Introduction to Hodge Theory

1 Introduction

In this notes we are going to introduce some of the concepts concerning with Hodge Theory and give some interesting applications of the main theorems. We will follow both real and complex case where we can see how the theory provides beautiful connections between geometry and topology of manifolds. In the final section we will discuss how the Hodge theorem can provide some topological restrictions for a complex manifold to be Kähler.

2 Background

2.1 Hodge star operator and Laplacian

Let (M^n,g) be a closed manifold, i.e., compact without boundary, Riemannian and oriented. We denote $\Lambda^k(M)$ the space of sections of the bundle of differential k-forms on the manifold, i.e., $\Lambda^k(M) := \Gamma(\Omega^k(M))$.

First we define the Hodge star operator:

$$\star : \Lambda^k(M) \to \Lambda^{n-k}(M)$$

by requiring $\alpha \wedge \star\beta =< \alpha, \beta > dVol_M$, where $<,>$ is the induced inner product defined on the bundles of k forms.

If we choose a positive orthonormal basis $\{e_1, \ldots, e_n\}$ for the tangent space T_xM at $x \in M$, then

$$\star(e^1 \wedge \ldots \wedge e^k) = e^{k+1} \wedge \ldots \wedge e^n.$$

In particular, in local coordinates around x we have $dvol_M = \sqrt{det(g_{ij})}dx_1 \wedge \ldots \wedge dx_n$.

Example 1. *Let $\omega \in \Lambda^1(M)$, $\omega = \omega^i dx_i$, then*

$$\star\omega = (-1)^{j-1} g^{ij} \omega^i dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n$$

.

Exercise 1. *Check that $\star\star = (-1)^{k(n-k)} Id : \Lambda^k(M) \to \Lambda^k(M)$.*

Our next step is to define the L^2 inner product of $\alpha, \beta \in \Lambda^k(M)$ by:

$$(\alpha, \beta) := \int_M < \alpha, \beta > dvol_M = \int_M \alpha \wedge \star\beta$$

Recall the exterior derivative $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$, the fundamental property of d is that $d \circ d = 0$. We say $\omega \in \Lambda^k(M)$ is *closed* if $d\omega = 0$ and when $\omega = d\eta$ then ω is called exact form. From $d \circ d = 0$ we see that exact forms are also closed forms, therefore we can take the quotient of the subspace of closed forms over the exact forms. This quotient is a R vector space called the $k-th$ de Rham cohomology group and denoted by

$$H^k_{DR}(M).$$

In the presence of the L^2 inner product $(,)$ we can ask for the adjoint of the operator d which we will denote by $d^\star$. In other words,

$$d^\star : \Lambda^k(M) \to \Lambda^{k-1}(M),$$

satisfying $(d\alpha, \beta) = (\alpha, d^\star \beta)$.

Lemma 1. *$d^\star : \Lambda^k(M) \to \Lambda^{k-1}(M)$ satisfies*

$$d^\star = (-1)^{n(k-1)-1} \star d \star .$$

Proof. Let α be a $k-1$ form and β a k form. Recall the Leibniz rule for the d operator:

$$d(\alpha \wedge \star\beta) = d\alpha \wedge \beta + (-1)^{k-1} \alpha \wedge d \star \beta.$$

Using Stokes theorem we get:

$$\int_M d\alpha \wedge \beta = \int_M (-1)^k \alpha \wedge d \star \beta$$

Now we observe that $d\star\beta$ is a $n-k+1$ form. Therefore $\star\star = (-1)^{(n-k+1)(k-1)} Id$.

$$(d\alpha, \beta) = \int_M d\alpha \wedge \star\beta = \int_M \alpha \wedge \star[(-1)^{(n-k+1)(k-1)}(-1)^k \star d \star \beta] =$$

$$= \int_M \alpha \wedge \star[(-1)^{n(k-1)-1} \star d \star \beta] = (\alpha, (-1)^{n(k-1)-1} \star d \star \beta).$$

Therefore, $d^\star = (-1)^{n(k-1)-1} \star d \star$. □

Definition 1. *The Laplace operator on $\Lambda^k(M)$ is the linear operator*

$$\Delta = dd^\star + d^\star d : \Lambda^k(M) \to \Lambda^k(M).$$

ω is called harmonic if $\Delta(\omega) = 0$.

Properties of the Laplacian

1. $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$, i.e, ($\Delta$ is self-adjoint).
2. $\star\Delta = \Delta\star$.
3. $\Delta\alpha = 0$, if and only if $d\alpha = d^\star\alpha = 0$.

   Proof. $(\Delta\alpha, \alpha) = (dd^\star\alpha, \alpha) + (d^\star d\alpha, \alpha)$
   $= (d^\star\alpha, d^\star\alpha) + (d\alpha, d\alpha)$
   $= |d^\alpha|^2 + |d\alpha|^2$ □

Example 2. *If $k = 0$ then $\Delta = d^{\star}d$ and in local coordinates we have for any $f \in C^{\infty}(M)$:*

$$\Delta f = -\frac{1}{\sqrt{det(g_{ij})}}\frac{\partial}{\partial x_i}[g^{ij}\sqrt{det(g_{ij})}\frac{\partial f}{\partial x_j}]$$

Example 3. *In the Euclidean case there is a natural expression for the laplacian of a k form. If $\omega = \omega^I dx_I$ then*

$$\Delta_e \omega = -\sum_{m=1}^{n} \frac{\partial^2 \omega^I}{\partial x_m^2} dx_I.$$

We have seen that any harmonic $k-$form is closed and therefore it represents a nontrivial cohomological class in $H^k_{DR}(M)$. Indeed, if ω is harmonic and exact then $|\omega|^2 = (\omega, \omega) = (\omega, d\eta) = (d^{\star}\omega, \eta) = 0$ which implies that $\omega = 0$. Let's denote the space of k harmonic forms by $H^k(M)$, hence we have constructed a injective map

$$H^k(M) \to H^k_{DR}(M).$$

We can try to find a minimizing element in each cohomological class with respect the L^2 inner product $(\cdot, \cdot)$.

If ω is a closed smooth k form and has minimal norm in its cohomological class then $d^{\star}\omega = 0$. Indeed, let $\eta \in \Lambda^{k-1}(M)$

$$0 = \frac{\partial}{\partial t}|\omega + td\eta|^2|_{t=0}$$

$$0 = \frac{\partial}{\partial t}(\omega + td\eta, \omega + td\eta) = 2(\omega, d\eta)$$

This is equivalent to $(d^{\star}\omega, \eta) = 0$ for all $\eta \in \Lambda^{k-1}(M)$ ie, $d^{\star}\omega = 0$.

Conversely, if ω is closed and $d^{\star}\omega = 0$ (i.e., ω is harmonic) then ω has minimal norm. It is enough to prove that $|\omega + d\eta|^2 > |\omega|^2$.

$$|\omega + d\eta|^2 = (\omega + d\eta, \omega + d\eta) = |\omega|^2 + |d\eta|^2 + 2(\omega, d\eta)$$

So we are done since the last term on the right hand side is zero.

A natural question is if it possible to minimize the norm in each cohomological class such that the minimum is attained by a smooth differential form? The answer is positive thanks to the following theorem:

Theorem 1 (Hodge)**.** *Let M^n be an oriented closed Riemannian manifold of dimension n.Then $H^k_{DR}(M)$ has finite dimension and in each cohomological class we have precisely one harmonic k form.*

Remark In general, the above theorem can also be seen in the following form:

$$\Lambda^k(M) = H^k(M) \bigoplus Im\Delta,$$

where $H^k(M) = Ker(\Delta)$ has finite dimension.

Exercise 2. *Check that the above decomposition implies the Hodge Theorem.*

The first application is the famous Poincaré duality:

Corollary 1 (Poincaré Duality)**.** *Let M^n a closed oriented differentiable manifold then we have an isomorphism between $H^k_{DR}(M)$ and $H^{n-k}_{DR}(M)$.*

Proof. First remember that any differentiable manifold can be equipped with a Riemannian metric. By Hodge Theorem it is enough to prove that we have a isomorphism between $\mathcal{H}^k(M)$ and $\mathcal{H}^{n-k}(M)$. For this we just use the property that the Hodge star operator sends harmonic forms into harmonic forms. Indeed, just recall

$$\star\Delta = \Delta\star$$

Therefore, $\star$ is the isomorphim we are looking for. □

Definition 2. *For each k from 0 to n we define the $k-th$ **Betti Number** as $b_k(M) := dim H^k_{DR}(M)$.*

By Poincaré Duality we have $b_k(M) = b_{n-k}(M)$

We would like to give another important application of the Hodge Theorem in the scenary of Riemannian Geometry. Let's consider a (M^n, g, ∇) a Riemannian manifold with the Levi Civita conection associated to its metric ($\nabla g = 0$). Recall the Cuvarture Tensor R which is a tensor ($C^\infty(M)$ linear in each variable):

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

defined by:

$$R(X, Y, Z) := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R(X, Y, Z, W) = g(R(X, Y, Z), Z).$$

Note that

$$R(X, Y, Z, W) = -R(Y, X, Z, W)$$

and

$$R(X, Y, Z, W) = R(Z, W, X, Y).$$

If we fixe $x, y \in T_pM$ then $R_{x,y} : T_pM \to T_pM$ defined by $R_{x,y}(z) = R(x, z, y)$ is a linear map on T_pM, so it makes sense to take its trace which we will denote by $Ric(x, y)$.

If we choose a orthonormal basis $\{e_1, \ldots, e_n\}$ at T_pM then we can define the Ricci curvature as:

$$Ric(X, Y) := \text{Trace}\, R(X, \cdot, Y) = \sum_{i=1}^{n} R(X, e_i, Y, e_i).$$

Since the Ricci curvature is symmetric, it defines on each tangent space an endomorphism which is selfadjoint with respect to the metric g, that we still denote by Ric. In particular, we can extent its action to an endormophism on $\Gamma(T^\star(M)) = \Lambda^1(M)$.

$$Ric(\alpha)(v) := \alpha(Ric(v)).$$

Now we can state a simple case of a famous identy in Riemannian Geometry, known as the **Weitzenböck formula** for 1-forms:

$$\Delta\alpha = \nabla^\star\nabla\alpha + Ric(\alpha),$$

where $\nabla^\star$ is the adjoint of ∇ with respect the L^2 inner product.

Theorem 2. *Let (M^n, g) is a closed oriented Riemannian manifold with non-negative Ricci curvature then the first betti number $b_1(M) \leq n$. Besides, if Ric is positive then $b_1(M) = 0$.*

Proof. If α is a harmonic 1-form then

$$0 = (\Delta\alpha, \alpha) = (\nabla\alpha, \nabla\alpha) + (Ric(\alpha), \alpha).$$

Now,

$$(Ric(\alpha), \alpha) = \int_M g(Ric(\alpha), \alpha) dv_g.$$

But,

$$\begin{aligned} g(Ric(\alpha), \alpha) &= \sum_i \alpha(e_i) Ric(\alpha)(e_i) = \\ &= \sum_i \alpha(e_i)\alpha(Ric(e_i)) = \\ &= \sum_i g(Ric(e_i), \alpha^\sharp) g(\alpha^\sharp, e_i) = \\ &= g(Ric(\alpha^\sharp), \alpha^\sharp) = Ric(\alpha^\sharp, \alpha^\sharp). \end{aligned}$$

Therefore, we have $|\nabla\alpha|^2 + \int_M Ric(\alpha^\sharp, \alpha^\sharp) = 0$.

An immediate consequence is that if $Ric > 0$ then $b_1 = 0$. If $Ric \geq 0$ then we get $\nabla\alpha = 0$, in other words, α is parallel and so its norm is constant and consequently if we have b_1 linearly indepedent harmonic 1 forms then in each tangent space we will have b_1 linearly indepedent vectors on $T^\star_p(M)$ and therefore $b_1 \leq n$. □

The above theorem gives a topological obstruction for the existence of metrics with positive Ricci curvature. For example, $T^n = \mathbb{R}^n/\mathbb{Z}^n$ and $\mathbb{S}^1 \times \mathbb{S}^2$ do not have a such metric since $b_1 = n$ and $b_1 = 1$ respectively.

3 Applications of Hodge Theory on Complex Geometry

For this section we are going to consider a compact complex manifold (M^{2n}, g, J), where g is a Riemannian metric compatible with the complex structure J. The complexification of tangent space $T_xM \otimes \mathbb{C}$ can be decomposed as $T^{1,0}_x(M) \oplus T^{0,1}_x(M)$, where $T^{1,0}_x(M)$ is the vector space generated by $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\}$ with $T_x 0, 1(M) = \overline{T^{1,0}_x(M)}$.

In the same way, we can express the complexification of the space of k forms as

$$(\mathcal{A}^k(M))_C = \mathcal{A}^k(M) \otimes \mathbb{C} = \oplus[\Lambda^p T^{1,0}(M) \otimes \Lambda^q T^{0,1}(M)].$$

From now on, we use the following notation for the decompsition above $\mathcal{A}^k = \oplus \mathcal{A}^{p,q}$. Recall the natural operators in this setting:

$$\partial : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p+1,q}(M)$$
$$\overline{\partial} : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M)$$

The action of $\overline{\partial}$ on a particular (p,q)-form $\omega = \omega^{IJ} dz^I \wedge d\overline{z}^J$ is given by:

$$\overline{\partial}\omega = \frac{\partial}{\partial \overline{z}_k}\omega^{IJ} d\overline{z}_k \wedge dz^I \wedge d\overline{z}^J$$

The action of ∂ is analogous. Recal that $d = \partial + \overline{\partial}$. Since $d \circ d = 0$, we have $\partial \circ \partial = 0$, $\overline{\partial} \circ \overline{\partial} = 0$.

Definition 3. *Denoting the space of $(p,q)-$closed forms wrt $\overline{\partial}$ as $Z^{p,q}_{\overline{\partial}}(M)$ then the groups*

$$H^{p,q}_{\overline{\partial}}(M) = \frac{Z^{p,q}_{\overline{\partial}}(M)}{\overline{\partial} A^{p,q-1}(M)}$$

are called the Dolbeault cohomology groups.

From our Riemannian metric g we can obtain a hermitian metric on $(T_xM)_C$ by extending this product linearly with respec to $\mathbb{C}$. We will denote this hermitian product by h.

Excercise Check the following properties of $h : (T_xM)_C \times (T_xM)_C \to \mathbb{C}$:

1. $h(\overline{Z}, \overline{W}) = \overline{h(Z,W)}$;
2. $h(Z, \overline{Z}) > 0$;
3. $h(Z,W) = 0$, if $Z, W \in T^{1,0}(M)$.

There is a natural isomorphism between T_xM and $T^{1,0}_x(M)$ given by $v \mapsto v - iJ(v)$. Therefore, using this isomorphism we can get the following formula:

$$h(v - iJv, w + iJw) = 2g(v,w) - 2ig(Jv, w).$$

That expression just means that we can regain the riemannian metric from the hermitian metric by taking the real part of it. Conversely, for any hermitian product (satisfying the properties $1, 2$ and 3) we can produce a riemannian metric on the manifold by taking the real part. In the same way, by taking the imaginary part of a hermitian product we get a 2-form $\omega \in \Lambda^2(M)$, in our case $\omega = g(J\cdot, \cdot)$.

In local coordinates, $h_{ij} = h(\partial z_i, \partial \overline{z}_j) = \overline{h_{ji}}$. The real 2-form $\omega(v,w) = g(Jv, w)$is called the fundamental form of the complex manifold.

Exercise 3. *In local coordinates $\omega = -ih_{ij} dz_i \wedge d\overline{z}_j$.*

Exercise 4. $\omega^n = n! dvol_M$.

Definition 4. *We say that a complex manifold (M^{2n}, g, J) is a Kähler manifold if the fundamental form $\omega = -\frac{i}{2} h_{ij} dz_i \wedge d\overline{z}_j$ is closed.*

Proposition 1. *If (M, g, J, ω) is a Kähler manifold then $[\omega]$ represents a nontrivial cohomological class in $H^2_{DR}(M)$. The same is true for $[\omega^i] \in H^{2i}_{DR}(M), i \in \{1, \ldots, n\}$.*

Proof. If $\omega^i = d\eta$ then $\omega^n = d\eta \wedge \omega^{n-i} = d(\eta \wedge \omega^{n-i}) - (-1)^{2i-1} \eta \wedge d\omega^{n-i} = d(\eta \wedge \omega^{n-i})$ because ω is closed. Therefore, applying Stokes Theorem we have $n! vol(M) = 0$ which is a contradiction. ☐

The Hodge star operator $\star$ can be naturally extend to a map $\star : \mathcal{A}^k(M) \to \mathcal{A}^{2n-k}$. It is possible to see that in this way the Hodge star satisfies:

$$\alpha \wedge \star\beta = h(\alpha, \beta) dVol_M.$$

We can observe that the hermitian product h is non-zero only if we apply it to forms of type (p, q) and (q, p) respectively. It is an easy exercise to see that

$$\star : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{n-q,n-p}(M).$$

As we did in the Riemannian case, let's define the L^2 inner product $(\cdot, \cdot)$ in the space $\mathcal{A}^k(M)$ as

$$(\alpha, \beta) = \int_M \alpha \wedge \star\overline{\beta}$$

Lemma 2. *The decomposition $\mathcal{A}^k(M) = \oplus_{\{p+q=k\}}\mathcal{A}^{p,q}$ is orthogonal with respect to $(\cdot, \cdot)_{L^2}$.*

Let's first recall that $d^\star = - \star d\star$ because the $\dim M = 2n$. The next proposition provides the ajoint of the operators ∂ and $\overline{\partial}$.

Proposition 2. *Consider $\partial^\star := - \star \partial\star$ and $\overline{\partial}^\star := - \star \overline{\partial}\star$, then*

$$(\partial\alpha, \beta) = (\alpha, \partial^\star\beta)$$

$$(\overline{\partial}\alpha, \beta) = (\alpha, \overline{\partial}^\star\beta)$$

Proof. The idea is again to use the Leibniz rule together with Stokes Theorem again, just remember that $d = \overline{\partial}$ when applied to form of type $(n, n-1)$. The details are left to the reader. □

The Laplace operators are defining by

$$\Delta_d = dd^\star + d^\star d \tag{1}$$

$$\Delta_{\overline{\partial}} = \overline{\partial}^\star\overline{\partial} + \overline{\partial}\overline{\partial}^\star \tag{2}$$

$$\Delta_\partial = \partial\partial^\star + \partial^\star\partial. \tag{3}$$

Definition 5. *Let $\omega \in \mathcal{A}^{p,q}(M)$, we say ω is $\overline{\partial}-$harmonic if $\Delta_{\overline{\partial}}\omega = 0$. The spaces of $\overline{\partial}-$harmonic (p, q)-form is denoted by $\mathcal{H}^{p,q}_{\overline{\partial}}(M)$.*

Remark I should mention here that I am making an abuse of notation by using $\mathcal{H}^k(M)$ for both the space of real k-harmonic forms and the complex k-harmonic forms.To be precise, the second space is just the complexification of the first one. The same remark should be applied to the De Rham cohomology groups in the real and complex case.

Lemma 3. *$\omega \in \mathcal{A}^{p,q}(M)$ is $\overline{\partial}-$harmonic if and only if $\overline{\partial}\omega = 0$ and $\overline{\partial}^\star\omega = 0$.*

For the sake of clarity, let's state again the Hodge theorem,

Theorem 3 (Hodge)**.** *We have the following orthogonal decomposition:*

$$\mathcal{A}^{p,q}(M) = \mathcal{H}^k_d(M) \oplus d\mathcal{A}^{k-1}(M) \oplus d^\star\mathcal{A}^{k+1}(M)$$

.

Corollary 2 (Hodge)**.** *In each cohomological class there is only one harmonic form. Moreover, there is a natural isomorphism between* $H_d^k(M) = \mathcal{H}_d^k(M)$.

Each $\overline{\partial}-$harmonic form associates a cohomological cass in the Dolbeault cohomology groups. The technique used to prove Hodge theorem can be used again to prove the following:

Theorem 4 (Dolbeault Theorem)**.** *We have the following orthogonal decomposition:*

$$\mathcal{A}^{p,q}(M) = \mathcal{H}^{p,q}_{\overline{\partial}}(M) \oplus \overline{\partial}\mathcal{A}^{p,q-1}(M) \oplus \overline{\partial}^{\star}\mathcal{A}^{p,q+1}(M).$$

Corollary 3. *In each Dolbault cohomological class there is only one* $\overline{\partial}-$*harmonic form. Moreover, there is a natural isomorphism between* $H^{p,q}_{\overline{\partial}}(M)$ *and* $\mathcal{H}^{p,q}_{\overline{\partial}}(M)$.

Exercise 5. *Let* C *be a smooth projective curve or equivalently a compact Riemann surface of genus* $g > 0$. *Prove using Hodge theory that any map* $\mathbb{P}^1 \to C$ *is constant.*

Proposition 3 (Kähler Identity)**.** *If* (M^{2n}, g, J, ω) *is a Kähler manifold then we have the following identity:*

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}.$$

The L^2 product gives a orthogonal decomoposition for $\mathcal{A}^k(M)$

$$\mathcal{A}^k(M) = \oplus_{p+q=k}\mathcal{A}^{p,q}(M).$$

It is clear from our definitions that the $\Delta_{\overline{\partial}}$ preserves such decomposition. The same is in general not true for the Δ_d. However, the above propostion says that this remarkable property holds when we considering a Kähler manifold.

In other words,

$$\mathcal{H}^k(M) = \oplus\mathcal{H}^{p,q}(M) = \oplus\mathcal{H}^{p,q}_{\overline{\partial}}(M)$$

is a orthogonal decomposition of the space of k-harmonic forms in harmonic forms of type $(p, q), p+q = k$. By $\mathcal{H}^{p,q}(M)$ we mean $\mathcal{H}^k(M) \cap \mathcal{A}^{p,q}(M)$. Observe that from the above proposition $\mathcal{H}^{p,q}(M) = \mathcal{H}^{p,q}_{\overline{\partial}}(M)$.

The Betti numbers $b_k(M) = \dim H^k_{DR}(M, \mathbb{R})$ is equal $\dim_{\mathbb{C}} \mathcal{H}^{p,q}(M)$. Hence,

$$b_k(M) = \sum_{p+q=k} b_{p,q}(M).$$

Because d is a real operator; thus α is harmonic iff $\overline{\alpha}$ is harmonic. As a consequence, in the Kähler case,

$$\mathcal{H}^{p,q}(M) = \overline{\mathcal{H}^{q,p}}.$$

Therefore, $b_{p,q}(M) = b_{q,p}(M)$. An immediate consequence is the following:

Corollary 4. *If* (M^{2n}, J, ω) *is a compact Kähler manifold,*

$$b_k(M) = \sum_{p+q=k} b_{p,q}(M) \tag{4}$$

$$b_{p,q}(M) = b_{q,p}(M) \tag{5}$$

$$b_k(M) \text{ is even for odd } k. \tag{6}$$

The above corollary is a strong topological obstruction for a complex manifold be Kähler. See the next example:

Example 4 (Hopf Manifold)**.** *Consider in* $\mathbb{C}^n \setminus \{0\}$ *the free and discrete* $\mathbb{Z}$*-action by:*

$$n \cdot (z_1, \ldots, z_n) = (\lambda^n z_1, \ldots, \lambda^n z_n),$$

where λ *is a complex number of* $|\lambda| < 1$*. The quotient of* $\mathbb{C}^n \setminus \{0\}$ *by this action is a complex manifold* X *called the Hopf manifold. One can easily show that* X *is diffeomorphic to* $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$*. In particular, we get* $b_1(X) = b_1(\mathbb{S}^1 \times \mathbb{S}^{2n-1}) = 1$*. Since the first Betti number of a Kähler manifold is even, a Hopf manidold does not admit a Kähler structure.*