Deformation theory

Abstract

These notes are meant to be a quick and quite informal introduction to "Deformation Theory". The focus is on some concrete simple examples, we refer to other sources for a systematic treatment of the subject.

Introduction

Deformation theory is concerned with the understanding of the local geometry of moduli spaces, which are among the most central objects in modern algebraic geometry. Moduli spaces naturally arise when we try to classify algebro-geometric objects: given a class $\mathcal{M}$ of algebro-geometric object, *e.g.*

$\mathcal{M} = \{$isomorphism classes of vector bundles of fixed rank over X$\}$

$\mathcal{M} = \{$isomorphism classes of subvarieties of a fixed varieties X with fixed Hilbert polynomial$\}$

$\mathcal{M} = \{$ isomorphism classes of regular maps $f\colon X \to Y\}$

we would like to describe $\mathcal{M}$ and to do so we look for some geometric structure on it. A priori, this is just a set, but imposing suitable (stability) condition on the class of objects we are studying, we can get something more. Hopefully $\mathcal{M}$ will have the structure of a scheme, but in most cases $\mathcal{M}$ we can only hope for a weaker structure.

In this notes we will not be concerned with the problem of the existence of a moduli space, but assuming there is one, or at least there is locally one, we want to understand its structure around a point studying infinitesimal deformations.

1 Global study: notion of family and the functor of points

In order to give a geometric structure to $\mathcal{M}$ it is not enough to know what its point are but we need to understand how they fit together in a geometric significant way. This is encoded by the notion of *family*.

In general, given ξ an algebro-geometric object, a family of objects of type ξ over S is the data of a morphism

$$\chi \to S,$$

flat over S, such that the fiber χ_s are "of same type of χ. Let us now list some down-to-earth examples of this general philosophy:

- Let $\mathcal{F}$ be a coherent sheaf on X, a flat family of sheaves over the base S is a coherent sheaf $\chi \to X \times S$, flat over S, such that χ_s is a coherent sheaf on X;

- Let $Y \subset X$ a subvariety of a variety over $\mathbb{C}$, a flat family of subvarieties over S is a scheme $\mathcal{Y} \in X \times S$, flat over S, such that for each $s \in S$ $\mathcal{Y}_s$ is a subvariety of $X \cong X \times_{\mathbb{C}} s$;
- Let E be a vector bundle over X. A family of vector bundles over S is a vector bundle $\mathcal{E} \to X \times S$ [1]
- Let X a nonsingular variety over $\mathbb{K}$ (*e.g.* a complex manifold), a family of non singular varieties over the base scheme S is a variety $X' \to S$, flat over S, such that for every $s \in S$ X'_s is a $\mathbb{K}(s)$ variety.

It comes out that "flatness" is the right condition to impose to avoid a series of wild situations. Note that as flatness is a stable property under base change, associating to a scheme S a flat family parametrised by S is functorial. So we can define a functor:

$$\begin{aligned} &\underline{\mathcal{M}}\colon \text{(Schemes)} \to \text{(Sets)} \\ &S \to \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{flat families over } S \end{array} \right\} \end{aligned}$$

and for each morphism of schemes we have:

$$f\colon S' \to S'' \quad \Rightarrow \quad \begin{array}{ccc} \chi \times_{S''} S' & \longrightarrow & \chi \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S''. \end{array}$$

Moreover, flat morphisms have a number of useful properties: *e.g.*, if $\mathcal{F} \to \mathbb{P}^r \times S$ is a coherent sheaf flat over S (here we actually need S reduced), then the Hilbert polynomial $h_{\mathcal{F}_s}(t)$ is locally constant in s; if $f\colon X \to Y$ a flat morphism of irreducible scheme of finite type over a field $\mathbb{K}$, then the relative dimension is constant.

As we have seen, the notion of family allows us to define a functor from our category, Schemes, to the category of Set, which in literature is called the **Moduli-Problem Functor**.

Let us recall that for any scheme M we can define its functor of points:

$$\begin{aligned} &\underline{M}\colon \text{(Schemes)} \to \text{(Sets)} \\ &S \to \mathrm{Hom}_{\mathrm{Sch}}(S, M) \end{aligned}$$

so called because $\underline{M}(\mathrm{Spec}(\mathbb{C}))$ is in bijection with the closed point of M. In the "Moduli Spaces and Stables Bundles" section we have learnt that **functor of points** determines uniquely the scheme structure on M up to isomorphism. To sum up, seeking for a geometric structure on the set $\mathcal{M}$ amounts to look for a scheme M whose functor of point is isomorphic to the Moduli-Problem Functor $\underline{\mathcal{M}}$. If such an M exist we say that $\underline{\mathcal{M}}$ is **representable**.

It is usually a difficult task finding such a scheme M and in many cases of geometric interest there is no hope for the existence of the moduli space in the category of schemes, and one is obliged to enlarge the category to more general objects as stacks. Moreover, even when we know that the moduli space exists and is a scheme M, this is quite difficult to describe: it will be reducible and

[1] We note that in this case $\mathcal{E}$ is automatically flat over S because is locally free.

very singular and with a lot of components[2]. So, the natural thing to do is starting study its local property..

2 Deformation Theory

The main tool we have to study the local properties of a moduli space is Deformation Theory. Suppose we have M and $[\chi] \in M$, where $[\chi]$ represents, for example, an isomorphism class of (stable) vector bundle or an isomorphism class of embedded varieties in a certain fixed variety X. We may wonder if $[\chi]$ is or not a smooth point of M and what is the dimension of M in $[\chi]$. To answer these questions its enough to study M "near" $[\chi]$. In pratice we will study a functor very similar to $\underline{\mathcal{M}}$, but we restrict our attention to consider families over fat points, rather than arbitrary schemes.

Definition 1. Denote by **Art** the category of Artin Local $\mathbb{C}$ with residue field $\mathbb{C}$. In particular we denote by $A_n = \mathbb{C}[x]/(x^{n+1})$ and by $B_n = \mathrm{Spec}(A_n)$.

Definition 2. Let F be a functor:

$$\begin{aligned} F\colon \mathbf{Art} &\to \mathbf{Sets} \\ A &\to F(\mathrm{Spec}(A)) \end{aligned}$$

we say F is a *deformation functor* if $F(\mathrm{Spec}(\mathbb{C}))$ = one point.

Example 1. 1. Let : **Schs** $\to$ **Sets** be the functor:

$$S \to \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{vector bundle } \mathcal{E} \to X \times S \end{array} \right\}.$$

Let us fix a vector bundle $\mathcal{E}_0 \to X$. The functor of infinitesimal deformation of $\mathcal{E}_0$ associated to F is defined by:

$$\begin{aligned} \mathrm{Def}_{\mathcal{E}_0}\colon \mathbf{Art} &\to \mathbf{Sets} \\ A &\to F(\mathrm{Spec}(A)) \times_{F(B_0)} \{\mathcal{E}_0\} \end{aligned}$$

where here $B_0 = \mathrm{Spec}(\mathbb{C})$ according to the notation of the previous definition.

2. Let $\mathcal{F}$ be a coherent sheaf over X, we define $\mathrm{Def}_{\mathcal{F}}$ in the following way: for each $A \in \mathbf{Art}$

$$A \to \left\{ \begin{array}{ll} \text{isomorphism classes of} & \\ \quad\mathcal{F}_A & \text{s.t } \mathcal{F}_A \text{ is flat over } A \\ \quad\big\downarrow & \text{and } \mathcal{F}_A \times_A \mathbb{C} \cong \mathcal{F} \\ X \times \mathrm{Spec}(A) & \end{array} \right\}$$

3. Let $Y \subset X$ be a closed subscheme of a scheme X. We define the functor of infinitesimal embedded deformations of Y in the following way: for each $A \in \mathbf{Art}$

[2] To be precise we should mention the difference between fine moduli space and coarse moduli space, but as we are not interested in the global study, we are not going into this.

$$A \to \left\{ \begin{array}{l} \text{isomorphism classes of} \\ Y_A \subset X \times \mathrm{Spec}(A) \end{array} \quad s.t. \quad \begin{array}{l} Y_A \text{ is flat over } A \\ \text{and } Y_A \times_A \mathbb{C} \cong Y \end{array} \right\}$$

4. Let X be a nonsingular variety over $\mathbb{C}$ (e.g. a complex manifold, a projective complex variety), we define the infinitesimal deformation of X in the following way: for each $A \in \mathbf{Art}$

$$A \to \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \begin{array}{ccc} X & \longrightarrow & X_A \\ & & \downarrow \\ & & \mathrm{Spec}(A) \end{array} \end{array} \quad s.t. \quad \begin{array}{l} X \to X_A \text{ is a closed embedding} \\ X_A \text{ is flat over } A \\ \text{and } X_A \times_A \mathbb{C} \cong X \end{array} \right\}$$

Remark 1. We need to remark that in Examples 1 and 2, and more general every time we look at infinitesimal deformations of an algebraic-geometric structure which has non trivial automorphisms, we have to pay attention that the one we gave is not quite the correct definition of deformation functor. To be precise we have to say that an infinitesimal deformation is the data (e.g. in 2) of a flat family $\mathcal{F}_A \to \mathrm{Spec}(A)$ together with a *fixed* isomorphism $\phi\colon \mathcal{F}_A \times_A \mathbb{C} \to \mathcal{F}$.; i.e. we think of associating to A an isomorphism class of triples (F, ϕ, F_A) such that the following diagram

$$\begin{array}{ccc} F \simeq^{\phi} F_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{C}) & \longrightarrow & F_A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

commutes. We say that two such triples (F', ϕ', F'_A) and (F'', ϕ'', F''_A) $\mathcal{F}'_A$ and $\mathcal{F}''_A$ are isomorphic if and only if there exists $f\colon \mathcal{F}'_A \to \mathcal{F}''_A$ isomorphism of sheaves over $X \times \mathrm{Spec}(A)$ such that $f|_0 \circ \phi' = \phi''$. We will see later in an example while it is important to pay attention.

3 First order deformations: the classical approach

We defined in the previous section the infinitesimal deformations of an object χ as the flat families over $\mathrm{Spec}(A)$ with $A \in \mathbf{Art}$. We moreover define:

Definition 3. curvilinear deformations the deformations over the rings A_n, or equivalently over the fat points B_n;

first order deformation the deformations over $A_1 = \mathbb{C}[x]/(x^2)$ or equivalently over B_1. These are often called deformations over the dual number.

Given a deformation functor F we call its firts order deformations tangent space to F and write $F(A_1) = TF$. The reason why it make sense to say "tangent space" is the following exercise:

Exercise 1. Let $\underline{M}$ be a representable functor and M its moduli space. Then

$$\underline{M}(B_1) = \{(m, v) \mid m \in M \text{ and } v \in T_m M\},$$

where $T_m M$ is the Zariski tangent space $(\mathfrak{m}_m/\mathfrak{m}_m^2)^*$ (Hint: an element of the Zariski tangent space is the same of a morphism.) $\varphi\colon \mathcal{O}_{M,m} \to \mathbb{C} \oplus \mathbb{C}x$.

Let's now study the first order deformations for the deformations functor listed above.

Proposition 1. *Let F denote the functor of embedded deformations $Z \subset X$ and let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z in X. Then the first order deformations are in one-to-one correspondence with the global sections of the normal sheaf:*

$$F(A_1) = \mathrm{H}^0(Z, \mathcal{N}_Z) = \mathrm{Hom}_X(\mathcal{I}, \mathcal{O}_X/\mathcal{I}).$$

Proof. [2] We want to describe the set:

$$F(A_1) = \left\{ \begin{array}{l} \text{isomorphism classes of} \\ Z_1 \subset X \times B_1 \end{array} \quad s.t. \quad \begin{array}{l} Z_1 \text{ is flat over } B_1 \\ \text{and } Z_1 \times_{A_1} \mathbb{C} \cong Z \end{array} \right\}.$$

First of all, we note that this set is non empty because it contains at least the trivial deformation $Z \times B_1 \subset X \times X_1$, which at the level of (sheaf) algebras is

$$\mathcal{I} \otimes_{\mathbb{C}} \mathbb{C}[t]/(t^2) = \mathcal{I}[t]/(t^2) \subseteq \mathcal{O}_X[t]/(t^2).$$

We can observe that as $\mathcal{O}_X$ module $\mathcal{O}_X[t]/(t^2) = \mathcal{O}_X \oplus t\mathcal{O}_X$, so any element[3] can be written as $x + ty$. Now, given a first order deformation $\mathcal{I}_{B_1} \subset \mathcal{O}_X[t]/(t^2)$, if this is not the trivial one, we are not able to associate to each $x \in \mathcal{I}$ an unique $y \in \mathcal{O}_X$ such that $x + ty \in \mathcal{I}_{B_1}$ reduces to x modulo t. However, to such a lifting which reduce to x modulo t differ by an element in $\mathcal{I}$, because by a chaising diagram the flatness of Z_{B_1} implies the exactness of the sequence

$$0 \to \mathcal{I} \to \mathcal{I}_{B_1} \to \mathcal{I} \to 0.$$

This means that for every first order deformation we have a well defined morphism of sheaves $\phi\colon \mathcal{I} \to \mathcal{O}_X/\mathcal{I}$, i.e. $\phi \in \mathrm{Hom}_X(\mathcal{I}, \mathcal{O}_X/\mathcal{I})$. [4] Conversely, given a map $\phi \in \mathrm{Hom}_X(\mathcal{I}, \mathcal{O}_X/\mathcal{I})$, we define

$$\mathcal{I}_{B_1} = \left\{ f + t\tilde{\phi}(f) \mid f \in \mathcal{I},\ \tilde{\phi}(f) \in \mathcal{O}_X \text{ is any lift of } \phi(f) \in \mathcal{O}_X/\mathcal{I} \right\}.$$

It is easy to verify the sheaf ideal so definied is flat over A_1 and reduces to $\mathcal{I}$ mod t. To conclude the proof is now enough to check this constructions are inverse each other. ☐

Proposition 2. *Let X be a nonsingular variety over $\mathbb{C}$, (a complex manifold) then the first order deformation are in one to one correspondence with the elements of $H^1(X, \mathcal{T}_X)$.*

Proof. [2] As we are supposing X is non singular, we can think to $H^1(X, \mathcal{T}_X)$ as the Čech cohomology. Given a deformation X' its restrictions to affine subsets of an open covering of X are deformations U_i' isomorphic to the trivial one, i.e.

[3] We should here interpret the word "element" stalkwise and each morphism is a morphism of sheaves

[4] We may observe that this is exactly the same construction that we do when we compute the tangent space of the Grasmannian in a point, indeed in that case we conclude $T_L \,\mathrm{Gr}(k, V) = \mathrm{Hom}(L, V/L)$

we can chose $\varphi_i\colon U_i \times_{\mathbb{C}} B_1 \to U_i'$. (This is non trivial to see) On the overlaps, if we denote by B the coordinate ring of U_{ij} we have $\varphi_j^{-1} \circ \varphi_i$ is an automorphism of $B[t]/(t^2)$ which reduces to the identity mod t. This means (after a bit of work) $\varphi_j^{-1} \circ \varphi_i(b_0 + tb_1) = b_0 + t(b_1 + \theta_{ij}(b_0))$ and it comes out that θ_{ij} has to belong to $H^0(U_{ij}, \mathcal{T}_X)$ and satisfy the cocycle condition on triple intersections. Moreover it is possible to check that chosing a different isomorphism with the trivial deformation on the opens of the covering, we get a collection of derivation on U_{ij} which differs from the previous one by an element in $\check{C}^0(\mathcal{U}, \mathcal{T}_X)$. Viceversa use the θ_{ij} defined on the overlaps to glue together local trivial deformation of X to a global nontrivial one. □

Proposition 3. *The first order deformations with fixed isomorphism with the central fiber of a coherent sheaf $\mathcal{F}$ or of a vector bundle $\mathcal{E}$ over X are in one to one correspondence with $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F})$. If $\mathcal{E}$ is a vector bundle this is just $H^1(X, \mathcal{E}nd(\mathcal{E}))$.*

Proof. [2] We give the proof in terms of coherent sheaves but the argument for vector bundles is the same. Given a first order deformation $\mathcal{F}'$, tensoring over A_1 by $\mathcal{F}'$ the exact sequence

$$0 \to \mathbb{C} \to A_1 \to \mathbb{C} \to 0$$

we get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F} \to 0. \tag{1}$$

As we are looking at the functor with fixed isomorphism with the central fiber, two deformations $\mathcal{F}' \to \mathcal{F}''$ are isomorphic if and only if the isomorphism of $\mathcal{O}_X \otimes A_1$ reduces to the identity. This condition ensures that the map from $F(B_1)$ and $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F})$ defined associating to each infinitesimal deformation (2) is well defined and injective. Viceversa, given an equivalence class of extension:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ f\ } \mathcal{F}' \xrightarrow{\ g\ } \mathcal{F} \longrightarrow 0$$

we define a first order deformation of $\mathcal{F}$ puttng a $\mathcal{O}_X \otimes A_1$-module structure on $\mathcal{F}$, i.e. defining the multiplication by t in the following way $t = f \circ g\colon \mathcal{F}' \to \mathcal{F}'$. Check that this map is well defined and injective to conclude the proof.

□

4 Obstructions

After studying the first order deformations, it is natural to look at the so called "obstructions" to deformations. I won't give the precise definition of obstruction and I won't be able to explain what they essentially are and where they come from. However, I will try to communicate an idea and to motivate why the study of obstructions is interesting. Given an element (infinitesimal deformation) $\xi_A \in \mathrm{Def}_\chi(A)$ for some $A \in \mathbf{Art}$ (e.g. ξ an infinitesimal deformation over A of a coherent sheaf or of a non singular variety or of an embedded variety into the ambient space) and a surjective morphism of local artinian rings $B \twoheadrightarrow A$, is it possible to find an element $\xi_B \in \mathrm{Def}_\chi(B)$ which extends ξ_A? For example, if X_A is an infinitesimal deformation of a nonsingular variety, does a deformation X_B which extends X_A exists? In other words, an X_B such that the following diagram is cartesian?

$$\begin{array}{ccc} X_A & \dashrightarrow & X_B \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(B) \end{array} \quad \Rightarrow \quad X_B \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) = X_A.$$

The answer depends from the obstructions. Given a resonable deformation functor, we always find a vector space V, somehow natural. such that for any surjective morphism $\varphi : B \twoheadrightarrow A$ there is a map v_φ: $\mathrm{Def}_\xi(A) \to V$ which is zero if and only if the deformation X_A admits an extension over B. In other words, $(V, \{v_\varphi\})$ tells when there is an obstruction to the lifting. Moreover, if my geometric object χ represents an isomorphism class which is a point in a certain moduli space M, we should interpret obstructions to Def_χ as a "measure of how singular is M at the point χ". More precisely,

$$\dim_{\mathbb{C}} T_\chi M \geq \dim M_\chi \geq \dim_{\mathbb{C}} T_\chi M - \dim_{\mathbb{C}} V$$

where according to the notation above V is the vector space which contains the obstructions to the infinitesimal deformations of χ.

Example 2. 1. Let $\mathcal{F}$ be a coherent sheaf on X and $\mathcal{F}_1$ a first order deformation. We want to extend this deformation to a 2-order deformation, i.e., we want a coherent sheaf

$$\begin{array}{c} \mathcal{F}_2 \\ \downarrow \\ X \times B_2 \end{array} \qquad \begin{array}{l} \mathcal{F}_2 \text{ flat over } B_2 \\ \text{s.t } \mathcal{F}_2 \times_{A_2} A_1 \cong \mathcal{F}_1 \\ \text{and } \mathcal{F}_2 \times_{A_2} \mathbb{C} \cong \mathcal{F} \end{array}$$

Such an $\mathcal{F}_2$ should fill in the following diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \xrightarrow{t} & \mathcal{F}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{F} & \longrightarrow & \mathcal{F} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \qquad \begin{array}{c} e \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \\ \uparrow \\ \tilde{e} \in \mathrm{Ext}^1(\mathcal{F}_1, \mathcal{F}) \end{array}$$

Clearly, the exsistence of $\mathcal{F}_2$ is equivalent to the exsistence of $\tilde{e}$, but the map between the ext-groups is not surjective, indeed from the Ext-long exact sequence, we have:

$$\cdots \longrightarrow \mathrm{Ext}^1(\mathcal{F}_1, \mathcal{F}) \longrightarrow \mathrm{Ext}^1(F, F) \xrightarrow{\delta} \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) \longrightarrow \cdots$$

which means we have an obstruction to extend a first order deformation e given by $\delta e \in \mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$, which is our vector space V in this example. Indeed it is possible to show that not only the obstructions to lift the first order deformations, but all obstructions lies in that vector space. Exactly the same argument holds for vector bundles and in that case the second Ext group is simply $H^2(\mathcal{E}nd(\mathcal{E}))$.

2. Let X be a non singular complex variety. We have seen that the first order deformations are in one-to-one correspondende with $\{\theta_{ij}\}_{i,j} \in \check{H}^1(X, \mathcal{T}_X)$, where each θ_{ij} is the data of an authomorphism $U_{ij} \times B_1 \to U_{ij} \times B_1$. For each U_{ij} we can chose a lifting $\theta'_{ij} \colon U_{ij} \times B_2 \to U_{ij} \times B_2$. Now the collection $\{\theta'\}_{i,j}$ define a lifting of our deformation if and only if θ'_{ij} restrict to θ_{ij} on B_1 for all ij $\theta'_{ij}\theta'_{jk} = \theta'_{ik}$. In general, the local liftings define a 2-cocycle

$$\theta_{ijk} = \theta'_{ij}\theta'_{jk}\theta'^{-1}_{ik} \in \Gamma(U_{ijk}, \mathcal{T}_X)$$

and chosing a different lifting we find a new 2-cocycle with the same cohomoly of this one. If its cohomoly class is trivial, then I have a canonical way to lift my first order deformation, otherwise $[\{\theta_{ijk}\} i, j, k] \in H^2(X, \mathcal{T}_X)$ represents the obsructions to the existence of the higher deformation.

3. Looking at embedded deformations $Z \subset X$, we have seen that the first order deformations of Z in X are in one-to-one correspondence with global sections of the normal bundle $H^0(Z, \mathcal{N}_Z)$. Here, the problem of lifting deformations comes from the fact that the normal bundle is a quotient and not a subbundle of $\mathcal{T}_X$.

Upshot: there exist a natural obstruction to lift deformation lies in $H^1(Z, \mathcal{N}_Z)$.

Summing up what we have seen in the examples:

non-singular varieties

$H^0(X, \mathcal{T}_X)$: automorphisms of X
$H^1(X, \mathcal{T}_X)$: first order deformations of X
$H^2(X, \mathcal{T}_X)$: obstructions of Def_X

embedded deformations

We don't have automorphisms of Z as embedded variety
$H^0(Z, \mathcal{N}_Z)$: first order deformations of $Z \subset X$
$H^1(Z, \mathcal{N}_Z)$: obstructions of $\mathrm{Def}_{Z\subset X} := \mathrm{Hilb_{Z,X}}$

vector bundles

$H^0(X, \mathcal{E}nd(\mathcal{E})$: automorphisms of $\mathcal{E}$
$H^1(X, \mathcal{E}nd(\mathcal{E})$: first order deformations of $\mathcal{E}$
$H^2(X, \mathcal{E}nd(\mathcal{E})$: obstructions of $\mathrm{Def}_{\mathcal{E}}$

Remark 2. We have to note that in Example 1 and Example 3, according to the definition of deformation functors, we fixed the isomorphism with the central fiber. This is exactly what we need to avoid problems with the automorphisms of the geometric structure we are deforming. Indeed, the automorphisms are the guys which mess up the situation and obstruct the existence of moduli space. Remember for example, studying vector bundle we restrict our attention to stables one, which don't have automorphisms, when we want a moduli space. Let's see what happen when we have automorphisms with an example.

Example 3. Consider the vector bundle $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1$. Then we have:

$$H^1(\mathcal{E}nd(\mathcal{E})) = H^1(\mathcal{O}(-2)) \cong \mathbb{C},$$

which is the space of first order deformation with fixed isomorphism with the central fiber. However, if we don't fix the isomorphism with the central fiber, each automorphism $\alpha \in \mathrm{Aut}(E)$ will bring a first order deformation into an isomorphic one! Then, to understend the first order deformation/tangent space of $\mathrm{Def}_{\mathcal{E}}$, we have to care about the action of $\mathrm{Aut}(E)$ on $H^1(\mathcal{E}nd(\mathcal{E}))$.

It comes out that there is a $\mathbb{C}^* \subset \mathrm{Aut}(E)$ acting non trivially on $H^1(\mathcal{E}nd(\mathcal{E}))$. More precisely, it acts with weight 1 via $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. It follows that

$$\mathrm{Def}_{\mathcal{E}}(B_1) = \mathbb{C}/\mathbb{C}^*.$$

5 DGLA philosophy

The DGLA approach to deformation problems is essentially based on the following principle: "in characteristic 0, each deformation problem is controlled by a differential graded Lie algebra". This is not a theorem in this version, however if we look to all the examples we have seen as far such a DGLA actually exists. This pratically means that there exists a differential graded Lie algebra $(L, d, [-,-])$ such that $\mathrm{Def}_L \cong \mathrm{Def}_\chi$, where the deformation functor associate to L is defined in the following way:

$$A \in \mathbf{Art} \to \frac{\left\{x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0\right\}}{\exp(L^0 \otimes \mathfrak{m}_A)}$$

where we are taking the quotien for a suitable action (called in literature gauge action) of $\exp(L^0 \otimes \mathfrak{m}_A)$ on the set of Maurer-Cartan elements.

As there exists a full theory about Def_L, once we know the DGLA which controls the deformation problem we are studying, we have a kind of general recipe to compute the first order deformations and the space where obstruction lie. Indeed, $\mathrm{Def}_L(A_1) \cong H^1(L)$ and the obstructions to lift a Maurer-Cartan element lie in $H^2(L)$, where the cohomology is taken with respect the differential d of L. In our examples:

deformations of non singular varieties Deformations are controlled by $\check{C}^*(\mathcal{U}, \mathcal{T}_x)$ where the bracket is the bracket of vector field and the differential is the *Čech* differential.

deformations of vector bundles Deformations are controlled by $\check{C}^*(\mathcal{U}, \mathcal{E}nd(\mathcal{E}))$ where the bracket is the commutator of the composition and the differential is the *Čech* differential.

deformations of smooth embedded subvarieties Deformations are controlled by $\check{C}^*(\mathcal{U}, \mathcal{T}_Z \to \mathcal{T}_X)$ where the DGLA here is the homotopy fiber of the inclusion, the bracket is the homotopy bracket and the differential is the *Čech* differential.

We won't say more about this approach, but the interested ones will find many references avaiable on the web about the subject.

6 Kuranishi-Model

Historically, the Kuranishi model was studied in the contest of deformations of complex manifolds. Here we call deformation of the complex manifold X the data of a map $\pi\colon \chi \to T$, proper and flat between complex connected spaces such that $\pi^{-1}(t_0) = X_0 \cong X$ and small deformation a germ of a deformation, i.e., $\pi\colon (\chi, X_0) \to (T, t_0)$.

If, given a complex manifold Y there exists a small deformation $(\mathcal{Y}, Y) \to (\mathcal{B}(Y), 0)$ such that each other small deformation is obtained by this one via pull back, we call it Kuranishi family. We can note that by the requested property, this is exactly what we would call "a germ of the moduli space." It comes out that, if Y ic compact complex, $(\mathcal{B}(Y), 0)$ is unique up to (non canonical) isomorphism, and is a germ of analytic subspace of the vector space $H^1(Y, \mathcal{T}_Y)$, inverse image of the origin under a local holomorphic map (called Kuranishi map and denoted by k)

$$k\colon H^1(Y, \mathcal{T}_Y) \to H^2(Y, \mathcal{T}_Y).$$

More generally, in the Kuranishi model we want to see "the germ of the moduli space" as the zero locus of the Kuranishi map

$$k\colon H^1(\cdots) = \text{"}\textit{tangent}\text{"} \to H^2(\cdots) = \text{"}\textit{obstructions.}\text{"}$$

It says us, morally, there are "intrinsic" local equation for the germ of moduli determined by the tangent and the obstructions. This approach is, roughly speaking, the local version of Behrend-Fantechi tangent-obstruction complex. See [3] or [4] for actual explanations!

Remark 3. An importante tool when doing deformation theory is the so called T^1-lifting technique. The philosophy is that, to extend an equivalence from a family over A_n to one over A_{n+1}, it is enough to understand deformations "sideways" from A_n to $A_n \times A_1$. For many reason we do not treat this argument here, but we reccomend to read the appendix of [1] for a good expository treatment.

References

[1] N. Addington, R.P: Thomas *Hodge Theory and Derived Category of cubic Fourfolds* Math. J. 163, no. 10 (2014), 1885-1927, arXiv:1211.3758

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[3] K. Kodaira: *Complex manifold and deformation of complex structures.* Springer-Verlag (1986).

[4] J. Lie and G. Tian *Virtual Moduli cycles and Gromov-Witten invariants of algebraic variety* arXiv:alg-geom/9602007

[5] E. Sernesi: *Deformations of Algebraic Schemes.* Grundlehren der mathematischen Wissenschaften, **334**, Springer-Verlag, New York Berlin, (2006).