Curves - Homework 4 Solutions

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1. Check that the monodromy homomorphism M_f defined with respect to the function $f : S \to T$ is indeed a homomorphism

$$\pi_1(S \setminus \{t_1, t_2, \ldots, t_n\}, t) \to \mathfrak{S}(f^{-1}(t)),$$

where $t_1, \ldots t_n$ are the branch values of f and $\mathfrak{S}(f^{-1}(t))$ is the group of permutations of the finite set $f^{-1}(t)$. Show that M_f is transitive (since S is connected).

Solution: Let $\alpha, \beta \in \pi_1(S\setminus\{t_1, t_2, \ldots, t_n\}, t)$, and consider the lifts $\tilde{\alpha} : [0, 1] \to S\setminus\{f^{-1}(\{t_1, \ldots, t_n\})\}$, $\tilde{\beta} : [0, 1] \to S \setminus \{t_1, \ldots, t_n\}$. If we label $f^{-1}(t) = \{p_1, \ldots, p_m\}$, then by the path lifting property, these maps are unique once we have stipulated what $\tilde{\alpha}(0)$ and $\tilde{\beta}(0)$ are. There are therefore $\deg f = m$ different lifts of each of α and β, and we can choose lifts such that the composition of maps is possible.

To see that M_f is indeed a homomorphism, recall that for $\alpha \in \pi_1(S \setminus \{t_1, t_2, \ldots, t_n\}, t)$, we have that $M_f(\alpha)(\tilde{\alpha}(1)) = \alpha(0)$. Consider now the composition $\alpha\beta \in \pi_1(S \setminus \{t_1, t_2, \ldots, t_n\}, t)$, and the lift $\tilde{\alpha}\tilde{\beta} : [0, 1] \to S \setminus \{f^{-1}(\{t_1, t_2, \ldots, t_n\})\}$ such that $\tilde{\alpha}\tilde{\beta}(0) = \tilde{\alpha}(0) = p_i$. This path is then unique, and so we label its endpoint $\tilde{\alpha}\tilde{\beta}(1) = \tilde{\beta}(1) = p_j$. Then $M_f(\alpha\beta)(p_j) = p_i$. If we choose the lifts of α and β such that they are composable, then this is the same as the lift of $\alpha\beta$. Let $p_k = \tilde{\alpha}(1) = \tilde{\beta}(0)$. Then

$$M_f(\alpha) \circ M_f(\beta)(p_j) = M_f(p_k) = p_i$$

Since this is true for any lift of $\alpha\beta$, we have that $M_f(\alpha\beta) = M_f(\alpha) \circ M_f(\beta)$.

To see that this action is transitive, note that because S is compact and $f^{-1}(\{t_1, \ldots, t_n\})$ is a finite set, we have that $S \setminus f^{-1}(\{t_1, \ldots, t_n\})$ is open. Since S is connected, so is $S \setminus \{f^{-1}(\{t_1, \ldots, t_n\})\}$, which means that it is also path connected, since it is open and connected. Therefore there are exists a path $\tilde{\gamma} : [0, 1] \to S \setminus \{f^{-1}(\{t_1, \ldots, t_n\})\}$ such that $\tilde{\gamma}(0) = p_\ell$ and $\tilde{\gamma}(1) = p_r$ for arbitrary $p_\ell, p_r \in S \setminus f^{-1}(t)$. This then projects to a path $p(\tilde{\gamma}) = \gamma$ such that $\gamma(0) = \gamma(1) = t$, and so represents an element of $\pi_1(S\setminus\{t_1, t_2, \ldots, t_n\}, t)$, and by construction we have that $M_f(\gamma)(p_r) = p_\ell$. Since p_r, p_ℓ were arbitrary, we see that the action is transitive.

2. Give a Belyi function for the Riemann surface associated to

$$f(z, w) = z^2 - w(w - 1)(w - 2).$$

Solution: Let S_f be the compact Riemann surface associated to f. f is defined over $\overline{\mathbb{Q}}$, hence by Belyi's theorem, there exists a Belyi function $g : S_f \to \mathbb{P}^1$, i.e. we have a function g such that $\text{branch}(g) \subset \{0, 1, \infty\}$. To construct g explicitly, first consider the projection $\hat{w} : S_f \to \mathbb{P}^1$. Then $\text{branch}(\hat{w}) \subset \hat{w}(S_f \setminus S_f^w)$ and S_f^w is given by

$$\begin{aligned} S_f^w &= \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0, f_z(z, w) \neq 0, \text{the coefficient of highest power of } z \text{ in } f \neq 0\} \\ &= \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0, 2z \neq 0\} \\ &= \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0, w \notin \{0, 1, 2\}\} \ . \end{aligned}$$

Hence $\hat{w}(S_f \setminus S_f^w) = \{0, 1, 2, \infty\}$ and $\text{branch}(\hat{w}) \subset \{0, 1, 2, \infty\}$.

We want to have the branching values as a subset of $\{0, 1, \infty\}$, and so we need to get rid of $2 \in \text{branch}(\hat{w})$. We can get rid of rational numbers between 0 and 1 using Belyi's polynomial, so we need to transform 2 using the Mobius transformation $t(w) = \frac{1}{w}$ which keeps the set $\{0, 1, \infty\}$ fixed. Hence, if we define $g_1 := t \circ \hat{w}$, then we get $\text{branch}(g_1) \subset \{0, 1, \infty, \frac{1}{2}\}$.

Now, to get rid of $\lambda = \frac{1}{2} = \frac{m}{m+n}$ (so $m = 1$ and $n = 1$), we consider $g := p_\lambda \circ g_1$ where p_λ is the Belyi's polynomial given by

$$p_\lambda = p_{m,n} = \frac{(m+n)^{m+n}}{m^m n^n} w^m (1-w)^n = 4w(1-w) \; .$$

This gives $\text{branch}(g) \subset \{0, 1, \infty\}$, i.e. $g\colon S_f \to \mathbb{P}^1$ is a Belyi function. To write explicitly, we have

$$\begin{aligned} g(z, w) &= p_\lambda \circ t \circ \hat{w}(z, w) \\ &= p_\lambda \circ t(w) \\ &= p_\lambda \left(\frac{1}{w} \right) \\ &= \frac{4}{w} \left(1 - \frac{1}{w} \right) \\ &= \frac{4(w-1)}{w^2} \end{aligned}$$

3. Let S_f be the compact Riemann surface defined by the irreducible polynomial

$$f(z, w) = z^2 - w(w-1)(w-\sqrt{2})$$

Construct a Belyi function on S_f.

Solution: f is defined over $\overline{\mathbb{Q}}$, hence by Belyi's theorem, we have a Belyi function $g\colon S_f \to \mathbb{P}^1$, i.e. we have a function g such that $\text{branch}(g) \subset \{0, 1, \infty\}$. To construct g explicitly, first consider the projection $\hat{w}\colon S_f \to \mathbb{P}^1$. Then $\text{branch}(\hat{w}) \subset \hat{w}(S_f \setminus S_f^w)$ and S_f^w is given by

$$\begin{aligned} S_f^w &= \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0, f_z(z, w) \neq 0, \text{the coefficient of highest power of } z \text{ in } f \neq 0\} \\ &= \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0, 2z \neq 0\} \\ &= \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0, w \notin \{0, 1, \sqrt{2}\}\} \; . \end{aligned}$$

Hence $\hat{w}(S_f \setminus S_f^w) = \{0, 1, \sqrt{2}, \infty\}$ and $\text{branch}(\hat{w}) \subset \{0, 1, \sqrt{2}, \infty\}$. Our first aim is to modify $\hat{w}$ to make its branching values all rational. For that, we need to get rid of $\sqrt{2}$. So consider the minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$, which is $m(w) = w^2 - 2$. Then define $g_1 := m \circ \hat{w}$ and we have $\text{branch}(g_1) = m(\text{branch}(\hat{w})) \cup \text{branch}(m)$ where

$$\text{branch}(m) = m(\{\text{roots of } m'\}) \cup \{\infty\} = m(\{\text{roots of } 2w\}) \cup \{\infty\} = m(\{0\}) \cup \{\infty\} = \{-2, \infty\} \; .$$

Therefore we get

$$\begin{aligned} \text{branch}(g_1) &= m(\text{branch}(\hat{w})) \cup \text{branch}(m) \\ &\subset m(\{0, 1, \sqrt{2}, \infty\}) \cup \{-2, \infty\} \\ &= \{-2, -1, 0, \infty\} \cup \{-2, \infty\} \\ &= \{-2, -1, 0, \infty\} \end{aligned}$$

which is all rational. Next, we want to have $\{0, 1, \infty\}$ as a subset of the branching values. For that it is easy to see that by considering $g_2 := s \circ g_1$ where s is the isomorphism $s(w) = -w$, we get

$$\text{branch}(g_2) = s(\text{branch}(g_1)) \cup \text{branch}(s) \subset s(\{-2. -1, 0, \infty\}) \cup \emptyset = \{0, 1, \infty, 2\} \; .$$

Note that we have $\text{branch}(s) = \emptyset$ since s is an isomorphism. Note also that we could have produced the desired s by finding the Mobius transformation which sends $w_1 = 0$ to 0, $w_2 = -1$ to 1, and $w_3 = \infty$ to ∞, given by the formula

$$s(w) = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(w - 0)(-1 - \infty)}{(w - \infty)(-1 - 0)} = \frac{w \times -\infty}{-\infty \times -1} = -w \; .$$

Finally, we want to have the branching values as a subset of $\{0, 1, \infty\}$. For that, we need to get rid of $2 \in \text{branch}(g_2)$. We can get rid of rational numbers between 0 and 1 using Belyi's polynomial. So, we need to transform 2 using the Mobius transformation $t(w) = \frac{1}{w}$ which keeps the set $\{0, 1, \infty\}$ as the same. Hence, if we define $g_3 := t \circ g_2$, then we get $\text{branch}(g_3) \subset \{0, 1, \infty, \frac{1}{2}\}$.

Now, to get rid of $\lambda = \frac{1}{2} = \frac{m}{m+n}$ (so $m = 1$ and $n = 1$), we consider $g := p_\lambda \circ g_3$ where p_λ is the Belyi's polynomial given by

$$p_\lambda = p_{m,n} = \frac{(m+n)^{m+n}}{m^m n^n} w^m (1 - w)^n = 4w(1 - w) \; .$$

In the end, we get $\text{branch}(g) \subset \{0, 1, \infty\}$, i.e. $g \colon S_f \to \mathbb{P}^1$ is a Belyi function. To write explicitly, we have

$$\begin{aligned} g(z, w) &= p_\lambda \circ t \circ s \circ m \circ \hat{w}(z, w) \\ &= p_\lambda \circ t \circ s \circ m(w) \\ &= p_\lambda \circ t \circ s(w^2 - 2) \\ &= p_\lambda \circ t(2 - w^2) \\ &= p_\lambda \left(\frac{1}{2 - w^2} \right) \\ &= 4 \left(\frac{1}{2 - w^2} \right) \left(1 - \frac{1}{2 - w^2} \right) \\ &= \frac{4(1 - w^2)}{(2 - w^2)^2} \; . \end{aligned}$$

4. Consider the Fermat curve $F_n = \{[X : Y : Z] \in \mathbb{P}^2 \; : \; X^n + Y^n = Z^n\}$. Let $f : F_n \to \mathbb{P}^1$ be given by $[X : Y : Z] \mapsto [X : Z]$. Compose this with the map $g : \mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto z^n$. Show that as a result we get a Belyi map of degree n^2.

   Solution: First note that f has degree n and $\text{branch}(f) = \{0, 1, \infty\}$. The map g has degree n and a single branch point at $z = 0$. Therefore $\deg(g \circ f) = n^2$. To see that $\text{branch}(g \circ f) \subseteq \{0, 1, \infty\}$, note that

   $$\begin{aligned} \text{branch}(g \circ f) &= g(\text{branch}(f)) \cup \text{branch}(g) \\ &= \{0, 1, \infty\}, \end{aligned}$$

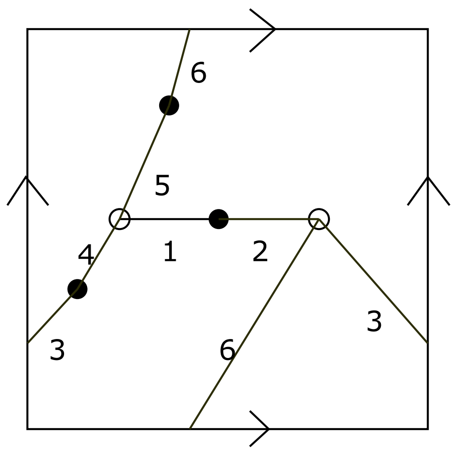
   and so $g \circ f$ is a Belyi function of degree n^2.

5. Let $\sigma_0 = (1, 5, 4)(2, 6, 3)$ and $\sigma_1 = (1, 2)(3, 4)(5, 6)$. Construct the corresponding surface and the dessin d'enfant on it.

Solution: First, note that $\sigma_1\sigma_0 = (1,6,4,2,5,3)$. The cycles in σ_0 are the white vertices and the cycles in σ_1 are the black vertices. The elements in a cycle are the edges around the corresponding vertex in counter-clockwise order. The cycles in $\sigma_1\sigma_0$ are faces, and the elements in a cycle are the half of the edges of the corresponding face in clockwise order. Then we have 2 white vertices, 3 black vertices, 6 edges, and 1 face. Then by the genus formula

$$2 - 2g = \#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\} = (2+3) - 6 + 1 = 0 \ .$$

Hence $g = 1$, which means our surface is a torus. Then we can draw the dessin d'enfant on a torus using the information above as follows:



6. Let E be the elliptic curve defined by the affine equation $z^2 = w^3 + 1$. Show that the rational map $f\colon E \to \mathbb{P}^1$ defined by
$$(z,w) \mapsto \frac{1+z}{2}$$
is a Belyi map. What is its degree? Show that the dessin associated to f has a unique white vertex and a unique black vertex. Show that $f^{-1}[0,1]$ consists of 3 edges connecting these vertices. Give two permutations σ_0, σ_1 describing this dessin. What is the monodromy group?

Solution: Let $g = w^3 + (1 - z^2)$, then $E = S_g$. Let $f : S_g \to \mathbb{P}^1$ be $f(z,w) = \frac{1+z}{2}$. Observe that $f = m \circ \hat{z}$ where $\hat{z}\colon S_g \to \mathbb{P}^1$ is the projection to z-coordinate and $m\colon \mathbb{P}^1 \to \mathbb{P}^1$ with $m(z) = \frac{1+z}{2}$. So m is an isomorphism and $\text{branch}(m) = \emptyset$. Then we get

$$\text{branch}(f) = m(\text{branch}(\hat{z})) \cup \text{branch}(m) = m(\text{branch}(\hat{z})) \ .$$

So, we need to find $\text{branch}(\hat{z})$. For that, we need to compute S_g^z:

$$\begin{aligned} S_g^z &= \{(z,w) \in \mathbb{C}^2 : g(z,w) = 0, g_w(z,w) \neq 0, \text{the coefficient of highest power of } w \text{ in } g \neq 0\} \\ &= \{(z,w) \in \mathbb{C}^2 : g(z,w) = 0, 3w^2 \neq 0\} \\ &= \{(z,w) \in \mathbb{C}^2 : g(z,w) = 0, w \neq 0\} \\ &= \{(z,w) \in \mathbb{C}^2 : g(z,w) = 0, z^2 \neq 1\} \\ &= \{(z,w) \in \mathbb{C}^2 : g(z,w) = 0, z \notin \{-1,1\}\} \end{aligned}$$

Remember we have $\text{branch}(\hat{z}) \subset \hat{z}(S_g \setminus S_g^z)$. It is clear that $\hat{z}(S_g \setminus S_g^z) = \{-1,1,\infty\}$, hence we have $\text{branch}(\hat{z}) \subset \{-1,1,\infty\}$. Then we get

$$\text{branch}(f) = m(\text{branch}(\hat{z})) \subset m(\{-1,1,\infty\}) = \{0,1,\infty\}$$

which shows that f is a Belyi map. So, we can draw the dessin associated to $f : S_g \to \mathbb{P}^1$. Observe that $\deg f = \deg \hat{z} = 3$, the degree of g seen as a polynomial in w. $E = S_g$ is an elliptic curve, hence our surface is a torus (we will show an alternative way to deduce this in the end). To draw the dessin on it, we need to find branch points of f and their ramification indices.

The branch points for 0 are given by $f^{-1}(0) = \hat{z}^{-1}(-1)$. To compute $|\hat{z}^{-1}(-1)|$, we will consider the circle $|z+1| = \varepsilon$ for a sufficiently small $\varepsilon > 0$. We can write $g(z,w) = 0$ as $w^3 + (1 - z^2) = 0$ which gives

$$w^3 = (z-1)(z+1)$$

We only care about the terms with $(z+1)$ in the right hand side, hence we can consider

$$w^3 = (z+1)$$

instead. If we draw the solutions of this equation in w for the circle $|z+1| = \varepsilon$, we will get $\gcd(3,1) = 1$ disjoint circle in the complex plane, where 3 is the power of w and 1 is the power of $(z+1)$ in the equation. Hence we get $|\hat{z}^{-1}(-1)| = 1$. This means $f^{-1}(0) = \hat{z}^{-1}(-1) = \{a\}$ for some a, and $v_f(a) = 3$ since the ramification indices in $f^{-1}(0)$ must add up to $\deg f = 3$. This shows the dessin associated to f has a unique white vertex with 3 edges attached to it.

The branch points for 1 are given by $f^{-1}(1) = \hat{z}^{-1}(1)$. To compute $|\hat{z}^{-1}(1)|$, we will consider the circle $|z-1| = \varepsilon$ for a sufficiently small $\varepsilon > 0$. We can write $g(z,w) = 0$ as $w^3 = (z-1)(z+1)$ as shown above. We only care about the terms with $(z-1)$ in the right hand side, hence we can consider

$$w^3 = (z-1)$$

instead. If we draw the solutions of this equation in w for the circle $|z-1| = \varepsilon$, we will get $\gcd(3,1) = 1$ disjoint circle in the complex plane, where 3 is the power of w and 1 is the power of $(z-1)$ in the equation. Hence we get $|\hat{z}^{-1}(1)| = 1$. This means $f^{-1}(1) = \hat{z}^{-1}(1) = \{b\}$ for some b, and $v_f(b) = 3$ since the ramification indices in $f^{-1}(1)$ must add up to $\deg f = 3$. This shows the dessin associated to f has a unique black vertex with 3 edges attached to it.

The branch points for ∞ are given by $f^{-1}(\infty) = \hat{z}^{-1}(\infty)$. To compute $|\hat{z}^{-1}(\infty)|$, we will consider the circle $|z| = N$ for a sufficiently large $N > 0$. We can write $g(z,w) = 0$ as $w^3 = (z-1)(z+1)$ as shown above. Since z is very large, we have $z - 1 \approx z$ and $z + 1 \approx z$ approximately, hence we approximately get

$$w^3 = z^2 \ .$$

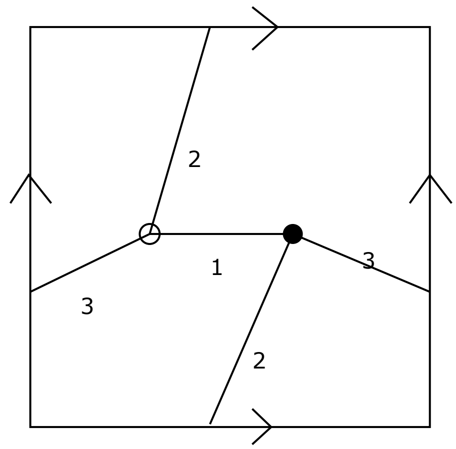
If we draw the solutions of this equation in w for the circle $|z| = N$, we will get $\gcd(3,2) = 1$ disjoint circle in the complex plane, where 3 is the power of w and 2 is the power of z in the equation. Hence we get $|\hat{z}^{-1}(\infty)| = 1$. This means $f^{-1}(\infty) = \hat{z}^{-1}(\infty) = \{c\}$ for some c, and $v_f(c) = 3$ since the ramification indices in $f^{-1}(\infty)$ must add up to $\deg f = 3$. This shows the dessin associated to f has a unique face with $3 \times 2 = 6$ edges.

We know our surface is torus, because S_f is an elliptic curve. But without this information, we could still deduce that S_f is torus. The total number of edges is $\deg f = 3$. By the genus formula

$$2 - 2g(S_f) = \#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\} = (1+1) - 3 + 1 = 0$$

hence $g(S_f) = 1$ which means S_f is a torus.

Now, we are ready to draw the dessin associated to f using the information above. We get a unique dessin (up to equivalence) on the torus, which is:



So the dessin, i.e. $f^{-1}[0,1]$, consists of 3 edges connecting the unique white and black vertices. By looking at the picture, we get $\sigma_0 = (1,2,3)$ and $\sigma_1 = (1,2,3)$.

Finally, the monodromy group $\mathrm{Mon}(f) \subset S_3$ can be calculated with the monodromy of the dessin, which is given by $< \sigma_0, \sigma_1 >=< (1,2,3) >= \{\mathrm{id}, (1,2,3), (1,3,2)\}$. Hence $\mathrm{Mon}(f) = \{\mathrm{id}, (1,2,3), (1,3,2)\}$.

7. **(i)** For any $n > 0$, consider the Belyi map on $\mathbb{P}^1$ defined by

$$f(z) = \frac{4z^n}{(z^n+1)^2}$$

Show that its dessin is the complete bipartite graph $K_{2,n}$.

Solution: First, note that

$$\begin{aligned}
\mathrm{branch}(f) &\subset f(\{\text{zeroes of } f'\}) \cup \{f(\infty)\} \\
&= f\left(\left\{\text{zeroes of } \frac{4nz^{n-1}(z^n+1)^2 - 4z^n(2nz^{n-1})(z^n+1)}{(z^n+1)^4}\right\}\right) \cup \{0\} \\
&= f(\{\text{zeroes of } 4nz^{n-1}(z^n+1)(1-z^n)\}) \cup \{0\} \\
&= f(\{0\} \cup \{z : z^n = 1\} \cup \{z : z^n = -1\}) \cup \{0\} \\
&= \{0, 1, \infty\}\ .
\end{aligned}$$

This means f is a Belyi function. So, we can draw the dessin associated to $f\colon \mathbb{P}^1 \to \mathbb{P}^1$. Observe that $\deg f = 2n$, the highest power of z in $f(z)$. Our surface is $\mathbb{P}^1$, i.e. sphere. To draw the dessin on it, we need to find branch points of f and their ramification indices.

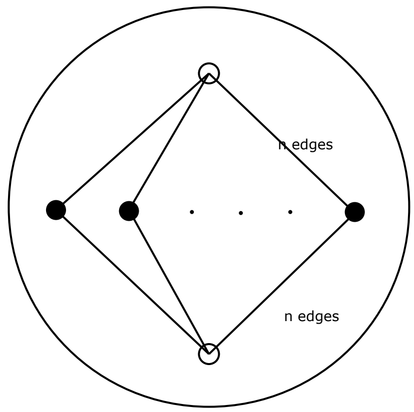
Since f is a rational function, we know that the only branch points with ramification index greater than 1 are the zeroes of f', which are given by the set $\{0\} \cup \{z : z^n = 1\} \cup \{z : z^n = -1\}$, and also we may have ∞, so they are $\{0, a_1, \ldots, a_n, b_1, \ldots, b_n, \infty\}$ where a_i are the solutions of $z^n = 1$ and b_i are the solutions of $z^n = -1$ for $i = 1, \ldots, n$. Using this information, we get the following data:

The branch points for 0 are given by $f^{-1}(0) = \{0, \infty\}$ and their ramification indices are $v_f(0) = n$ (since z has the power n in the nominator of $f(z)$) and $v_f(\infty) = n$ (since around ∞, $z^n + 1 \approx z^n$ and $f(z) \approx \frac{4z^n}{(z^n)^2} = \frac{4}{z^n}$, where z has the power n in the denominator).

We know $\{a_1, \ldots, a_n\} \subset f^{-1}(1)$ and $v_f(a_i) > 1$ for $i = 1, \ldots, n$. Also, we know that the sum of the ramification indices in $f^{-1}(1)$ add up to $\deg f = 2n$. Therefore we must have $v_f(a_i) = 2$ for all $i = 1, \ldots, n$ and the branch points for 1 are given by $f^{-1}(1) = \{a_1, \ldots, a_n\}$.

The branch points for ∞ are given by $f^{-1}(\infty) = \{b_1, \ldots, b_n\}$ and we know $v_f(b_i) > 1$ for $i = 1, \ldots, n$. Since the sum of the ramification indices in $f^{-1}(\infty)$ add up to $\deg f = 2n$, we must have $v_f(b_i) = 2$ for $i = 1, \ldots, n$.

Now, we are ready to draw the dessin associated to f. The branch points for 0 correspond to the white points, and the ramification index of each branch point is the number of edges adjacent to the corresponding vertex. The branch points for 1 correspond to the black points, and the ramification index of each branch point is the number of edges adjacent to the corresponding vertex. The branch points for ∞ correspond to the faces, and the ramification index of each branch point is half the number of edges of the corresponding face. The total number of edges is $\deg f = 2n$. Then we get a unique dessin (up to equivalence) on the sphere, which is:



(ii) For any $n > 0$, consider the Belyi map on $\mathbb{P}^1$ defined by

$$f(z) = \frac{(z^n + 1)^2}{4z^n}$$

Show that its dessin is the circular graph C_{2n} with $2n$ vertices.

Solution: First, note that

$$\begin{aligned}
\text{branch}(f) &\subset f(\{\text{zeroes of } f'\}) \cup \{f(\infty)\} \\
&= f\left(\left\{\text{zeroes of } \frac{4z^n(2nz^{n-1})(z^n+1) - 4nz^{n-1}(z^n+1)^2}{(4z^n)^2}\right\}\right) \cup \{\infty\} \\
&= f(\{\text{zeroes of } 4nz^{n-1}(z^n+1)(z^n-1)\}) \cup \{\infty\} \\
&= f(\{0\} \cup \{z : z^n = 1\} \cup \{z : z^n = -1\}) \cup \{\infty\} \\
&= \{0, 1, \infty\} \ .
\end{aligned}$$

This means f is a Belyi function. So, we can draw the dessin associated to $f\colon \mathbb{P}^1 \to \mathbb{P}^1$. Observe that $\deg f = 2n$, the highest power of z in $f(z)$. Our surface is $\mathbb{P}^1$, i.e. sphere. To draw the dessin on it, we need to find branch points of f and their ramification indices.

Since f is a rational function, we know that the only branch points with ramification index greater than 1 are the zeroes of f', which are given by the set $\{0\} \cup \{z : z^n = 1\} \cup \{z : z^n = -1\}$, and also we may have ∞, so they are $\{0, a_1, \ldots, a_n, b_1, \ldots, b_n, \infty\}$ where a_i are the solutions of $z^n = 1$ and b_i are the solutions of $z^n = -1$ for $i = 1, \ldots, n$. Using this information, we get the following data:

The branch points for 0 are given by $f^{-1}(0) = \{b_1, \ldots, b_n\}$ and we know $v_f(b_i) > 1$ for $i = 1, \ldots, n$. Since the sum of the ramification indices in $f^{-1}(0)$ add up to $\deg f = 2n$, we must have $v_f(b_i) = 2$ for $i = 1, \ldots, n$.

We know $\{a_1, \ldots, a_n\} \subset f^{-1}(1)$ and $v_f(a_i) > 1$ for $i = 1, \ldots, n$. Also, we know that the sum of the ramification indices in $f^{-1}(1)$ add up to $\deg f = 2n$. Therefore we must have $v_f(a_i) = 2$ for all $i = 1, \ldots, n$ and the branch points for 1 are given by $f^{-1}(1) = \{a_1, \ldots, a_n\}$.

The branch points for ∞ are given by $f^{-1}(\infty) = \{0, \infty\}$ and their ramification indices are $v_f(0) = n$ (since 0 is a pole of order n in $f(z)$) and $v_f(\infty) = n$ (since around ∞, $z^n + 1 \approx z^n$ and $f(z) \approx \frac{(z^n)^2}{4z^n} = \frac{z^n}{4}$, where z has the power n in the nominator).

Now, we are ready to draw the dessin associated to f. The branch points for 0 correspond to the white points, and the ramification index of each branch point is the number of edges adjacent to the corresponding vertex. The branch points for 1 correspond to the black points, and the ramification index of each branch point is the number of edges adjacent to the corresponding vertex. The branch points for ∞ correspond to the faces, and the ramification index of each branch point is half the number of edges of the corresponding face. The total number of edges is $\deg f = 2n$. Then we get a unique dessin (up to equivalence) on the sphere, which is: