Curves - Homework 4

Yankı Lekili, Autumn 2019

1) Check that the monodromy homomorphism M_f defined with respect to a function $f : S \to T$ is indeed a homomorphism

$$\pi_1(T \setminus \{t_1, \dots, t_n\}, t) \to \mathfrak{S}(f^{-1}(t))$$

where $\{t_1, \dots, t_n\}$ are branching values of f and $\mathfrak{S}(f^{-1}(t)$ is the group of permutations of the finite set $f^{-1}(t)$. Show that M_f is transitive (since S is connected).

2) Give a Belyi function for the Riemann surface associated to

$$z^2 = w(w-1)(w-2)$$

3) Let S_f be the compact Riemann surface defined by the irreducible polynomial

$$f(z, w) = z^2 - w(w-1)(w-\sqrt{2})$$

Construct a Belyi function on S_f.

4) Consider the Fermat curve $F_n = \{[X, Y, Z] \in \mathbb{P}^2 : X^n + Y^n = Z^n\}$. Let $f : F_n \to \mathbb{P}^1$ given by $[X, Y, Z] \to [X, Z]$. Compose this with the map $g : \mathbb{P}^1 \to \mathbb{P}^1$ given by $z \to z^n$. Show that as a result we get a Belyi map of degree n^2.

5) Let $\sigma_0 = (1, 5, 4)(2, 6, 3)$ and $\sigma_1 = (1, 2)(3, 4)(5, 6)$. Construct the corresponding surface and the dessin d'enfant on it.

6) Let E be the elliptic curve defined by the affine equation $z^2 = w^3 + 1$. Show that the rational map $f : E \to \mathbb{P}^1$ defined by

$$(z, w) \mapsto \frac{1+z}{2}$$

is a Belyi map. What is its degree? Show that the dessin associated to f has a unique white vertex and a unique black vertex. Show that $f^{-1}[0, 1]$ consists of 3 edges connecting these vertices. Give two permutations σ_0, σ_1 describing this dessin. What is the monodromy group?

7) (i) For any $n > 0$, consider the Belyi map on $\mathbb{P}^1$ defined by

$$f(z) = \frac{4z^n}{(z^n + 1)^2}$$

Show that its dessin is the complete bipartite graph $K_{2,n}$.

(ii) For any $n > 0$, consider the Belyi map on $\mathbb{P}^1$ defined by

$$f(z) = \frac{(z^n + 1)^2}{4z^n}$$

Show that its dessin is the circular graph C_{2n} with $2n$ vertices.

(Hint: Write the above functions as $f(z) = g(z^n)$ for some g, and consider the dessin for g first.)