# Complex manifolds and the Kähler condition

## 1 Almost Complex structures

**Definition 1.1.** Consider a $2m$-dimensional real manifold, $M$. A *complex chart* on M is a pair $(U, \psi)$, with $U$ open in $M$ and $\psi : U \to \mathbb{C}^m$ a diffeomorphism between $U$ and some open set in $\mathbb{C}^m$. In this way, $\psi$ defines a set of complex coordinates $z^1, \ldots, z^m$ on $U$. If $(U_1, \psi_1)$ and $(U_2, \psi_2)$ are two complex charts, the *transition function* between them is $\psi_{12} : \psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$, defined by $\psi_{12} = \psi_2 \circ \psi_1^{-1}$. $M$ is a *complex manifold* if it has an atlas of complex charts $(U, \phi)$, with all the transition functions *holomorphic.*

**Definition 1.2.** An *almost complex structure* on a smooth even-dimensional real manifold $M$ is a smoothly varying endomorphism, $J$, on each tangent space, satisfying $J^2 = -Id$.

*Example* 1.3. A complex manifold $M$ has a canonical almost complex structure. Choose holomorphic coordinates $z^\alpha = x^\alpha + iy^\alpha$ about $p$. The smooth tangent space of $M$ is generated by $\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}$, define $J\left(\frac{\partial}{\partial x^\alpha}\right) = \frac{\partial}{\partial y^\alpha}$, $J(\frac{\partial}{\partial y^\alpha}) = -\frac{\partial}{\partial x^\alpha}$ and extend linearly.

**Exercise 1.** *Prove that the canonical almost complex structure, $J$, on a complex manifold is independent of the holomorphic coordinates chosen.*

**Definition 1.4.** Let M be an almost complex manifold. Given any $\mathbb{R}$-vector bundle over $M$, say $(E, M, \pi)$, we may consider the bundle with fibres $\pi^{-1}(p) \otimes \mathbb{C}$. The complexified bundle $E_{\mathbb{C}}$ is a $\mathbb{C}$-vector bundle with rank equal to $rank(E)$.

For any almost complex manifold we have the rank $2n$ $\mathbb{R}$-bundles $TM$ and it's dual $T^\star M$. We call $TM_{\mathbb{C}}$ and $T^\star M_{\mathbb{C}}$ the *complexified tangent and cotangent bundles* respectively.

Recall that $J$ is an endomorphism of $T_pM$ for each $p$. The map $J(\alpha)(X) := \alpha(J(X))$ for a $1-$form $\alpha$, is an endomorphism of $T^\star_p M$ satisfying $J^2 = -Id$. $J$ extends by linearity to endomorphisms of $TM_{\mathbb{C}}$ and $T^\star M_{\mathbb{C}}$ (still with $J^2 = -Id$). Crucially the complexified cotangent space has a direct sum decomposition by $J$-eigenspaces.

**Proposition 1.5.** *Let $J$ be an endomorphism of $T^\star M_{\mathbb{C}}$ with $J^2 = -Id$. Let $\Lambda^{(1,0)}$, $\Lambda^{(0,1)}$ be the $J$-eigenspaces of $i, -i$ respectively. Then*

$$T^\star M_{\mathbb{C}} = \Lambda^{(1,0)} \oplus \Lambda^{(0,1)}$$

*Proof.* Define $\Sigma : T^\star M_{\mathbb{C}} \to \Lambda^{(1,0)} \oplus \Lambda^{(0,1)}$ by $\Sigma(X) = \frac{1}{2}(X - iJX, X + iJX)$ and $\Phi : \Lambda^{(1,0)} \oplus \Lambda^{(0,1)} \to T^\star M_{\mathbb{C}}$ by $\Phi(X, Y) = X + Y$. $\Sigma$ and $\Phi$ are inverse linear maps. □

*Remark* 1.6. The complexified tangent space $(TM)_{\mathbb{C}}$ also admits a decomposition into $i, -i$ eigenspaces by the same argument. We denote this decomposition by $TM_{\mathbb{C}} = T^{(1,0)} \oplus T^{(0,1)}$.

**Proposition 1.7.** $\Lambda^{(1,0)}$ *is the annihilator of* $T^{(0,1)}$.

*Proof.* Take $X \in T^{(0,1)}$ and $\alpha \in \Lambda^{(1,0)}$. Then $(J\alpha)X = i\alpha X$ since $\alpha$ is in the $i$ eigenspace. But we also have $(J\alpha)X := \alpha(JX) = \alpha(-iX) = -i\alpha X$ since $X$ is in the $-i$ eigenspace. So we have that $\alpha X = 0$. $\square$

We get a corresponding decomposition of the $kth$ wedge power of $T^\star M_{\mathbb{C}}$:

$$\Lambda^k(T^\star M_{\mathbb{C}}) = \bigoplus_{p+q=k} \Lambda^p(\Lambda^{(1,0)}) \otimes \Lambda^q(\Lambda^{(0,1)})$$

To study forms on almost complex manifolds we need to build up quite a bit of notation:

$\Lambda^{(p,q)} := \Lambda^p(\Lambda^{(1,0)}) \otimes \Lambda^q(\Lambda^{(0,1)})$ is the *(p,q)-cotangent bundle* of $M$

$\mathcal{A}^k(M) := \Gamma(\Lambda^k(T^\star M_{\mathbb{C}}))$ is the space of *k-forms* on $M$

$\mathcal{A}^{(p,q)}(M) := \Gamma(\bigwedge^{(p,q)})$ is the space of *(p,q)-forms* on $M$.

## 2 Integrability

We have studied forms on an almost complex manifold in some detail. Now it is time to analyse when an almost complex manifold is induced by a complex structure, and in this case define Dolbeault cohomology. First some useful definitions.

The *Lie bracket* of vector fields from differential geometry will be used to study almost complex structures. If we think of vector fields as smooth derivations then we may define it as a commutator by $[X,Y](f) = X(Y(f)) - Y(X(f))$. One geometric interpretation of the Lie bracket is that it measures how much the flow lines for $X$ and $Y$ fail to commute (see [1, Chapter 20]).

**Definition 2.1.** The *Nijenhuis tensor* of an almost complex manifold $(M, J)$ is defined

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

We can think of this as somehow measuring the "torsion" of the almost complex structure.

We saw in Section 1 that every complex manifold admits a canonical almost complex structure. The question we now ask is, *given* some almost complex structure on a manifold, when has it arisen from holomorphic co-ordinates in this way?

Well, if it has we know that $\mathcal{A}^{(0,1)}$ is spanned locally by $\alpha = \sum f_i \mathrm{d}z^i$ for smooth functions $f_i$. Then $\mathrm{d}\alpha = \sum \mathrm{d}f_i \wedge \mathrm{d}z^i$ which is in

$$\left(\mathcal{A}^{(1,0)} \oplus \mathcal{A}^{(0,1)}\right) \wedge \mathcal{A}^{(1,0)} = \mathcal{A}^{(2,0)} \oplus \mathcal{A}^{(1,1)}$$

So for complex manifolds, $M$,

$$\mathrm{d}\left(\mathcal{A}^{(1,0)}(M)\right) \subseteq \mathcal{A}^{(2,0)}(M) \oplus \mathcal{A}^{(1,1)}(M)$$

**Proposition 2.2.** *For an almost complex manifold $(M, J)$, TFAE:*

1. $\mathrm{d}\left(\mathcal{A}^{(1,0)}(M)\right) \subseteq \mathcal{A}^{(2,0)}(M) \oplus \mathcal{A}^{(1,1)}(M)$

2. $\Gamma\left(T^{(1,0)}\right)$ *is closed under taking Lie brackets.*

3. $N \equiv 0$

*Proof.* Recall that $X \mapsto (X - iJX, X + iJX)$ gives an isomorphism $\Sigma : (TM)_{\mathbb{C}} \to T^{(1,0)} \oplus T^{(0,1)}$ (see the proof of proposition 1.5 and following remark).

(1) $\iff$ (2) *Exercise 2.*
Hint: let $\alpha \in \mathcal{A}^{(1,0)}$ and use Cartan's formula $2\mathrm{d}\alpha(X, Y) = X(\alpha Y) - Y(\alpha X) - \alpha[X, Y]$.

(2) $\iff$ (3) Let $X, Y \in \Gamma(TM_{\mathbb{C}})$, then $X - iJX, Y - iJY \in \Gamma(T^{(1,0)})$ and by assumption so is $[X - iJX, Y - iJY]$, equivalently $[X - iJX, Y - iJY] + iJ[X - iJX, Y - iJY] \equiv 0$. Expanding out the terms of each Lie bracket shows that $[X - iJX, Y - iJY] + iJ[X - iJX, Y - iJY] = -N(X, Y) - iJ(N(X, Y))$. So $N \equiv 0$ is equivalent to the expression on the left vanishing for all $X - iJX, Y - iJY \in \Gamma(TM_{\mathbb{C}})$, *i.e.* (2).

□

If these conditions hold, we say that $J$ is *integrable.*

**Theorem 2.3** (Newlander-Nierenberg)**.** *An integrable almost complex structure is induced by a complex structure. That is, the above conditions are sufficient (as well as necessary) for a manifold to be complex.*

Recall that the usual exterior derivative operator for smooth manifolds $d$ maps $k$ forms to $k + 1$ forms. We extend this linearly to sections of the complexified cotangent bundle to get an operator.

$$d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$$

Let $\Pi^{(p,q)}$ be the projection of $\Lambda^k(T^\star M)_{\mathbb{C}}$ to $\Lambda^{(p,q)}$. Define $\partial = \Pi^{(p+1,q)} \circ d$ and $\bar{\partial} = \Pi^{(p,q+1)} \circ d$, then

$$\partial : \mathcal{A}^{(p,q)}(M) \to \mathcal{A}^{(p+1,q)}(M)$$

$$\bar{\partial} : \mathcal{A}^{(p,q)}(M) \to \mathcal{A}^{(p,q+1)}(M)$$

For all $k \geq 0$, $d = \partial + \bar{\partial} : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$

3) $\Leftrightarrow$ 4) For the right to left implication note that by definition $\partial(\mathcal{A}^{(1,0)}(M)) \subseteq \mathcal{A}^{(2,0)}(M)$ and $\bar{\partial}(\mathcal{A}^{(1,0)}(M)) \subseteq \mathcal{A}^{(1,1)}(M)$. For the reverse implication see [2, 2.6.15].

By expanding $d^2 = (\partial + \bar{\partial})^2 = 0$ on $\mathcal{A}^{(p,q)}$ and decomposing the image into $(p+2, q)$, $(p+1, q+1)$ and $(p, q+2)$ forms, we see that for an integrable almost complex structure

$$\partial^2 = \partial\bar{\partial} - \bar{\partial}\partial = \bar{\partial}^2 = 0$$

In this scenario we may take the cohomology of the following sequence:

$$\ldots \xrightarrow{\bar{\partial}} \mathcal{A}^{(p,q-1)}(M) \xrightarrow{\bar{\partial}} \mathcal{A}^{(p,q)}(M) \xrightarrow{\bar{\partial}} \mathcal{A}^{(p,q+1)}(M) \xrightarrow{\bar{\partial}} \ldots$$

i.e we define the quotient vector spaces: $H^{(p,q)}(M) = \frac{ker(\bar{\partial}:\mathcal{A}^{(p,q)}(M)\to\mathcal{A}^{(p,q+1)}(M))}{Im(\bar{\partial}:\mathcal{A}^{(p-1,q)}(M)\to\mathcal{A}^{(p,q)}(M))}$ called the **Dolbeault cohomology** of $M$, which depends on the complex structure.

*Remark* 2.4. Not all even-dimensional smooth manifolds admit an almost complex structure, $S^4$ with the standard smooth structure is a counterexample.

# 3 The Kähler Condition

Let $(M, J)$ be an almost complex manifold and suppose in addition we have a Riemannian metric $g$ such that $J$ is an orthogonal transformation with respect to $g$. In symbols: For any vector fields $X$ and $Y$ we have $g(X, Y) = g(JX, JY)$. In this case the triple $(M, g, J)$ is called a **Hermitian Manifold**.

Now we define a non-degenerate 2-form $\omega(X, Y) = g(JX, Y)$ ($\omega$ is called non-degenerate if $\omega(X, Y) = 0\ \forall X \Rightarrow Y = 0$). This allows us to state the **Kähler condition**:

**Definition 3.1.** Suppose that $(M, g, J)$ is a Hermitian manifold if the associated 2-form $\omega$ is closed then we call $M$ a Kähler manifold.

For Kähler (and most generally symplectic) manifolds the existance of such an $\omega$ imposes topological restrictions on even-dimensional (orientable) manifolds admitting such structures.

**Proposition 3.2.** *Let $M$ be a 2n-dimensional manifold with a closed non-degenerate 2-form $\omega$. Then the even dimensional de Rham cohomology groups have strictly positive dimension.*

*Proof.* The 2-form omega is non-degenerate, which is equivalent to the form $\omega^n$ being everywhere non-zero, and hence a volume form. If this form were $d\gamma$ for an (n-1)-form $\gamma$ then by Stoke's theorem the integral of $\omega^n$ over $M$ would be 0, hence this form represents a non-trivial cohomology class. Recall that the wedge map on forms descends to a product of cohomology groups so that $[\omega^n] = [\omega]^n$, in particular we must have $[\omega]^k \neq 0$ for each $k$. We have a non-zero element of $H^{2k}(M)$ for $k = 1, .., n$. □

# 4 Exercises

1. Prove that the canonical almost complex structure, $J$, on a complex manifold is independent of the holomorphic coordinates chosen.

2. Show that the following two conditions for an almost complex structure to be a complex structure are equivalent:

$$\mathrm{d}\left(\Gamma\left(\Lambda^{0,1}\right)\right) \subseteq \Gamma\left(\Lambda^{2,0} \oplus \Lambda^{1,1}\right) \tag{1}$$

$$X, Y \in \Gamma\left(T^{1,0}\right) \implies [X, Y] \in \Gamma\left(T^{1,0}\right) \tag{2}$$

3. Consider an almost complex structure on $\mathbb{R}^4$ given by $\Lambda^{1,0} = \langle\sigma^1, \sigma^2\rangle$ where $\sigma^1 = \mathrm{d}z^1 + a\,\mathrm{d}\bar{z}^2$, $\sigma^2 = \mathrm{d}z^2 - a\,\mathrm{d}\bar{z}^1$, and $a$ is a smooth function of $z^1$ and $z^2$.

(a) Check $\Lambda^{1,0} \cap \overline{\Lambda^{1,0}} = 0$, so this does define an almost complex structure

(b) Show that condition (1) in exercise 2 is equivalent to

$$\frac{\partial a}{\partial \bar{z}^1} + a\frac{\partial a}{\partial z^2} = 0 = \frac{\partial a}{\partial \bar{z}^2} - a\frac{\partial a}{\partial z^1}$$

(c) Deduce that $T^{0,1} = \left\langle \frac{\partial}{\partial \bar{z}^1} + a\frac{\partial}{\partial z^2}, \frac{\partial}{\partial \bar{z}^2} - a\frac{\partial}{\partial z^1} \right\rangle$

(d) The metric associated to the standard inner product on $\mathbb{R}^4$ is almost Hermitian, i.e. $g(JX, JY) = g(X, Y)$ for all $X, Y \in T_m\mathbb{R}^4$. Show that this is equivalent to $\Lambda^{1,0}$ being *isotropic* for the complexification of $g$, i.e. $g(\sigma^i, \sigma^j) = 0$ for all $i, j$.

(e) Express $J$ as a $4 \times 4$ matrix relative to $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^2}$

4. Show that, for $\omega$ a Hermitian form, $\omega(X - iJX, Y - iJY) = 0$.

5. Prove the following lemma:
If $\nabla_X\omega \in \Lambda^2$ with $\omega \in \Lambda^{1,1}$ then $\nabla_X\omega \in \Lambda^{2,0} \oplus \Lambda^{0,2}$, i.e. $\nabla_X\omega(JX, JY) = -\nabla_X\omega(X, Y)$
Starting hint: $(\nabla_X\omega)(Y, Z) = g\left((\nabla_X J)\, Y, Z\right)$

6. Show that projective manifolds are Kähler. The idea is to show that (complex) projective space is Kähler, which requires us to define the Fubini-Study Metric structure on $\mathbb{P}^n$.

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