperplane theorem

Our goal is to understand the topology of a projective variety by analysing its "hyperplane slices".

Remark 1.1. In those notes, we will usually assume that considered manifolds are complex projective, but except for the Hard Lefschetz theorem everything works well also for symplectic manifolds.

All the cohomologies in the notes will be defined over $\mathbb{Q}$. First, we state the celebrated Lefschetz hyperplane theorem.

Theorem 1.2 (Lefschetz hyperplane theorem)**.** *Let $X \subseteq \mathbb{P}^m$ be a smooth projective variety of dimension n, and let H be a hyperplane such that $M := H \cap X$ is smooth. Then the map*

$$H^i(X, \mathbb{Q}) \to H^i(M, \mathbb{Q})$$

induced by the inclusion of $i\colon M \subseteq X$ is an isomorphism for $i \leq n-2$ and is injective for $i = n-1$.

It implies the following corollary.

Corollary 1.3. *With the assumptions as above, the following holds:*

$$\begin{aligned} H_k(H) &\overset{i_*}{\underset{\simeq}{\to}} H_k(X), \text{ for } k < n-1, \\ H_{n-1}(H) &\overset{i_*}{\twoheadrightarrow} H_{n-1}(X), \text{ for } k = n-1, \\ H_k(H) &\overset{i^*}{\underset{\simeq}{\leftarrow}} H_{k+2}(X), \text{ for } k > n-1, \end{aligned}$$

where the third map is defined via Poincare duality.

Exercise 1.4. *Prove the corollary, using the Lefschetz hyperplane theorem.*

Sketch of the proof based on Morse theory. Let $H = \{s = 0\}$, where $s \in H^0(X, \mathcal{O}_X(1))$. Since $\mathcal{O}_X(1)$ is ample, there exists a hermitian metric h on it (the pullback of the Fubini-study metric), such that $\frac{i}{2\pi} F_\nabla = \frac{i}{2\pi}\partial\overline{\partial}\log h$ is positive, where F_∇ is the curvature. One can prove that it is possible to choose some other metric so that $\frac{i}{2\pi}\partial\overline{\partial}\log|s|^{-2}$ is positive.

Then, easy calculation shows, that

$$\phi(x) := \log|s|^{-2}$$

has at least n negative eigenvalues at critical points. Morse theory implies that X is obtained from H by adding handles of dim $\leq n$. This concludes the proof. $\square$

Let $f\colon X \to C$ be a holomorphic function. We say that $x \in X$ is an *ordinary double point* (ODP), if $\mathrm{Hess}(f)$ is nondegenerate at x.

Definition 1.5. Let L be a holomorphic line bundle on X. A *Lefschetz pencil* on X is a family of distinct hypersurfaces $H_t = \{p \in X \mid s_t(p) = 0\}$ for $t \in \mathbb{P}^{\Bbbk}$ and $s_t \in H^0(X, L)$, satisfying the following conditions

- $s_t = s_0 + ts_\infty$ for $t \in \mathbb{P}^1$.
- Only finitely many hypersurfaces H_t are singular. Any singular H_t may have only one singularity and it must be an ODP.
- The base locus $\mathbb{B} := \bigcap_{t\in\mathbb{P}^{\Bbbk}} X_t$ is smooth.

The picture, which is useful to have in mind, is that of the pencils for an embedded variety $X \subseteq \mathbb{P}^n$ as collections of hyperplanes in $\mathbb{P}^n$.

The first condition says that the Lefschetz pencils correspond to lines in $\mathbb{P}(H^0(X, L)^\wedge)$. The second and the third one say that Lefschety pencils are families of hypersurfaces, which are 'reasonably' general.

Note that the equation $s_t = s_0 + ts_\infty$, implies

$$\mathbb{B} = X_0 \cap X_\infty.$$

Using a dimension counting argument and the discriminant variety, one can show that every smooth hyperplane section $H \cap X \subseteq X$ gives a Lefschetz pencil such that $X_0 = H \cap X$.

A Lefschetz pencil gives us a natural rational map $\phi\colon X \dashrightarrow \mathbb{P}^1$ defined by

$$x \mapsto (s_0(x) : s_\infty(x)).$$

It is defined outside of the subscheme $\mathbb{B}$. Note that the closure of the fiber $\phi^{-1}(t)$ for $t \in \mathbb{P}^1$ is exactly the hypersurface H_t. One could define Lefschetz pencils as rational maps $X \dashrightarrow \mathbb{P}^1$, with the singularities of fibers satisfying beforementioned properties.

In order to make the map well-defined, we consider the blow-up $\widetilde{X}$ of X along B. Then ϕ extends to a well defined map $\widetilde{\phi}\colon \widetilde{X} \to \mathbb{P}^1$.

Exercise 1.6. *Show that* $H_*(\widetilde{X}) = H_*(X) \oplus H_{*-2}(B)$.

2 Local study of ordinary double points

Analogously to real Morse theory, every holomorphic function around an ordinary double points is of the form

$$f = \sum z_i^2,$$

where z_i are suitable coordinates. Let us consider the fiber around a critical point $X_t = \{\sum z_i^2 = t\}$ for some $t \in \mathbb{C}$.

Definition 2.1. We define a *vanishing cycle* for $t = se^{i\phi} \in \mathbb{C}$, where $s \in \mathbb{R}$, to be the sphere

$$L_t := \{(z_1, \ldots, z_n) \mid z_i = \sqrt{s}e^{i\frac{\phi}{2}}, x_i \in \mathbb{R}, \sum x_i^2 = 1\} \subseteq X_t.$$

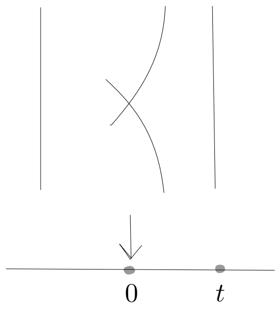

Figure 1: Map f

We define a *Lefschetz thimble* Δ_i to be the union of all vanishing spheres at some real path between 0 and t. Note that Δ_i depends on the choice of the path.

One can check that a vanishing cycle L_t is Lagrangian in X_t.

Proposition 2.2. *Assume that* 0 *and* ∞ *are regular values of* f*. Let* $x_1, \dots, x_k \in \mathbb{P}^1$ *be all critical points. Connect* 0 *with* $x_1, \dots, x_k$ *by some real paths, and define* L_i *together with* Δ_i *for* $1 \leq i \leq k$ *to be the Lefschetz pencil and the Lefschetz thimble at* 0*, respectively, with respect to the ordinary double point* p_i*. Then* $\widetilde{X} \backslash X_\infty$ *is homotopic to* $\bigcup X_0 \cup_{L_i} \Delta_i$.

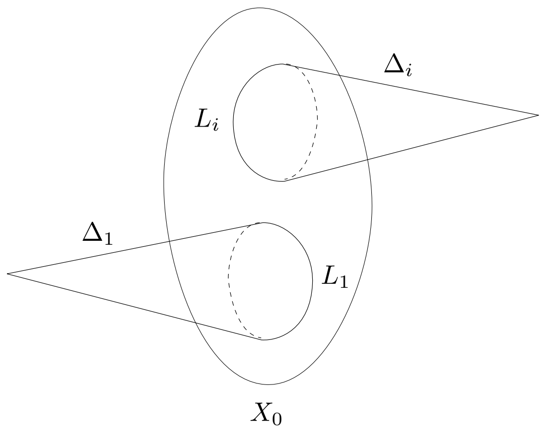

Figure 2: An example of $\bigcup X_0 \cup_{L_i} \Delta_i$

Sketch of the proof. Let p_i be the path connecting 0 and x_i. Define a skeleton $S := \bigcup p_i$. Then, the standard "run a flow" argument implies that $\widetilde{X} \backslash X_\infty$ is homotopic to $\widetilde{X}|_S$. Similarily, we retract every fiber X_t over $t \in S$ to $\bigcup X_0 \cup_{L_i} \Delta_i$. $\square$

In the case when $X = \widetilde{X}$, the proposition above implies the Lefschetz hyperplane theorem. In general, one can easily reprove Lefschetz hyperplane theorem by a careful inductive argument comparing X and $\widetilde{X}$.

3 Monodromy

Choose a regular point $t \in \mathbb{P}^1$ of f. Assume that $t \neq \infty$. Let $P \subseteq \mathbb{P}^1$ be the subset of critical points. We would like to construct the *monodromy action*. It would be an action of the fundamental group $\pi(\mathbb{P}^1 \backslash P)$ on $H^i(X_t)$.

Definition 3.1. Take a path $p \in \pi(\mathbb{P}^1 \backslash P)$ such that $p(0) = p(1) = t$ and a cycle $Z \in H^i(X_t)$. We define $p(Z) \in H^i(X_t)$, the image of Z under the action of p, in the following way.

We move the cycle Z along the path p to obtain a family of cycles $Z_r \in H^i(X_{p(r)})$ for $r \in [0,1]$, where $Z_0 = Z$. We can do it by e.g. considering a lifted normalized geodesics flow (or by using Ehremann's theorem). We define $p(Z)$ to be the class $[Z_1]$ in $H^i(X_t)$. This class does not depend on the choice of cycles Z_r or of the representative of the class $[p]$ in $\pi(\mathbb{P}^1 \backslash P)$, because any two choices would be homotopic.

In other words, we move a cycle around a path and observe which cycle it deforms to, when we come back. If a path can be contracted, then the monodromy action of it is trivial. Note that it is crucial, that paths do not cross critical points, because for such points the fibers X_t are not manifolds (they are singular).

Remark 3.2. In the symplectic world, it is important to use the normalized flow for "moving" cycles, to make sure that the symplectic form is preserved.

Let n be the complex dimension of X. One can use vanishing cycles to express the action of monodromy on $H^{n-1}(X_t)$ explicitly. Note that X_t has real dimension $2n-2$.

Theorem 3.3 (Picard-Lefschetz formula)**.** *Take a class $A \in H^{n-1}(X_t)$ on a path $p \in \pi(\mathbb{P}^1 \backslash P)$ such that the interior of p contains only one critical point. Let L_t be the vanishing cycle corresponding to this ordinary double point. Then*

$$p(A) = A + (A \cap [L_t])[L_t],$$

where $[L_t] \in H_{n-1}(X_t)$. Note that $A \cap [L_t]$ is a number.

The Picard-Lefschetz formula shows that the monodromy acts by just adding a certain multiple of a vanishing cycle. In particular, the monodromy action on a cycle is either trivial or has an infinite order.

Also, note the following fact.

Fact 3.4. *Choose a fiber X_t. Then, the monodromy acts transitively on vanishing cycles of X_t obtained with respect to all ordinary double points.*

4 Hard Lefschetz Theorem

Theorem 4.1 (Hard Lefschetz theorem)**.** *Let X be a projective variety, and let $\omega \in H^2(X)$ be the class of a hyperplane section. Then the map $H^{n-1}(X) \xrightarrow{\wedge\omega} H^{n+1}(X)$ is an isomorphism.*

Let H be a hyperplane section of X. Extend H to a Lefschetz pencil $\widetilde{X} \to \mathbb{P}^1$, so that $H = X_0$. We define $H_{n-1}(X_t)_{\mathrm{van}} \subseteq H_{n-1}(X_t)$ to be a subgroup generated by vanishing cycles. Using Hard Lefschetz theorem we would like to prove the following two propositions.

Proposition 4.2. *Let X be a projective variety and H a hyperplane section. Then*

$$H_{n-1}(H) = j^* H_{n+1}(X) \oplus H_{n-1}(X_t)_{\mathrm{van}},$$

with $j^* \colon H_{n+1}(X) \xrightarrow{\cap H} H_{n-1}(H)$.

Proposition 4.3. *The intersection product*

$$H_{n-1}(X_t)_{\mathrm{van}} \times H_{n-1}(X_t)_{\mathrm{van}} \xrightarrow{\cap} H_0(X_t) \simeq \mathbb{Q}$$

is nondegenerate.

First, we consider the following diagram.

Exercise 4.4. *Confirm yourself that the following diagram commutes*

$$\begin{tikzcd} H_{n+1}(X) \arrow[r, "\cap H"] \arrow[d, "\simeq"] & H_{n-1}(H) \arrow[r, "i_*"] & H_{n-1}(X) \arrow[d, "\simeq"] \\ H^{n-1}(X) \arrow[rr, "\wedge\omega"] & & H^{n+1}(X) \end{tikzcd}$$

where i is the inclusion $H \subseteq X$.

Since $\wedge\omega$ is an isomorphism, we get that

$$H_{n-1}(H) = j^* H_{n+1}(X) \oplus \ker(i_*).$$

Thus, the first proposition follows from the following lemma.

Lemma 4.5. *We have $H_{n-1}(H)_{\mathrm{van}} = \ker(i_*)$.*

Proof. First, I claim it is enough to show that the restriction map

$$H^{n-1}(\widetilde{X}) \to H^{n-1}(\widetilde{X} \backslash X_\infty) \tag{1}$$

is surjective. Indeed, if we knew it, then we would have that $\ker(H_{n-1}(H) \xrightarrow{i_*} H_{n-1}(X)) = \ker(H_{n-1}(H) \xrightarrow{i_*} H_{n-1}(\widetilde{X} \backslash X_\infty))$. The latter group is generated by vanishing cycles by Proposition 2.2.

In order to show (1), we consider the long exact sequence of cohomologies for pairs:

$$\ldots \longrightarrow H^{n-1}(\widetilde{X}) \longrightarrow H^{n-1}(\widetilde{X} \backslash X_\infty) \longrightarrow H^n(\widetilde{X}, \widetilde{X} \backslash X_\infty) \longrightarrow H^n(\widetilde{X}) \longrightarrow \ldots.$$

Let NX_∞ be the normal bundle of X_∞ inside $\widetilde{X}$. We have

$$H^n(\widetilde{X}, \widetilde{X} \backslash X_\infty) = H^n(NX_\infty, NX_\infty \backslash X_\infty) = H^{n-2}(X_\infty),$$

where the first equality follows from the excision, and the second one from Thom's isomorphism theorem. Recall that Thom's isomorphism theorem states that $H^k(E) \simeq H^{k+r}(E, E \backslash X)$, where E is a real vector bundle of rank r over a CW-complex X.

By a careful application of Proposition 2.2, we know that

$$H^{n-2}(X_\infty) \to H^n(\widetilde{X} \backslash X_0),$$

induced by the inclusion $X_\infty \subseteq \widetilde{X}$, is injective (be careful: we switched the role of 0 and ∞!). In particular, $H^{n-2}(X_\infty) \to H^n(\widetilde{X})$ is also injective, and the long exact sequence of cohomologies shows that $H^{n-1}(\widetilde{X}) \to H^{n-1}(\widetilde{X}\backslash X_\infty)$ is surjective. This concludes the proof of the lemma.

□

Now, consider the following lemma.

Lemma 4.6. *It holds that $j^* H_{n+1}(X) = (H_{n-1}(H)_{\text{van}})^\perp$, with $j^* \colon H_{n+1}(X) \xrightarrow{\cap H} H_{n-1}(H)$ and $\perp$ taken with respect to the intersection pairing $H_{n-1}(X_t) \times H_{n-1}(X_t) \to H_0(X_t)$.*

Together, with Proposition 4.2, this implies that

$$H_{n-1}(H) = H_{n-1}(H)_{\text{van}} \oplus H_{n-1}(H)_{\text{van}}^\perp.$$

Hence $H_{n-1}(H)_{\text{van}} \cap H_{n-1}(H)_{\text{van}}^\perp = \emptyset$, and so the intersection product on $H_{n-1}(H)_{\text{van}}$ is nondegenerate. This concludes the proof of Proposition 4.3.

The idea of the proof of the lemma is quite natural. By the Picard-Lefschetz theorem, the group $(H_{n-1}(H)_{\text{van}})^\perp$ consists of exactly those cycles which are invariant under the monodromy action. Such cycles can be uniformly swept out on the manifold to give us a global class.

The reader familiar with local systems should notice that applying the aforemontioned strategy to $R^{n-1}\phi_*\mathbb{Q}$, where $\phi \colon X \to \mathbb{P}^1$ is the Lefschetz pencil, gives a straightforward proof of the lemma.

Sketch of the proof of the lemma. First take $A \in j^* H_{n+1}(X) \subseteq H_{n-1}(H)$. Since A is the restriction of a globally defined cycle on X, it must be invariant under the action of monodromy. Thus, the Picard-Lefschetz formula shows that $A \cap [L] = 0$ for all vanishing cycles L. This concludes one direction of the proof.

Now, take $A \in (H_{n-1}(H)_{\text{van}})^\perp$. Without loss of generality, we may assume that $t = \infty$, and we sweep out this cycle along all "straight lines" from ∞ to 0, to obtain an $n+1$-chain $\widetilde{A}$. One can show that

$$\partial \widetilde{A} = \sum \langle \tau_{i-1} \circ \ldots \circ \tau_1(A), L_i \rangle \, A_i + C,$$

where τ_i and L_i are the monodromy action and the vanishing cycle, respectively, corresponding to the i-th ordinary double point, and $C \in H_{n-1}(X_0)$. One can show that in our case $C = 0$, thus, since A is orthogonal to all vanishing cycles, we get

$$\partial \widetilde{A} = 0.$$

Hence $\widetilde{A} \in H_{n+1}(X)$, and $A = \widetilde{A} \cap H$, i.e. $A \in j^* H_{n+1}(X)$. □

Exercise 4.7. *Assume that we don't know that the Hard Lefschetz theorem holds. Show that the intersection product on $H_{n-1}(H)_{\text{van}}$ is nondegenerate, if and only if $H_{n-1}(H) \simeq H_{n-1}(H)_{\text{van}} \oplus j^* H_{n+1}(X)$.*

Remark 4.8. The exercise shows that in the case of sympletic manifolds, the Hard Lefschetz theorem is equivalent to non degeneracy of the intersection product on $H_{n-1}(H)_{\text{van}}$. This doesn't hold in general for symplectic manifolds.