# Koszul Duality

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# 1 Koszul Duality in Algebra

## 1.1 Semisimple Algebras

Let $A$ be a finite-dimensional algebra over $\mathbb{C}$. From the perspective of representation theory, the easiest class of algebras to understand are (left) simple algebras, whose (left) ideals are zero and itself. Next easiest are semisimple algebras: direct sums of simple algebras. Classically, these were classified by Wedderburn and then generalized by Artin to general fields of characteristic zero.

**Theorem 1.1.** *(Artin-Wedderburn) Let $A$ be a (left) semisimple ring over a field $k$, with decomposition $A = \oplus_i n_i V_i$ into distinct simple algebras. Then*

$$A \cong End_A(A)^{op} \cong \bigoplus_i End_A(n_i V_i)^{op} \cong \bigoplus_i (End_A(V_i)^{op})^{n_i \times n_i} \cong \bigoplus_i Mat_{n_i \times n_i}(D_i)$$

*where $D_i \cong End_A(V_i)^{op}$ is a division ring, by Schur's lemma and $A^{op}$ denotes the algebra with same underlying vector space as $A$ but $b \cdot^{op} a := a \cdot b$. Here elements of $A$ are viewed as an endomorphism by left multiplication.*

**Remark 1.2.** Taking $k = \mathbb{C}$ or more generally $k = \overline{k}$, then any finite algebraic extension is trivial and hence $D = k$ and the theorem merely states that finite-dimensional semisimple algebras over $\mathbb{C}$ are products of matrix algebras, and in particular left and right modules are equivalent. A commutative semisimple algebra is a finite sum of copies of the ground field.

**Example 1.3.** The group algebra $k[G]$ of a finite group $G$ is semisimple if the characteristic of $k$ does not divide the order of the group. Finding such a decomposition as a sum of matrix algebras in practice requires determining the irreducible representations of $G$. For $G = S_3$ permutations on three letters, the group algebra with coefficients in the complex numbers decomposes as

$$\mathbb{C}[S_3] \cong End_{\mathbb{C}[S_3]}(1) \oplus End_{\mathbb{C}[S_3]}(sgn) \oplus End_{\mathbb{C}[S_3]}(std) \cong \mathbb{C} \oplus \mathbb{C} \oplus Mat_{2\times 2}(\mathbb{C})$$

where 1 is the trivial representation, $sgn$ maps each permutation to its sign, and $std$ is the two-dimensional representation given explicitly by $\{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0\}$ with permutations acting by permuting the indices.

Semisimplicity of an algebra is apriori a strong condition on its modules. In particular, all short exact sequences of modules split so there are no non-trivial extensions between modules of semisimple rings. Today we'll define and study Koszul algebras, which are graded algebras with a slightly weaker condition on extensions of graded modules.

## 1.2 Koszul Algebras

**Definition 1.4.** Let $A = \oplus_{i \geq 0} A_i$ be a positively graded algebra. A is *Koszul* if

(1) $A_0$ is semisimple.

(2) There exists a graded projective resolution $P_\bullet$ of $A_0$ by $A$-modules such that $P_i = AP_i^i$.

**Remark 1.5.** We will mainly consider the case where $A_0 = k$ is a field. Everything will hold in the generality of $A_0$ semisimple and augmented. The second condition is saying that the diagonal part of the resolution generates the rest.

**Example 1.6.**
- As a first example consider the polynomial ring $\mathbb{C}[x]$ with $x$ in degree 1. Then one has the following two-term projective resolution of $\mathbb{C}$ by free $\mathbb{C}[x]$-modules,

$$0 \longrightarrow \mathbb{C}[x][-1] \xrightarrow{m_x} \mathbb{C}[x] \xrightarrow{ev_{x=0}} \mathbb{C}.$$

  Here we use the notation $M[n]$ is the graded module with $j$th graded piece $M[n]_j = M_{n+j}$. These shifts are careful bookkeeping so each map respects the grading. Indeed the $i$th module is generated in degree i, for $i = 0, 1$. We conclude that $\mathbb{C}[x]$ is Koszul.

- Next consider the ring $\mathbb{C}[x]/(x^2)$ with $x$ in degree 1. Here one has an infinite length projective resolution of $\mathbb{C}$ by free $\mathbb{C}[x]/(x^2)$-modules,

$$\cdots \mathbb{C}[x]/(x^2)[-2] \xrightarrow{m_x} \mathbb{C}[x]/(x^2)[-1] \xrightarrow{m_x} \mathbb{C}[x]/(x^2) \longrightarrow \mathbb{C},$$

  hence $\mathbb{C}[x]/(x^2)$ is Koszul.

Semisimplicity is too stringent a condition for graded algebras as it implies they are concentrated in degree 0. One way to weaken this condition is to allow for a small class of non-trivial extensions. Koszulity is equivalent to such a weakening.

**Proposition 1.7.** *Consider $k$ to be an $A$-module by the augmentation $A \to A_0 \cong k$. Then $A = \bigoplus_{i \geq 0} A_i$ is Koszul if and only if $Ext^i_A(k, k[-s]) = 0$ unless $s = i$.*

**Remark 1.8.** The Ext algebra is bigraded by cohomological grading and internal grading. That is,

$$\mathrm{Ext}_A(M, N) = \oplus_{r,s} Ext^{r,s}(M, N) = \oplus_{r,s} Ext^r_A(M, N[-s])$$

where $s$ denotes internal and $r$ cohomological grading.

**Definition 1.9.** Let $A$ be a Koszul algebra with augmentation $\epsilon : A \to k$. Then the *Koszul dual* algebra $A^!$ is

$$A^! = \mathrm{Ext}_A(k, k),$$

which is also an augmented algebra since its zero graded piece is $\mathrm{Hom}_A(k, k) \cong k$.

**Remark 1.10.** The Koszul dual of a Koszul algebra is Koszul and $(A^!)^! \cong A^!$ for Koszul algebras.

**Example 1.11.** Consider the polynomial algebra $\mathbb{C}[x]$ as before, with $|x| = 1$ and augmentation given by evaluating $x$ at 0. Then

$$\begin{aligned}
(\mathbb{C}[x])^! &= Ext_{\mathbb{C}[x]}(\mathbb{C}, \mathbb{C}) \\
&= H^*(\mathrm{Hom}_{\mathbb{C}[x]}(\mathbb{C}[x][-1] \xrightarrow{m_x} \mathbb{C}[x], \mathbb{C})) \\
&= H^*(\mathrm{Hom}_{\mathbb{C}[x]}(\mathbb{C}[x][-1], \mathbb{C}) \xleftarrow{m_x} \mathrm{Hom}_{\mathbb{C}[x]}(\mathbb{C}[x], \mathbb{C})) \\
&= H^*(\mathbb{C} \xleftarrow{0} \mathbb{C}) \\
&= \mathbb{C}[1] \oplus \mathbb{C} \\
&= \mathbb{C}[y]/(y^2) \quad |y| = 1
\end{aligned}$$

Here $m_x$ denotes multiplication by $x$ so $m_x^*$ is pre-multiplication by $x$. The algebra operation on cohomology is the cup product, which allows for the final identification. Tracking the identifications one has $y = x^*$. We leave it as a worthwhile exercise to show that $(\mathbb{C}[x]^!)^! \cong \mathbb{C}[x]$.

**Remark 1.12.** (Choice of augmentation)
Consider $\mathbb{C}[x]$, with a general augmentation $\epsilon : \mathbb{C}[x] \to \mathbb{C}$. Such a map is determined by the image of $x$ since $\epsilon$ is the identity on the ground field by definition. Now one can build the same resolution as before replacing $m_x$ with $m_{x-\epsilon(x)}$, and the same computation produces an isomorphic answer with generator $y = (x - \epsilon(x))^*$.

By the proposition, Koszulity of $A$ can be interpreted as saying the bigraded Ext algebra

$$Ext_A(k,k) \cong \oplus_i Ext_A^{i,i}(k,k)$$

is concentrated on the diagonal. For computations it is often useful to encode this condition in a polynomial.

**Definition 1.13.** The Poincare polynomial for a graded algebra $A$ is,

$$P_A(u,v) = \sum_{i,j} u^i v^j dim_k \mathrm{Ext}_A^{i,j}(k,k) \in bZ[[u,v]].$$

Notice, if $A$ is Koszul then $P_A(-1,t) = h_{A^!}(-t)$ where

$$h_A(t) = \sum_{i \geq 0} t^i dim_k A_i \in \mathbb{Z}[[t]]$$

is the Hilbert series of $A$. The equality $h_A(t)P_A(-1,t) = 1$ implies in the Koszul setting that $h_A(t)h_{A^!}(-t) = 1$. This provides a computationally nice way to show an algebra is *not* Koszul. To make this practical, one needs a way of finding potential Koszul dual candidates. Quadradic duality provides such a method.

## 1.3 Quadratic Duality

**Definition 1.14.** $A$ is *quadratic* if $A_0$ is semisimple and $A$ is generated by $A_1$ over $A_0$ with quadratically-generated relations.

For comfort, take $A_0 = k$ and $A_1 = V$ a finite-dimensional vector space and this is merely saying $A$ is isomorphic to $T(V)/(R)$ where $R \subset V \otimes V$.

More concretely, a quadratic presentation of $A$ is a pair $(V, R)$ with $R \subset V \otimes V$ with a choice of isomorphism $A \cong T(V)/(R)$. Given two quadratic presentations of $A$, $(V_1, R_1)$ and $(V_2.R_2)$ then the composition

$$\phi_2 \circ \phi_1^{-1} : T(V_1)/(R_1) \to A \to T(V_2)/(R_2)$$

identifies $V_1$ with $V_2$ and $R_1$ with $R_2$. Hence a quadratic algebra is one admitting a quadratic presentation and such a presentation is unique, up to equivalence of quadratic data.

**Remark 1.15.** Quadratic algebras are the simpliest quotients of tensor algebras, in the following sense. Quotienting $T_k(V)$ by a relation in degree zero produces $T_{k/k'}(V)$, a tensor algebra over the quotient ring. Quotienting by a relation of degree one produces $T_k(V/V')$, a tensor algebra on the quotient vector space. Therefore, quotienting by quadratic relations is required to achieve a larger class of algebras than just tensor algebras.

Koszul duality for quadratic algebras has a particularly simple description, called quadratic duality. Given a quadratic presentation $A \cong T(V)/(R)$, as above, one can form a dual quadratic algebra $A^{quad} \cong T(V^*)/(R^\perp)$ where $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $R^\perp = Ann(R) \subset V^* \otimes V^*$. As shown above, $A^{quad}$ does not depend on the chosen quadratic data.

In fact, Koszul algebras are necessarily quadratic and these two notions of duality coincide for Koszul algebras.

**Exercise 1.16.** Let $A$ be a graded, associative algebra with augmentation given by projecting to $A_0 \cong k$

(1) $Ext^1_A(k, k[-s]) \neq 0$ implies $s = 1$ if and only if $A_1$ generates $A$ as an algebra over $k$.

(2) $Ext^1_A(k, k[-s]) \neq 0$ implies $s = 1$ and $Ext^2_A(k, k[-s]) \neq 0$ implies $s = 2$ if and only if $A$ is a quadratic algebra.

**Proposition 1.17.** *Let $A$ be Koszul with augmentation $A \to A_0 \cong k$. Then $A^{dual} \cong A^!$.*

**Remark 1.18.** Quadratic algebras form a large class including the symmetric algebra and exterior algebra. Koszul algebras form a nice subclass that is easier to work with in part because they always have a reasonable sized projecitve resolution called the Koszul resolution.

**Example 1.19.**

- If $A = k < x > /(x^2)$ then $V \cong k$ a 1-dimensional vector space generated by $x$ and $R = (x^2) = V \otimes V$. We denote the generator of its dual by $x^*$. Then, $A^! = k < x^* >$ is the full tensor algebra on the dual vector space, since $R^\perp$=0. Computing the Hilbert series one gets $h_A(t)h_{A^!}(t) = (1+t)(1-t+t^2-t^3+\cdots) = 1$.

- If $A = k[x, y]$ is polynomial in two variables. Then $V = k \otimes k$ is the two-dimensional vector space generated by $x$ and $y$ and $R = xy - yx$, often written $x \otimes y - y \otimes x$ to emphasize the isomorphism to a quotient of a tensor algebra. This is a one-dimensional subspace of $V \otimes V$ with $R^\perp = xy + yx$. Hence $A^! = \wedge(x^*, y^*)$ is the exterior algebra on the dual generators. More generally, $S(V)$ and $\wedge V^*$ are Koszul dual.

- If $A = k < x, y, z > /(x^2, xy - yx, xz + zx)$. Then quadratic data for $A$ is given by $V = \mathrm{Span}\{x, y, z\}$ and $R = (x \otimes x, x \otimes y - y \otimes x, x \otimes z + z \otimes x)$ is a three-dimensional subspace of $V \otimes V$. It's annihilator in $V^* \otimes V^*$ is

$$R^\perp = (x^* \otimes y^* + y^* \otimes x^*, x^* \otimes z^* - z^* \otimes x^*, y^* \otimes y^*, z^* \otimes y^*, y^* \otimes z^*, z^* \otimes z^*).$$

  So $A^! = T(V^*)/(R^\perp)$.

## 1.4 Koszul Resolution

In general, a duality can be phrased in two ways: (1) an assignment which squares to the identity, (2) a perfect pairing: the equivalence given by a suitable relationship between functions and products. Dualizing a finite-dimensional vector space, for instance, can be described as (1) $Hom_k(-, k) : Vect \to Vect$ or (2) $(,) : Vect \otimes Vect \to k$: the hom-tensor adjunction relating the two. For Koszul duality, we have established (1) using Ext (or alternatively quadratic data). What is the perfect pairing? It should be a quasi-isomorphism $A \otimes A^! \to k$.

Although merely a shift in perspective, the utility of this map cannot be understated. For instance, this gives a very convenient resolution of $A$ by $A$-bimodules, useful for Hochschild cohomology computations and therefore for understanding the deformation theory of $A$.

**Example 1.20.** We continue with the motivating example of the symmetric and exterior algebras. Here the Koszul complex is,

$$S(V) \otimes \wedge^{dim(V)}(V^*) \to S(V) \otimes \wedge^{dim(V)-1}(V^*) \to \cdots S(V) \otimes V^* \to S(V)$$

with

$$x_1 x_2 \cdots x_p \otimes y_1 \wedge y_2 \wedge \cdots \wedge y_q \mapsto \sum_{i=1}^{q} y_i x_1 \cdots x_p \otimes y_1 \wedge \cdots \wedge \widehat{y_i} \wedge \cdots \wedge y_q.$$

Here we regard $S(V)$ as products of elements of $V^*$ and hence the grading on $S(V)$ raises by one while the grading on $\wedge(V^*)$ falls by one. The map

$$x_1 \cdots x_p \otimes y_1 \wedge \cdots \wedge y_q \mapsto \sum_{i=1}^{p} x_1 \cdots \widehat{x}_i \cdots x_p \otimes \sum_{i=1}^{p} x_i \wedge y_1 \wedge \cdots \wedge y_q$$

is an explicit chain homotopy equivalence between $(p+q)$id and zero, hence the complex is acyclic.

## 1.5 Derived Koszul Duality

Warning: The remainder of the talk is derived! A few remarks about what this means, why its happening, and where you can go for support:

- From a practical standpoint, replace the symbol $\mathrm{Ext}_A(k,k)$ with the symbol $\mathrm{RHom}_A(k,k)$, the latter being a complex which computes $\mathrm{Ext}_A(k,k)$. Such a complex is not unique: it depends on a choice of projective resolution for $k$. However, the identity map on $k$ lifts to a chain homotopy equivalence between any two choices of projective resolution. Therefore one must pass to the derived category of the category of $A$-modules for $\mathrm{RHom}_A(k,k)$ to be well-defined.

- From a theoretical standpoint, the algebra $\mathrm{Ext}_A(k,k)$ 'knows' much less about $A$ than the complex $\mathrm{RHom}_A(k,k)$. In a topological setting, for instance, one can recover the weak-homotopy type of a space from its singular complex but not from the homology of this complex. Working with the complex, albeit bulky, is often convenient.

- In flavor, working derived cleans up messy statements and consequently clarifies certain phenomenon. For instance the natural transformation between the derived functor of a composition and the composition of derived functors is classically (i.e. cohomologically) the convergence of a spectral sequence. Warning: this doesn't really simplify computations, just theory.

- Still unconvinced? In the introduction to "Derived Algebraic Geometry" Jacob Lurie gives a deficiency in classical algebraic geometry (namely non-tranverse intersections in Bezout's theorem) which is illucidated from a derived perspective. In "Derived Categories and their uses" Keller gives an historical account. Quoting Verdier, he refers to derived categories as 'formalism for hyperhomology.' A nice introduction to this material is "Derived Categories for the Working Mathematician" by Richard Thomas.

**Definition 1.21.** Let $A$, $B$ be augmented differential graded algebras. They are *Koszul dual* if

$$B \cong \mathrm{RHom}_A(k,k) \qquad A \cong \mathrm{RHom}_B(k,k)$$

are isomorphisms of differential graded algebras.

**Remark 1.22.** $\mathrm{RHom}_A(k,k)$ has the structure of a differential graded algebra where the differential is coming from the choice of projective resolution $(P_\bullet, d)$ for $k$ and the algebra structure is the Yoneda algebra

Koszul dual dg-algebras have equivalent dg-modules. More precisely,

**Proposition 1.23.** *If $A$ is a finitely-generated $k$-module then the functor $(-)^! : D(A - mod) \to D(A^! - mod)$ taking $M$ to $M^! := RHom_A(M, k)$ induces an equivalence of derived categories.*

**Remark 1.24.** This is *not* an equivalence preserving the standard t-structure on the derived category. Additionally, for an equivalence of bounded derived categories one needs in addition that $A^!$ be left Noetherian.

**Example 1.25.** Returning to the running example of the symmetric and exterior algebras, one has

$$D^b(\wedge V^*) \cong D^b(S(V))$$

called the BGG correspondence. Since the latter is a quotient of the category of coherent sheaves on projective space by a theorem of Serre, the correspondence is often restated to give an equivalence between the stable category of finitely-generated modules over $\wedge V^*$ and the bounded derived category of coherent sheaves on $\mathbb{P}(V)$.

# 2 Koszul Duality in Topology

Let $X$ be a simply-connected topological space[1]. Choose $p : pt \to X$ a point in $X$, later written $p \in X$ for sanity.

Consider the following two differential graded algebras associated to the pair $(X, p)$:

- Let $C^*(X)$ denote the graded vector space with $C^n(X)$ the space of (say simplicial) n-cochains valued in $\mathbb{Q}$. Then $(C^*(X), d, \cup, C^*(p))$ is a differential graded augmented algebra where $d$ is the coboundary map induced from an alternating sum of face maps, $\cup$ is the usual cup product, and the augmentation is the pullback of $C^*(-)$ along $p \to X$, together with the fact that $C^*(p)$ is a chain complex with differentials alternating between zero and the identity on $\mathbb{Q}$ that is quasi-isomorphic to $\mathbb{Q}$ concentrated in degree 0.
- Let $\Omega_p(X)$ denote the topological space of all continuous maps $(S^1, (1,0)) \to (X, p)$ with the compact-open topology. Then $(C_{-*}(\Omega_p(X)), d, \text{Pont}, C_{-*}(e_p))$ is a differential graded augmented algebra where $C_{-*}(-)$ denotes the graded vector space of simplicial chain valued in $\mathbb{Q}$ and concentrated in non-positive degree, $d$ is the boundary map, Pont is the Pontryagin product, meaning the product induced from the composition of loops in $\Omega_p(X)$ on homology, and the augmentation is given by considering only chains on the constant path $e_p$ at $p$.

We will commit the usual sin of writing only the graded vector space to mean this entire structure.

The loop space is rather large and so one would like to understand it in terms of something smaller. The path-loop space fibration gives such a description linking the homotopy groups of $X$ to its loop space. So one might expect the two differential graded algebras are related. In fact, they are Koszul dual.

**Theorem 2.1.** *(Adams, Eilenberg-MacLane) Cochains on $X$ and chains on the loop space $\Omega_p(X)$ are Koszul dual*

$$C_{-*}(\Omega_p(X)) \cong RHom_{C^*(X)}(C^*(p), C^*(p))$$

$$C^*(X) \cong RHom_{C_{-*}(\Omega_p(X))}(C_{-*}(cont_p), C_{-*}(cont_p))$$

[1] One needs some weak finitness and seperability conditions on $X$. Compactly-generated Hausdorff is probably enough, but the skeptical reader should take $X$ to be a finite CW-complex.

**Example 2.2.** Consider $X = S^2$, in which case one can replace $C^*(S^2)$ with the smaller model $H^*(S^2)$ using the fact that $S^2$ is formal.

$$C^*(S^2) \cong H^*(S^2) \cong k[x]/(x^2)$$

with $\deg(x) = 2$. Hence the Koszul dual is

$$C_{-*}(\Omega S^2) \cong C^*(S^2)^! \cong (k[x]/(x^2))^! \cong k[y]$$

with $\deg(y) = -1$.

**Example 2.3.** It's instructive to see what goes wrong when $X$ is not simply connected, so consider $X = S^1$. By formality of $S^1$,

$$C_*(S^1) \cong H^*(S^1) \cong k[x]/(x^2)$$

with degree of $x$ equal to 1. We've already computed its Koszul dual to be $k[y]$ with $y$ in degree 2. However, $\Omega_p(S^1) \simeq \mathbb{Z}$, since the fibers under the degree map are contractible. Hence

$$C_{-*}(\Omega_p(S^1)) \cong C_{-*}(\mathbb{Z}) \cong H_{-*}(\mathbb{Z}) \cong k[\mathbb{Z}]$$

. with elements of $\mathbb{Z}$ concentrated in degree 0. The latter is *not* Koszul

Since $C^*(X)$ and $C_{-*}(\Omega_{pt}(X))$, with appropriately chosen augmentation, are Koszul dual, one gets a relationship between their modules (and bimodules) and hence Hochschild cohomology. That is,

$$HH^*(C^*(X), C^*(X)) \cong HH^*(C_{-*}(\Omega_{pt}(X)), C_{-*}(\Omega_{pt}(X))).$$

It was shown by Cohen and Jones in A Homotopy Theoretic Realization of String Topology that

$$H_{n-*}(LX) \cong HH^*(C_{-*}(\Omega X, \Omega X))$$

and Tradler, in his PhD thesis, showed the Chas-Sullivan product corresponds to the cup product in Hochschild cohomology. For simply-connected spaces, Koszul duality gives a means of computing the homology of the free loop space in terms of the Hochschild cohomology of $C^*(X)$.

Additionally, this instance of Koszul duality unified two perspectives of rational homotopy theory.

# 3 Rational Homotopy Theory

In this section we consider only simply connected spaces with the homotopy type of a CW-complex.

## 3.1 Introduction

Rational homotopy theory is the study of rational homotopy equivalence classes of topological spaces.

**Definition 3.1.** $X$ is rationally homotopy equivalent to $Y$ if there exists a continuous map $f : X \to Y$ such that the induced maps $f_n : \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_n(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphisms for all $n > 1$.

Recall, tensoring with $\mathbb{Q}$ has the effect of killing torsion since $\mathbb{Q}$ is divisble. This greatly simplifies the theory. In particular all rational homotopy groups of sphere were computed by Serre to be,

$$\pi_k(S^{2n}) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2n, 4n-1 \\ 0 & \text{otherwise} \end{cases} \qquad \pi_k(S^{2n+1}) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2n+1 \\ 0 & \text{otherwise} \end{cases}$$

where the generators are the identity maps and higher dimensional analogs of Hopf maps, in the even case. Alternatively, the integral homotopy groups of $S^2$ are not all known.

We restrict attention to simply-connected spaces with the homotopy type of a CW-complex. The former condition ensures a characterization of rational homotopy equivalence by maps inducing isomorphisms on homology (a.k.a. quasi-isomorphisms) using the Hurewicz's map. The later condition gives a characterization of contractible spaces as being those with vanishing homotopy groups, by Whitehead's theorem. Unwanted counterexamples to these phenomena include the Poincare homology sphere and the topologist's sine curve, respectively.

A broad goal of algebraic topology is to produce algebraic invariants of a topological space. We've stripped the subject of many of its difficulties by considering rational homotopy type of simply-connected, locally-contractible spaces. One might now wonder whether there are *complete* algebraic invariants. Indeed Quillen and Sullivan separately discovered two such complete algebraic invariants of rational homotopy types, linked by Koszul duality. That is,

**Theorem 3.2.** *(Quillen)*
*The homotopy category of simply connected rational topological spaces is equivalent to the homotopy category of 1-connected differential graded Lie algebras. The equivalence descends from a map on topological spaces*

$$\gamma : X \mapsto \gamma(X)$$

*whose homology is the graded Lie algebra* $(\pi(X) \otimes_{\mathbb{Z}}, [-,-]_{Wh})$*, defined below.*

**Theorem 3.3.** *(Sullivan)*
*The homotopy category of simply connected rational topological spaces is equivalent to the homotopy category of 1-connected commutative algebras.*

$$A_{PL} : X \mapsto A_{PL}(X; \mathbb{Q})$$

*whose cohomology is the graded-commutative algebra* $H^*(X; \mathbb{Q})$ *.*

**Theorem 3.4.** *(Berglund)*
*If* $X$ *is formal and coformal (meaning each model is quisi-isomorphic to its homology) then*

$$H^*(X; \mathbb{Q})^\vee \cong \pi_*(\Omega X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

*where* $^\vee$ *is the Koszul dual Lie algebra of rational cohomology.*

The goal of the remainder of the talk is to spell out these statements in more detail.
A few consequences include the following 'realization' results:

Any graded 1-connected

$$\left\{ \begin{array}{c} \text{commutative algebra} \\ \text{cocommutative coalgebra} \\ \text{Lie algebra} \end{array} \right\} \text{ over } \mathbb{Q} \text{ can be realized as } \left\{ \begin{array}{c} H^*(X, \mathbb{Q}) \\ H_*(X, \mathbb{Q}) \\ (\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}, [-,-]) \end{array} \right\}$$

for some $X$ simply connected.

Another interesting consequence is that one can find additional structures on the cohomology ring, coming from transferring the structure under the quasi-isomorphism $A_{PL}(X) \to C^*(X;\mathbb{Q})$, such that the cohomology ring with this structure is a complete invariant. This is the approach taken by Kadeishvil, who shows the $C_\infty$ structure on cohomology is a complete invariant.

## 3.2 Quillen's Model

Quillen's approach follows that of classical Lie theory; so we recall some basics here. One has the following relationships between a Lie group, its compactly supported functions, and its tangent space:

$$\textbf{Grps} \underset{G(-)}{\overset{\mathbb{Q}[-]}{\rightleftarrows}} \textbf{Hopf} \underset{U(-)}{\overset{P(-)}{\rightleftarrows}} \textbf{Lie}$$

One can extract the underlying group of a cocommutative Hopf algebra by considering group-like elements G(-)(i.e non-zero elements with comultiplication $\Delta(g) = g \otimes g$), and one can extract the underlying Lie algebra of a graded connected cocoummtuative Hopf algebra by considering primitive elements P(-) (i.e. elements with comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$.) The functor $\mathbb{Q}[-]$ takes a group to its group algebra and $U(-)$ takes a lie algebra to it universal enveloping algebra. (Probably these functors should be replaced by completed versions to obtain complete Hopf algebras.) A fundamental theorem of Lie is that for simply-connected Lie groups nothing is lost in this construction. More precisely, the exponential map allows one to recover the group structure from its tangent space at the identity.

Quillen proves a simplicial version of this correspondence, which is the key ingredient of his algebraic model for rational homotopy theory. Given a space $X$ one can form the Kan loop group $\Omega_{Kan}(X)$, which is a simplicial group on the nose! Then utilizing simplicial lie theory, Quillen associates a simplicial Lie algebra, which can be viewed as a differential graded Lie algebra by taking its normalzied complex. Quillen does not explicitly produce this differential graded Lie algebra, but its homology is the Whitehead Lie algebra.

**Definition 3.5.** The Whitehead product

$$[-,-]_{Wh} : \pi_*(X) \times \pi_*(X) \to \pi_{*-1}(X)$$

is defined by

$$[f,g]_{Wh} : S^{n+m-1} \to S^n \vee S^m \overset{f \vee g}{\to} X$$

where the first map is the attaching map needed to build $S^n \times S^m$ from its $(n+m-1)$-skeleton, $S^n \vee S^m$. The wedge point lies in the image of this map, allowing for a choice of basepoint in $S^{n+m-1}$ needed to induce maps on homotopy groups.

**Remark 3.6.** The Whitehead product makes the rational homotopy groups $\pi_{*+1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ into a graded Lie bracket, the Jacobi identity is proven using Massey products. A topological way of interpreting the shift is to consider the homotopy groups of $\Omega(X)$, in light of the path-loop fibration whose corresponding long exact sequence of homotopy groups identifies $\pi_{*+1}(X) \cong \pi_*(\Omega(X))$. Here our assumption that $\pi_1(X)$ is simply-connected means $\Omega(X)$ is path-connected.

**Exercise 3.7.**

- If $n = 1$ then this product is the action of $\pi_1(X)$ on $\pi_m(X)$ for each $m \geq 1$. In particular, if $m = 1$ the action is given by conjugation.
- Compute $[id, id] \in \pi_3(S^2)$ where $id : S^2 \to S^2$ is the identity. Generalize the computation to conclude that the rational homotopy groups of spheres are generated by the identity map as Lie algebras with the Whitehead product.

- Distinguish the spaces $X = S^2$ and $Y = S^3 \times \mathbb{C}P^\infty$ using the Whitehead bracket. (They have the same homotopy groups because both spaces are homotopy equivalent to the orbits of an action of $S^1$ on $S^3 \times S^\infty$.)

Sometimes the Whitehead Lie algebra (not the differential graded version in Quillen's construction) is sufficient, in the sense that it is quasi-isomorphic to Quillen's construction. In these cases one calls the original space *coformal*.

### 3.3 Sullivan's Model

We now turn to the construction of Sullivan.

**Definition 3.8.**

- A *minimal algebra* $(A, d)$ is a commutative differential graded algebra with $A \cong Sym(V)$ the symmetric algebra on a graded vector space $V = \oplus V_i$ where $V_1 = 0$ and $d(V) \cap V = 0$. (Warning: in the literature the notation $\wedge V$ is more standard.)
- Let $(A, d)$ be a differential graded algebra. A *model* for $(A, d)$ is a quasi-isomorphism $\varphi : (A, d) \to (A', d')$.
- A *minimal Sullivan model* for $(A, d)$ is a model by a minimal algebra.
- A *minimal Sullivan model* for a space $X$ is a a Suliivan model for $(A_{PL}(X), d)$, the piecewise linear polynomial differential forms on $X$. When $X$ is a smooth manifold, $A_{PL}(X) \otimes_{\mathbb{Q}} \mathbb{R}$ is the usual de Rham forms.

We will understand these definitions in the context of some examples, using the fact that minimal Sullivan models are unique to demonstrate our choices are correct.

**Example 3.9.**

(1) Let $X = S^{2n+1}$, and consider the map

$$(Sym(\mathbb{Q}a), da = 0) \to C^*(X; , \mathbb{Q}) \qquad a \mapsto \omega$$

where $\omega$ is a volume form, and $a^2 = 0$ is automatic by the Koszul sign rule. In this case, the image of the quasi-isomorphism lands in cohomology and hence the odd sphere is said to be formal.

(2) Let $X = S^{2n}$ and consider

$$(Sym(\mathbb{Q}a \oplus \mathbb{Q}b), a^2 = db, da = 0) \to C^*(X; \mathbb{Q}) \qquad a \mapsto \omega$$

where $a^2$ is a boundary and hence zero on cohomology.

(3) Let $X = \mathbb{C}P^n$ be complex projective space. Then

$$Sym(\mathbb{Q}a \oplus \mathbb{Q}b), a^{n+1} = db, da = 0) \to C^*(X; \mathbb{Q})$$

where $|a| = 2, |b| = 2n + 1$.

Sullivan finds a convenient model for the space of piecewise linear polynomial differential forms on $X$, $A_{PL}(X)$. The key features of $A_{PL}(X)$ are (1) $H^*(A_{PL}(X)) \cong H^*(X; \mathbb{Q})$ and (2)it is commutative. Basically, Sullivan noted that while the cup product on chains is not commutative, one could find a model where it is.

Sometimes the Sullivan model is quasi-isomorphic to $(H^*(X; \mathbb{Q}), 0)$ in which case one calls $X$ *formal*.

### 3.4 Koszul Duality of the Quillen and Sullivan Approaches

How are these two approaches related?
One can already see similarities for $S^2$, which is both formal and coformal and hence the models are,

$$(\mathbb{Q} \cdot a \oplus \mathbb{Q} \cdot b, [a, a] = b) \qquad (\mathbb{Q} \cdot a \oplus \mathbb{Q}b, db = a^2).$$

More generally, one has the following:

$$V^* := Hom_{\mathbb{Q}}(V, \mathbb{Q}) \cong \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and the quadratic part of the differential $d_2 : V \to \wedge^2 V$ has dual

$$d_2^* = [-, -]_{Wh} : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \otimes \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This is an instance of Koszul duality! Namely, $(\wedge V, d_2)$ is a (Koszul) commutative differential graded algebra and its Koszul dual is the Whitehead Lie algebra.

**Proposition 3.10.** *The (truncated) Sullivan model is Koszul dual to the homology of the Quillen model. That is,* $(\wedge V, d_2)^! \cong \pi_*(\Omega(X)) \otimes_{\mathbb{Z}} \mathbb{Q}$. *In particular, these algebras are Koszul.*

In the case that $X$ is both formal and coformal, then these are precisely the Suillivan and Quillen models. Moreover, in general there is a quasi-isomorphism of differential graded Lie algebras between Quillen's model and the Koszul dual of Sullivan's these are shadows of the more general statement that Quillen's model is Koszul dual to Sullivan's.