Elliptic Curves - Coursework 2

Due 6 December

You can earn at most 5 points. Choose a combination that adds up to 5.

1) (**1 point**) Determine $E(k)$ as an abelian group for the following:

1. $y^2 = x^3 + x$ over $k = \mathbb{F}_5$.
2. $y^2 = x^3 + 2x$ over $k = \mathbb{F}_5$.
3. $y^2 = x^3 + x$ over $k = \mathbb{F}_3$.
4. $y^2 + y = x^3 + x^2$ over $k = \mathbb{F}_2$.

2) (**1 point**) Compute torsion subgroups in $E(\mathbb{Q})$ for the following elliptic curves:

1. $y^2 - y = x^3 - x^2$
2. $y^2 + xy - 5y = x^3 - 5x^2$
3. $y^2 + xy = x^3 - 45x + 81$

3) **(1 point)** Let E be the elliptic curve defined by $y^2 = x^3 - 73x + 72$. (i) Compute the torsion subgroup of $E(\mathbb{Q})$. (ii) Show that rank of $E(\mathbb{Q})$ is at least 2.

4) **(1 point)** Construct an example of an elliptic curve over $\mathbb{Q}$ such that the points of order 2 are all rational and the restriction of the group homomorphism $\delta : E(\mathbb{Q}) \to (\mathbb{Q}^\times/(\mathbb{Q}^\times)^2)^3$ to the 2-torsion points is not injective. What does that imply for the torsion subgroup of the elliptic curve?

5) (**2 points**) (i) Construct an elliptic curve with a torsion element of order 8. (ii) Show that no torsion element can have order 16.

6) (**2 points**) Work out $E(\mathbb{Q})/2E(\mathbb{Q})$ for the following:

1. $y^2 = x^3 - 12x^2 + 20x$
2. $y^2 = x^3 + x^2 - 24x + 36$

7) **(r points)** Find an elliptic curve over $\mathbb{Q}$ such that $E(\mathbb{Q})$ has rank r.[1]

[1] You get r points for proving that the rank of your elliptic curve is r using the techniques that we developed in class. Moreover, your elliptic curve should not be one of those that we discussed in class (or one of the above!). If you get to $r > 29$, then you won't need to take the final exam.