Algebraic Curves - Homework 3 Solutions

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1. Given a lattice $\Lambda \subset \mathbb{C}$. Let $C_\Lambda \subset \mathbb{P}^2$ be the projective curve defined by

$$Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3 = 0$$

Show that there is a well-defined map $\mathbb{C}/\Lambda \to C_\Lambda$ given by

$$[z+\Lambda] \mapsto \begin{cases} [\wp(z), \wp'(z), 1] \text{ if } z \notin \Lambda \\ [0,1,0] \text{ if } z \in \Lambda \end{cases}$$

which is an isomorphism of Riemann surfaces.

Solution: First, note that C_Λ is a smooth curve, hence it is a Riemann surface (see the note in the end of the solution of Problem 2(i)). Define the map $f\colon \mathbb{C}/\Lambda \to C_\Lambda$ such that

$$f([z]) = \begin{cases} [\wp(z), \wp'(z), 1] \text{ if } z \notin \Lambda \\ [0,1,0] \text{ if } z \in \Lambda \end{cases}$$

To show that f is well-defined, first show $f([z]) \in C_\Lambda$: If $z \notin \Lambda$, then $f([z]) = [\wp(z), \wp'(z), 1]$ and we know $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3 = 0$ (see Theorem 39 in lecture notes), hence $f([z]) \in C_\Lambda$. If $z \in \Lambda$, then $f([z]) = [0,1,0] \in C_\Lambda$. Next, pick $[z] = [w]$. If $z \notin \Lambda$, then $w \notin \Lambda$ and $f([z]) = [\wp(z), \wp'(z), 1] = [\wp(w), \wp'(w), 1] = f([w])$ since $\wp(z) = \wp(w)$ and $\wp'(z) = \wp'(w)$ by doubly periodicity of $\wp$ and $\wp'$ and $z - w \in \Lambda$. If $z \in \Lambda$, then $w \in \Lambda$ and $f([z]) = [0,1,0] = f([w])$. Hence, f is well-defined.

To show f is injective, first observe that $f^{-1}([0,1,0]) = \{[0]\}$, therefore it is enough to show $f|_{(\mathbb{C}/\Lambda)\setminus\{[0]\}}$ is injective. Then assume $f([z]) = f([w])$ for $z, w \notin \Lambda$ (i.e. $[z], [w] \neq [0]$). We have $[\wp(z), \wp'(z), 1] = [\wp(w), \wp'(w), 1]$ which gives $\wp(z) = \wp(w)$ and $\wp'(z) = \wp'(w)$. Since $\wp$ (as a map with domain $\mathbb{C}/\Lambda$) has degree 2, $\wp([z]) = a$ has at most 2 solutions for any a, and $\wp$ is an even function, so all solutions are $[z]$ and $[-z]$. Then $\wp(z) = \wp(w) = a$ implies $[w] = [z]$ or $[w] = [-z]$. If $[w] \neq [z]$, then $[w] = [-z]$ and we have $\wp'(w) = \wp'(-z) = -\wp'(z)$ since $\wp'$ is an odd function. But $\wp'(z) = \wp'(w)$ also, hence $\wp'(z) = \wp'(-z) = 0$. But in this case $v_\wp([z]) \geq 2$ and $v_\wp([-z]) \geq 2$, and

$$2 = \deg \wp = \sum_{[b] \in \wp^{-1}(a)} v_\wp([b]) = v_\wp([z]) + v_\wp([-z]) \geq 4$$

which is a contradiction. Then $[w] = [z]$, which shows f is injective.

To show f is surjective, pick $[w_1, w_2, w_3] \in C_\Lambda$. If $w_3 = 0$, then $4w_1^3 = 0$, hence $w_1 = 0$. Therefore, $[w_1, w_2, w_3] = [0, w_2, 0] = [0,1,0] = f([0])$. If $w_3 \neq 0$, by scaling we can assume $w_3 = 1$. We want to find z such that $f([z]) = [\wp(z), \wp'(z), 1] = [w_1, w_2, 1]$, i.e. $\wp(z) = w_1$ and $\wp'(z) = w_2$. First, since $\wp\colon \mathbb{C}/\Lambda \to \mathbb{P}^1$ is holomorphic between compact connected spaces, it is surjective. Hence there exists z such that $\wp(z) = w_1$. Since $[w_1, w_2, 1] = [\wp(z), w_2, 1] \in C_\Lambda$, we have $w_2^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$. This equation has two solutions in w_2 (since it has degree 2 in w_2): One is $w_2 = \wp'(z)$ (by Theorem 33), the other is $w_2 = -\wp'(z)$. If $w_2 = \wp'(z)$, then $f([z]) = [\wp(z), \wp'(z), 1] = [w_1, w_2, 1]$. If $w_2 = -\wp'(z)$, then $f([-z]) = [\wp(-z), \wp'(-z), 1] = [\wp(z), -\wp'(z), 1] = [w_1, w_2, 1]$ since $\wp$ is even and $\wp'$ is odd function. Hence, f is surjective.

Lastly, to see $f\colon \mathbb{C}/\Lambda \to C_\Lambda$ is holomorphic, consider the charts of $\mathbb{C}/\Lambda$ which are

$$\phi_{i,j} = (\pi|_{U'_{i,j}})^{-1}\colon U_{i,j} \to U'_{i,j}$$

for $i,j = 0, 1/2$ where $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ is the projection (see Hw 1, Problem 3). If $(i,j) \neq (1/2,1/2)$, then $z \in U'_{i,j}$ implies $z \notin \Lambda$ and

$$f \circ \phi_{i,j}^{-1}(z) = f \circ \pi|_{U'_{i,j}}(z) = [\wp(z), \wp'(z), 1]$$

whose components are all holomorphic, since $\wp$ and $\wp'$ are holomorphic, hence $f \circ \phi_{i,j}^{-1}$ is holomorphic for $(i,j) \neq (1/2,1/2)$. If $(i,j) = (1/2,1/2)$, then there exists $z \in U'_{1/2,1/2}$ such that $z \in \Lambda$ and

$$f \circ \phi_{1/2,1/2}^{-1}(z) = f \circ \pi|_{U_{1/2,1/2}}(z) = \begin{cases} [\wp(z), \wp'(z), 1] \text{ if } z \notin \Lambda \\ [0,1,0] \text{ if } z \in \Lambda \end{cases} = \left[\frac{\wp(z)}{\wp'(z)}, 1, \frac{1}{\wp'(z)}\right]$$

since $\wp'(z) \neq 0$ for $z \in U'_{1/2,1/2}$ (in lecture notes, it is shown that $\wp'([z]) = 0$ only for $[z] = [1/2], [\tau/2], [(1+\tau)/2]$ and we know $[1/2], [\tau/2], [(1+\tau)/2] \notin U_{1/2,1/2}$ where $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$). All the components are holomorphic, since $\wp$ and $\wp'$ are holomorphic, hence $f \circ \phi_{1/2,1/2}^{-1}$ is holomorphic. This shows f is holomorphic.

As a conclusion, f is an isomorphism and $\mathbb{C}/\Lambda$ and C_Λ are isomorphic.

Note: One can show f is bijective by showing $\deg f = 1$ (degree is well-defined for f since f is a holomorphic map between compact, connected Riemann surfaces).

2. **(i)** Given a curve C_Λ by the equation

$$Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3 = 0$$

we define

$$J(\Lambda) = \frac{g_2^3}{g_2^3 - 27g_3^2}$$

Show that $g_2^3 - 27g_3^2 \neq 0$ so that $J(\Lambda)$ is well-defined.

Solution: Let $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$. We know $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ are the solutions of $\wp'(z) = 0$ (see the lecture notes). Also we know $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ for every z, by putting $z = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ we see that the equation $4w^3 - g_2w - g_3 = 0$, or equivalently

$$w^3 - \frac{g_2}{4}w - \frac{g_3}{4} = 0$$

has the solutions $w = \wp(\frac{1}{2}), \wp(\frac{\tau}{2}), \wp(\frac{1+\tau}{2})$. In Hw 2, Problem 10, we have shown that they are all distinct, so they are all solutions for this degree 3 polynomial. Now we will use the fact that the discriminant of a polynomial is zero if and only if it has a repeated root. Here, the polynomial has no repeated root, hence the discriminant Δ is non-zero. Also, for a cubic polynomial $w^3 + pw + q$

$$\Delta = -4p^3 - 27q^2$$

therefore for our polynomial $w^3 - \frac{g_2}{4}w - \frac{g_3}{4}$, we have

$$\Delta = \frac{1}{16}(g_2^3 - 27g_3^2) \neq 0$$

hence $g_2^3 - 27g_3^2 \neq 0$ and $J(\Lambda)$ is well-defined.

Alternative Solution: If you don't know the discriminant formula, you can proceed as follows: $h(w) := w^3 - \frac{g_2}{4}w - \frac{g_3}{4} = 0$ has 3 distinct roots, i.e. has no repeated root. Hence there exists no w

such that $h(w) = 0$ and $h'(w) = 0$. If we assume there exists such w, then $h'(w) = 0$ implies $w^2 = \frac{g_2}{12}$, and together with $h(w) = 0$ we get

$$\begin{aligned} w^3 - \frac{g_2}{4}w - \frac{g_3}{4} &= 0 \\ \frac{g_2}{12}w - \frac{g_2}{4}w - \frac{g_3}{4} &= 0 \\ w &= -\frac{3g_3}{2g_2} \\ w^2 = \left(-\frac{3g_3}{2g_2}\right)^2 &= \frac{9g_3^2}{4g_2^2} \ . \end{aligned}$$

But we also have $w^2 = \frac{g_2}{12}$, hence $g_2^3 - 27g_3^2 = 0$. Conversely, if $g_2^3 - 27g_3^2 = 0$, then we would have $h(w) = h'(w) = 0$ for $w = -\frac{3g_3}{2g_2}$ and w would be a double root. Since h has no double root, we have $g_2^3 - 27g_3^2 \neq 0$ and $J(\Lambda)$ is well-defined.

Note: $C_\Lambda = \{F = 0\}$ for $F = Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3$ is a smooth curve since if it is not smooth, then there exists $[X : Y : Z] \in \mathbb{P}^2$ such that

$$\begin{aligned} F_X &= -12X^2 + g_2Z^2 = 0 \\ F_Y &= 2YZ = 0 \\ F_Z &= Y^2 + 2g_2XZ + 3g_3Z^2 = 0 \ . \end{aligned}$$

If $Z = 0$, then $X = Y = 0$, which is not possible. Then $Z \neq 0$ and by scaling we can assume $Z = 1$. Then $Y = 0$, $12X^2 = g_2$ and $2g_2X = -3g_3$ which implies $4g_2^2X^2 = 9g_3^2$ and hence $4g_2^2(\frac{g_2}{12}) = 9g_3^2$ and $g_2^3 = 27g_3^2$. But we have shown that $g_2^3 \neq 27g_3^2$, which gives a contradiction. Therefore C_Λ is smooth and it is a Riemann surface.

(ii) Given two lattice $\Lambda, \tilde{\Lambda} \subset \mathbb{C}$. Show that if $J(\Lambda) = J(\tilde{\Lambda})$ then the corresponding projective curves

$$Y^2Z - 4X^3 + g_2(\Lambda)XZ^2 + g_3(\Lambda)Z^3 = 0 \text{ and } Y^2Z - 4X^3 + g_2(\tilde{\Lambda})XZ^2 + g_3(\tilde{\Lambda})Z^3 = 0$$

are projectively equivalent in $\mathbb{P}^2$.

Solution: Suppose $J(\Lambda) = J(\tilde{\Lambda})$. Then

$$\frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27g_3(\Lambda)^2} = \frac{g_2(\tilde{\Lambda})^3}{g_2(\tilde{\Lambda})^3 - 27g_3(\tilde{\Lambda})^2}$$

which implies $g_2(\Lambda)^3g_3(\tilde{\Lambda})^2 = g_2(\tilde{\Lambda})^3g_3(\Lambda)^2$. Assuming $g_2(\tilde{\Lambda}) \neq 0$ and $g_3(\tilde{\Lambda}) \neq 0$ (they imply $g_2(\Lambda) \neq 0$ and $g_3(\Lambda) \neq 0$ also, since otherwise we would get $g_2^3 - 27g_3^2 = 0$) we get

$$\left(\frac{g_2(\Lambda)}{g_2(\tilde{\Lambda})}\right)^3 = \left(\frac{g_3(\Lambda)}{g_3(\tilde{\Lambda})}\right)^2 = a^{12}$$

which gives $g_2(\Lambda) = a^4g_2(\tilde{\Lambda})$ and $g_3(\Lambda) = a^6g_3(\tilde{\Lambda})$ for some $a \in \mathbb{C}^\times$ (note that if $g_2(\tilde{\Lambda}) = 0$ or $g_3(\tilde{\Lambda}) = 0$, we get the same relations). Define $A\colon \mathbb{C}^3 \to \mathbb{C}^3$ such that $A(X, Y, Z) = (X, \frac{1}{a}Y, a^2Z)$. It is obvious to see that $A \in \mathrm{GL}(3, \mathbb{C})$. Also we have $A(C_\Lambda) = C_{\tilde{\Lambda}}$ since

$$\left(\left(\frac{1}{a}\right)Y\right)^2 a^2Z - 4X^3 + g_2(\tilde{\Lambda})X(a^2Z)^2 + g_3(\tilde{\Lambda})(a^2Z)^3 = Y^2Z - 4X^3 + g_2(\Lambda)XZ^2 + g_3(\Lambda)Z^3$$

hence C_Λ and $C_{\tilde{\Lambda}}$ are projectively equivalent in $\mathbb{P}^2$ (see Hw 2, Problem 4 for details).

(iii) Given two lattices $\Lambda, \tilde{\Lambda} \subset \mathbb{C}$. Show that the following are equivalent:

(a) $\mathbb{C}/\Lambda$ is biholomorphic to $\mathbb{C}/\tilde{\Lambda}$.

(b) $\Lambda = c\tilde{\Lambda}$ for some $c \in \mathbb{C}^\times$.

(c) $J(\Lambda) = J(\tilde{\Lambda})$.

Solution: (b) $\Rightarrow$ (c): Suppose $\Lambda = c\tilde{\Lambda}$ for some $c \in \mathbb{C}^\times$. Then

$$g_2(\Lambda) = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^4} = 60 \sum_{\lambda \in c\tilde{\Lambda} \setminus \{0\}} \frac{1}{\lambda^4} = 60 \sum_{\lambda \in \tilde{\Lambda} \setminus \{0\}} \frac{1}{(c\lambda)^4} = \frac{1}{c^4} g_2(\tilde{\Lambda})$$

and

$$g_3(\Lambda) = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^6} = 140 \sum_{\lambda \in c\tilde{\Lambda} \setminus \{0\}} \frac{1}{\lambda^6} = 140 \sum_{\lambda \in \tilde{\Lambda} \setminus \{0\}} \frac{1}{(c\lambda)^6} = \frac{1}{c^6} g_3(\tilde{\Lambda}) \ .$$

Then we have

$$J(\tilde{\Lambda}) = \frac{g_2(\tilde{\Lambda})^3}{g_2(\tilde{\Lambda})^3 - 27 g_3(\tilde{\Lambda})^2} = \frac{(c^4 g_2(\Lambda))^3}{(c^4 g_2(\Lambda))^3 - 27(c^6 g_3(\Lambda))^2} = \frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27 g_3(\Lambda)^2} = J(\Lambda) \ .$$

(c) $\Rightarrow$ (a): Suppose $J(\Lambda) = J(\tilde{\Lambda})$. Then we have shown in part (ii) that C_Λ and $C_{\tilde{\Lambda}}$ are projectively equivalent in $\mathbb{P}^2$, which implies C_Λ and $C_{\tilde{\Lambda}}$ are isomorphic as Riemann surfaces (see Hw 2, Problem 4). In Problem 1, we have shown that C_Λ and $\mathbb{C}/\Lambda$ are isomorphic, hence we get $\mathbb{C}/\Lambda$ and $\mathbb{C}/\tilde{\Lambda}$ are isomorphic (i.e. biholomorphic).

(a) $\Rightarrow$ (b): Suppose we are given an isomorphism $f \colon \mathbb{C}/\Lambda \to \mathbb{C}/\tilde{\Lambda}$ of Riemann surfaces. By composing it with a translation, we can assume that $f([0]) = [0]$. First, we want to find a map $\hat{f} \colon \mathbb{C} \to \mathbb{C}$ such that the following diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\hat{f}} & \mathbb{C} \\ \downarrow{\scriptstyle \pi_\Lambda} & & \downarrow{\scriptstyle \pi_{\tilde{\Lambda}}} \\ \mathbb{C}/\Lambda & \xrightarrow[f]{} & \mathbb{C}/\tilde{\Lambda} \end{array}$$

commutes and $\hat{f}(0) = 0$, where $\pi_\Lambda : \mathbb{C} \to \mathbb{C}/\Lambda$ and $\pi_{\tilde{\Lambda}} : \mathbb{C} \to \mathbb{C}/\tilde{\Lambda}$ are the projections. We know the following lifting theorem: Let X be a simply connected and locally path-connected space, and let $p : Y \to Z$ be a covering map. Then given $g : X \to Z$ continuous, $x_0 \in X$, $y_0 \in Y$ such that $p(y_0) = g(x_0)$, there is a unique continuous function $\hat{g} \colon X \to Y$ such that the following diagram

$$\begin{array}{ccc} & \overset{\hat{g}}{\nearrow} & Y \\ & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[g]{} & Z \end{array}$$

commutes and $\hat{g}(x_0) = y_0$.

In our case, by setting $X = \mathbb{C}$ (which is simply connected and path connected), $Y = \mathbb{C}$, $Z = \mathbb{C}/\tilde{\Lambda}$, $p = \pi_{\tilde{\Lambda}}$ (which is a covering map), $g = f \circ \pi_\Lambda$, $x_0 = 0$, $y_0 = 0$ (which satisfy $p(y_0) = g(x_0)$ since $f([0]) = [0]$), we get the unique continuous function $\hat{g} = \hat{f}$ such that the following diagram

$$\begin{array}{ccc} & \overset{\hat{f}}{\nearrow} & \mathbb{C} \\ & & \downarrow{\scriptstyle \pi_{\tilde{\Lambda}}} \\ \mathbb{C} & \xrightarrow[f \circ \pi_\Lambda]{} & \mathbb{C}/\tilde{\Lambda} \end{array}$$

commutes and $\hat{f}(0) = 0$. Observe that this implies the first square diagram above commutes.

Next, we want to show that $\hat{f}$ is holomorphic. Note that $\pi_{\tilde{\Lambda}}$ is invertible locally, so for each point $z \in \mathbb{C}$, by choosing sufficiently small open neighbourhood U_z of z, we have $\hat{f}|_{U_z} = \pi_{\tilde{\Lambda}}^{-1} \circ f \circ \pi_\Lambda$. Since $\pi_{\tilde{\Lambda}}$ is holomorphic, its (local) inverse $\pi_{\tilde{\Lambda}}^{-1}$ is holomorphic. Also, f and π_Λ are holomorphic. Hence $\hat{f}|_{U_z}$ is holomorphic for all $z \in \mathbb{C}$ and that implies $\hat{f}$ is holomorphic.

Next, we claim $\hat{f}(z) = cz$ for some $c \in \mathbb{C}^\times$. For that, we will show that $\hat{f}'(z)$ is constant. By inspecting the commutative square, we have $\hat{f}(z+\lambda) - \hat{f}(z) \in \tilde{\Lambda}$ for any $z \in \mathbb{C}$ and $\lambda \in \Lambda$. Fix $z_0 \in \mathbb{C}$ and $\lambda \in \Lambda$. Then

$$(\hat{f}(z+\lambda) - \hat{f}(z)) - (\hat{f}(z_0+\lambda) - \hat{f}(z_0)) \in \tilde{\Lambda}$$

for any $z \in \mathbb{C}$. The above expression is continuous in z, and it takes value in the discrete set $\tilde{\Lambda}$, hence it is constant. By putting $z = z_0$ it becomes 0, therefore it is identically zero. Then

$$\hat{f}(z+\lambda) - \hat{f}(z_0+\lambda) = \hat{f}(z) - \hat{f}(z_0)$$

for any $z \in \mathbb{C}$. But then

$$\hat{f}'(z_0+\lambda) = \lim_{z \to z_0} \frac{\hat{f}(z+\lambda) - \hat{f}(z_0+\lambda)}{z - z_0} = \lim_{z \to z_0} \frac{\hat{f}(z) - \hat{f}(z_0)}{z - z_0} = \hat{f}'(z_0) \ .$$

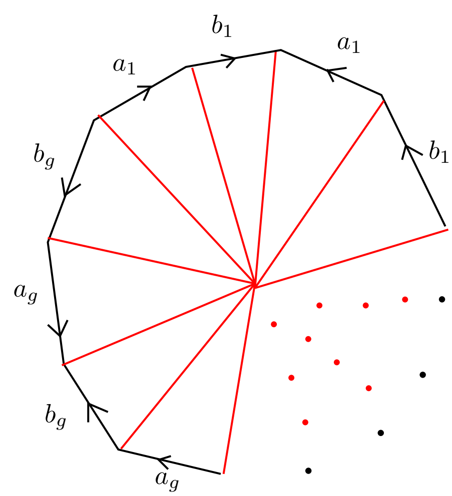
Hence, $\hat{f}'$ is Λ-periodic (i.e. $f'(z_0+\lambda) = f'(z_0)$ for any $z_0 \in \mathbb{C}$ and $\lambda \in \Lambda$). Now, consider a fundamental parallelogram $P \subset \mathbb{C}$ for Λ. Since $\hat{f}'$ is Λ-periodic, the domain P is enough to understand all of its values, i.e. $\hat{f}'(\mathbb{C}) = \hat{f}'(P)$. But we know $\bar{P}$ is compact, hence $\hat{f}'(\bar{P})$ is compact and obviously $\hat{f}'(\mathbb{C}) = \hat{f}'(\bar{P})$ also, so $\hat{f}'(\mathbb{C})$ is compact, which implies $\hat{f}'(\mathbb{C})$ is bounded (i.e. $\hat{f}'$ is bounded). $\hat{f}' \colon \mathbb{C} \to \mathbb{C}$ is also holomorphic, since $\hat{f}$ is holomorphic, then by Liouville's theorem $\hat{f}'$ is constant. We can write $\hat{f}'(z) = c$ for some $c \in \mathbb{C}$ for all $z \in \mathbb{C}$. Hence we get $\hat{f}(z) = a + cz$ for some $a \in \mathbb{C}$. Recall that we have $\hat{f}(0) = 0$, so $\hat{f}(z) = cz$. Also $c \neq 0$, since otherwise we can see by inspecting the commutative square that f would be constant, but we know f is an isomorphism. Therefore, $\hat{f}(z) = cz$ for $c \in \mathbb{C}^\times$.

Finally, we will show that $\hat{f}(\Lambda) = \tilde{\Lambda}$. We know $\hat{f}(z+\lambda) - \hat{f}(z) \in \tilde{\Lambda}$ for any $z \in \mathbb{C}$ and $\lambda \in \Lambda$. By setting $z = 0$ we get $\hat{f}(\lambda) - \hat{f}(0) = \hat{f}(\lambda) \in \tilde{\Lambda}$ for any $\lambda \in \Lambda$, which implies $\hat{f}(\Lambda) \subset \tilde{\Lambda}$. Conversely, pick $w \in \tilde{\Lambda}$. Since $\hat{f}(z) = cz$ for $c \in \mathbb{C}^\times$, $\hat{f}$ is an isomorphism, hence there exists $z \in \mathbb{C}$ such that $\hat{f}(z) = w$. By inspecting the commutative square, we see that $z \in \Lambda$. Therefore $\tilde{\Lambda} \subset \hat{f}(\Lambda)$ and $\hat{f}(\Lambda) = \tilde{\Lambda}$.

In conclusion, since $\hat{f}(z) = cz$ we get $\tilde{\Lambda} = \hat{f}(\Lambda) = c\Lambda$ where $c \in \mathbb{C}^\times$, which concludes the proof.

3. Compute the Euler characteristic of a compact orientable surface of genus g.

   Solution: We can present the compact orientable surface Σ_g of genus g as in the following figure (the black part)



where the edges with the same label are identified as shown with arrows. Then we have the triangulation drawn in the above figure (red and black parts). We get

$$\begin{aligned} \#V &= 2 \\ \#E &= 6g \\ \#F &= 4g \end{aligned}$$

hence the Euler characteristic of Σ_g is $\chi(\Sigma_g) = \#V - \#E + \#F = 2 - 2g$.

4. Consider the curve $C = \{[X : Y : Z] \in \mathbb{P}^2 \ : \ Y^2Z^2 = X^4 + Y^4 + Z^4\}$. Show that this curve is smooth. Consider the holomorphic map

$$\begin{aligned} f : C &\to \mathbb{P}^1 \\ [X : Y : Z] &\mapsto [X : Y]. \end{aligned}$$

Check that f is a well-defined holomorphic map when restricted to C. What is the degree of f? Compute the branching index b_f, and use this, together with the Riemann-Hurwitz forula, to compute the genus of C.

Solution: To check smoothness, we need to show that the only point (X_0, Y_0, Z_0) such that $F(X_0, Y_0, Z_0) = F_X(X_0, Y_0, Z_0) = F_Y(X_0, Y_0, Z_0) = F_Z(X_0, Y_0, Z_0) = 0$ is $(0, 0, 0)$. We have that

$$\begin{aligned} F_X &= 4X^3 \\ F_Y &= 4Y^3 - 2YZ^2 \\ F_Z &= 4Z^3 - 2Y^2Z. \end{aligned}$$

Therefore, for a singular point, we must have $X = 0$. If either of $Y, Z = 0$ also, then we must have $Z = Y = 0$, which is not a point in $\mathbb{P}^2$, so we can assume that $Y, Z \neq 0$. We then have

$$\begin{aligned} 4Z^2 - 2Y^2 &= 0 \\ 4Y^2 - 2Z^2 &= 0, \end{aligned}$$

which leads to $Y = Z = 0$, so the curve is smooth.

We are told that the map is holomorphic, and it is clearly well-defined away from the point $[0 : 0 : 1] \in \mathbb{P}^2$. Note, however, that $[0 : 0 : 1] \notin C$, and so the map is well-defined when restricted to C.

To calculate the degree, we must calculate how many points are in the pre-image of a generic point in $\mathbb{P}^1$. Suppose $[a : b] \in \mathbb{P}^1$. Then $f^{-1}([a : b]) = \{[a : b : Z] \in \mathbb{P}^2 \ : \ b^2Z^2 = a^4 + b^4 + Z^4\}$. This is a degree four polynomial in Z, and so there are 4 points in the preimage of f for any non-branch point $[a : b] \in \mathbb{P}^1$. To calculate the branching index, we need to find how many points there are in $\mathbb{P}^1$ such that the preimage contains less than four points. Firstly, note that if $b = 0$ then $a^4 + Z^4 = 0$, which has four solutions. Therefore there are no branching points in the plane $b = 0$, and so without loss of generality, we can set $b = 1$. To find the branching points, note that by the quadratic formula

$$Z^2 = \frac{1 \pm \sqrt{1 - 4(a^4 + 1)}}{2}.$$

There are two possible ways that this can have less than four solutions. Firstly, if $1 \pm \sqrt{1 - 4(a^4 + 1)} = 0$, then $a^4 + 1 = 0$, and there will be only three points in the preimage. There are four solutions to $a^4 + 1 = 0$, and so there will be four branch points with three points in the preimage. The points in the preimage of these branch points are $[\xi : 1 : 0]$, $[\xi : 1 : 1], [\xi : 1 : -1]$, where $\xi^4 = -1$.

The second way that there can be less than four points in the preimage is if $1 = 4(a^4 + 1)$. There are also four solutions to this equation, and in this case there will be two points in the preimage. Therefore, there are four points with two points in the preimage. The preimages of these branch points will be $[\eta : 1 : \frac{1}{\sqrt{2}}], [\eta : 1 : \frac{-1}{\sqrt{2}}]$, where $1 = 4(\eta^4 + 1)$.

$$b_f = \sum_{[X:Y] \in \text{branch}(f)} (\deg(f) - |f^{-1}([X : Y])|) = 4 * 2 + 4 * 1 = 12.$$

Since $g(\mathbb{P}^1) = 0$, we have by the Riemann-Hurwitz formula that

$$\begin{aligned} 2g(C) - 2 &= \deg(f)(g(\mathbb{P}^1) - 2) + b_f \\ &= 4(0 - 2) + 12 \\ &\implies g(C) = 3. \end{aligned}$$

5. Show that a proper, local homeomorphism between locally compact Hausdorff spaces is a covering map. Give an example of a local homeomorphism that is not a covering map.

   Solution: Let $p : E \to B$ be a proper, local homeomorphism between locally compact Hausdorff spaces. Pick a point $y \in B$. First, we want to show that $p^{-1}(y)$ is a finite set. p is proper and $\{y\}$ is compact, hence $p^{-1}(y)$ is compact. Also, for each $x \in p^{-1}(y)$, there is an open neighbourhood U of x such that $p|_U : U \to p(U)$ is a homeomorphism (since p is a local homeomorphism). In particular $p|_U$ is injective, which means $(p|_U)^{-1}(y) = \{x\}$, i.e. $p^{-1}(y) \cap U = \{x\}$. This means $p^{-1}(y)$ is discrete. So $p^{-1}(y)$ is compact and discrete, hence $p^{-1}(y)$ is finite.

   We have $p^{-1}(y) = \{x_1, \ldots, x_n\}$ for some $x_1, \ldots, x_n$. Since p is a local homeomorphism, there exists an open neighbourhood U_i of x_i such that $p|_{U_i}$ is a homeomorphism, in particular an open map (we can choose U_i such that they are pairwise disjoint, since E is Hausdorff). Then $V_i := p(U_i)$ is open and $V := \bigcap_{i=1}^n V_i$ is open. Since B is locally compact Hausdorff, there is an open neighbourhood W of y such that $\overline{W}$ is compact and $\overline{W} \subset V$.

   Let $Z = p^{-1}(\overline{W}) \setminus \bigsqcup_{i=1}^n U_i$. Then Z is compact since p is proper. Define $W' = W \setminus p(Z)$. p is continuous, hence $p(Z)$ is compact, and B is Hausdorff, hence $p(Z)$ is closed, therefore W' is open. Note that $y \in W'$. Let $U_i' = p^{-1}(W') \cap U_i$, then U_i' is open and all U_i''s are pairwise disjoint. Also, we have $p^{-1}(W') = \bigsqcup_{i=1}^n U_i'$ where $p|_{U_i'} : U_i' \to W'$ is homeomorphism (since $p|_{U_i}$ is homeomorphism and $p(U_i') = W'$). Hence p is a covering map.

   For the example, consider the map $p \colon (0, 2\pi) \to S^1$ with $p(t) = e^{it}$. Obviously, for all $t_0 \in (0, 2\pi)$, it has the open neighbourhood $(0, 2\pi)$ such that $p|_{(0,2\pi)} \colon (0, 2\pi) \to p((0, 2\pi)) = S^1 \setminus \{1\}$ is a homeomorphism. Hence p is a local homeomorphism. But p is not a covering map, since if you pick $1 \in S^1$ and any open neighbourhood U of 1 in S^1, then $p^{-1}(U)$ is obviously non-empty, and if V is a connected component of $p^{-1}(U)$, then $p|_V : V \to U$ is not a homeomorphism, since it is not surjective (that is because $p^{-1}(1) = \emptyset$).

   Note: Riemann surfaces are Hausdorff and they are locally compact, since they locally look like $\mathbb{C}$ which is locally compact. Hence this theorem applies to Riemann surfaces.

6. Let S be the compact Riemann surface associated to the equation $z^{2a} - 2w^b z^a + 1 = 0$, for fixed positive integers a, b with $\gcd(a, b) = 1$. Identify the branch points of the covering of the Riemann sphere defined by the projection to the z co-ordinate, and hence show that the genus of S is $ab - a$.

   Solution: Let $S = S_f$ be the compact Riemann surface associated to the irreducible polynomial $f(z, w) = (-2z^a)w^b + (1 + z^{2a})$. Assume $b \neq 1$ for now. We have the map $\hat{z} \colon S_f \to \mathbb{P}^1$, which is the extension of the projection map $z\colon S_f^z \to \mathbb{P}^1$, where S_f^z is given by

$$\begin{aligned} S_f^z &= \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0, f_w(z,w) \neq 0, \text{the coefficient of highest power of } w \text{ in } f \neq 0\} \\ &= \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0, -2bz^a w^{b-1} \neq 0, -2z^a \neq 0\} \\ &= \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0, z \neq 0, w \neq 0\} \\ &= \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0, z \neq 0, z^{2a} \neq -1\} \\ &= \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0, z \neq 0, z \neq z_i \text{ for } i = 1, \dots, 2a\} \end{aligned}$$

where z_i are the distinct solutions of $z^{2a} = -1$. Remember for the branching values of $\hat{z}$ (i.e. $\mathrm{branch}(\hat{z})$), we have $\mathrm{branch}(\hat{z}) \subset \hat{z}(S_f \setminus S_f^z)$. It is clear that $\hat{z}(S_f \setminus S_f^z) = \{0, z_1, \dots, z_{2a}, \infty\}$, hence we have

$$\mathrm{branch}(\hat{z}) \subset \{0, z_1, \dots, z_{2a}, \infty\} \ .$$

To find the genus of S_f (i.e. $g(S_f)$), we will use the Riemann-Hurwitz formula for $\hat{z}\colon S_f \to \mathbb{P}^1$, which is

$$2g(S_f) - 2 = \deg(\hat{z})(2g(\mathbb{P}^1) - 2) + b_{\hat{z}}$$

where $b_{\hat{z}}$ is the branching index of $\hat{z}$, given by

$$b_{\hat{z}} = \sum_{z \in \mathbb{P}^1} (\deg(\hat{z}) - |\hat{z}^{-1}(z)|) = \sum_{z \in \mathrm{branch}(\hat{z})} (\deg(\hat{z}) - |\hat{z}^{-1}(z)|) \ .$$

Note that we can sum over any set containing $\mathrm{branch}(\hat{z})$, since the points which are not branching values won't contribute to the sum. Hence we have

$$b_{\hat{z}} = \sum_{z \in \{0, z_1, \dots, z_{2a}, \infty\}} (\deg(\hat{z}) - |\hat{z}^{-1}(z)|)$$

since $\mathrm{branch}(\hat{z}) \subset \{0, z_1, \dots, z_{2a}, \infty\}$. In the Riemann-Hurwitz formula, we know $g(\mathbb{P}^1) = 0$, and $\deg(\hat{z})$ is just the degree of f seen as a polynomial in w. Hence $\deg(\hat{z}) = b$. Only thing left to calculate is $b_{\hat{z}}$, or $|\hat{z}^{-1}(z)|$ for $z \in \{0, z_1, \dots, z_{2a}, \infty\}$. For that we need to do analysis, since we deal with compactification S_f of S_f^z, not with S_f^z. To find $|\hat{z}^{-1}(z)|$ for $z \in \{0, z_1, \dots, z_{2a}, \infty\}$, we need to count the number of circles in the preimage of a small circle around z under the map $\hat{z}$.

To compute $|\hat{z}^{-1}(z_i)|$ for a fixed $i \in \{1, \dots, 2a\}$, we will consider the circle $|z - z_i| = \varepsilon$ for a sufficiently small $\varepsilon > 0$. We can write $f(z,w) = 0$ as $(-2z^a)w^b + (1 + z^{2a}) = 0$ which gives

$$\begin{aligned} w^b &= \frac{1}{2} z^{-a} (1 + z^{2a}) \\ &= \frac{1}{2} z^{-a} \prod_{j=1}^{2a} (z - z_j) \ . \end{aligned}$$

We only care about the terms with $(z - z_i)$ in the right hand side, hence we can consider

$$w^b = (z - z_i)$$

instead. If we draw the solutions of this equation in w for the circle $|z - z_i| = \varepsilon$, we will get $\gcd(b, 1) = 1$ disjoint circle in the complex plane, where b is the power of w and 1 is the power of $(z - z_i)$ in the equation. Hence we get $|\hat{z}^{-1}(z_i)| = 1$.

To compute $|\hat{z}^{-1}(0)|$, we will consider the circle $|z| = \varepsilon$ for a sufficiently small $\varepsilon > 0$. We can write $f(z,w) = 0$ as $w^b = \frac{1}{2} z^{-a} \prod_{j=1}^{2a} (z - z_j)$ as shown above. We only care about the terms with z in right hand side, hence we can consider

$$w^b = z^{-a}$$

instead. If we draw the solutions of this equation in w for the circle $|z| = \varepsilon$, we will get $\gcd(b, -a) = 1$ disjoint circle in the complex plane, where b is the power of w and $-a$ is the power of z in the equation. Hence we get $|\hat{z}^{-1}(0)| = 1$.

To compute $|\hat{z}^{-1}(\infty)|$, we will consider the circle $|z| = N$ for a sufficiently large $N > 0$. We can write $f(z, w) = 0$ as $w^b = \frac{1}{2} z^{-a} \prod_{j=1}^{2a} (z - z_j)$ as shown above. Since z is very large, we have $z - z_j \approx z$ approximately for any $j \in \{1, \ldots, 2a\}$, hence we approximately get

$$w^b \approx \frac{1}{2} z^{-a} \prod_{j=1}^{2a} z = \frac{1}{2} z^a \ .$$

If we draw the solutions of this equation in w for the circle $|z| = N$, we will get $\gcd(b, a) = 1$ disjoint circle in the complex plane, where b is the power of w and a is the power of z in the equation. Hence we get $|\hat{z}^{-1}(\infty)| = 1$.

Now we have everything to compute the branching index $b_{\hat{z}}$. We have

$$b_{\hat{z}} = \sum_{z \in \{0, z_1, \ldots, z_{2a}, \infty\}} (\deg(\hat{z}) - |\hat{z}^{-1}(z)|) = (2a + 2)(b - 1) = 2ab - 2a + 2b - 2 \ .$$

Using this, we can compute the genus $g(S_f)$ of S_f by the Riemann-Hurwitz formula as follows:

$$\begin{aligned} 2g(S_f) - 2 &= \deg(\hat{z})(2g(\mathbb{P}^1) - 2) + b_{\hat{z}} \\ 2g(S_f) - 2 &= b(2 \times 0 - 2) + (2ab - 2a + 2b - 2) \\ g(S_f) &= ab - a \end{aligned}$$

where we assumed $b \neq 1$ in the beginning.

Lastly, if $b = 1$, then $\deg(\hat{z}) = b = 1$ and $\hat{z} : S_f \to \mathbb{P}^1$ is an isomorphism, hence $g(S_f) = g(\mathbb{P}^1) = 0 = ab - a$. This concludes the solution.

7. (i) Find a compact Riemann surface S associated to the curve $C \subseteq \mathbb{C}^2$ defined by the irreducible polynomial $f(x, y) = x^3 + y^3 - 1$. How many points are there in $S \setminus C$. Show that the projection onto x induces a map $S \to \mathbb{P}^1$ of degree 3. Using the Riemann-Hurwitz formula for this projection to show that the genus of S is 1.

   Solution: Firstly, we need to check if the projectivisation is singular. If not, then we have found the compact and connected Riemann surface associated to f. The projectivisation is given by $F(X, Y, Z) = X^3 + Y^3 - Z^3$, which is non-singular. The number of points in $S \setminus C$ will be the number of points on S such that $Z = 0$. This is given by $X^3 + Y^3 = 0$, or rather, $\left(\frac{X}{Y}\right)^3 = -1$. Since points in projective space are defined by a ratio, this gives three points. Therefore, there are three points in $S \setminus C$. Consider the projection $x : S_f^x \to \mathbb{C}$ which extends to a map $\hat{x} : S \to \mathbb{P}^1$.

$$\begin{aligned} S_f^x &= \{(x, y) \in \mathbb{C}^2 | \ f(x, y) = 0, \ f_y \neq 0\} \\ &= \{(x, y) \in \mathbb{C}^2 | \ f(x, y) = 0, y \neq 0\} \\ &= \{(x, y) \in \mathbb{C}^2 | \ f(x, y) = 0, \ x^3 - 1 \neq 0\}. \end{aligned}$$

   Let $\{x_1, x_2, x_3\}$ be the cubed roots of unity. Then this extends to a map $\hat{x}^{-1} : S \to \mathbb{P}^1$ such that branch$(\hat{x}) \subseteq \{x_1, x_2, x_3, \infty\}$. To calculate $|\hat{x}(x_i)|$, we must count the number of circles in the preimage of $\hat{x}$ of a small enough circle around x_i. Let $|x - x_i| = \varepsilon$. Then

$$y^3 = (x - x_1)(x - x_2)(x - x_3).$$

Since we only care about terms with $x - x_i$ in it (the other terms are approximately constant), we are left with

$$y^3 \approx (x - x_i)$$

for $i \in \{1, 2, 3\}$. There is $\gcd(3, 1) = 1$ solutions to this equation, and so each x_i is a branch point. To calculate $|\hat{x}^{-1}(\infty)|$, consider a sufficiently large circle $|x| = N > 0$. Since N is very large, we have that $x - x_i \approx x$, and so we get

$$y^3 \approx x^3,$$

which has $\gcd(3, 3) = 3$ solutions. Just as a reality check, this makes sense, since it corresponds to the number of points in $S \setminus C$. This is because the points which get mapped to ∞ in the projection by $\hat{x}$ will be those which lie on the curve at $Z = 0$. We now have that

$$b_{\hat{x}} = \sum_{x \in \{x_1, x_2, x_3, \infty\}} (\deg(\hat{x}) - |\hat{x}^{-1}(x)|) = 6.$$

Since $g(\mathbb{P}^1) = 0$, we have by the Riemann-Hurwitz formula that

$$\begin{aligned} 2g(S) - 2 &= \deg(\hat{x})(2g(\mathbb{P}^1) - 2) + b_{\hat{x}} \\ &= 3(0 - 2) + 6 = 2, \\ g(S) &= 1. \end{aligned}$$

as required.

(ii) Describe the compact connected Riemann surface S associated to the affine curve $C \subseteq \mathbb{C}^2$ defined by the irreducible polynomial $f(x, y) = x^2 - x^6 + 1$. How many points are there in $S \setminus C$? Show that the projection to x induces a map $S \to \mathbb{P}^1$ of degree 2. Use the Riemann-Hurwitz formula to show that the genus of S is 2.

Solution: The projectivisation of f is given by $F(X, Y, Z) = X^2Y^4 - X^6 + Z^6$, and this has a singularity at $[0 : 1 : 0]$, so we need to normalise the curve in order to find the compact connected Riemann surface defined by f. Let

$$\begin{aligned} S_f^x &= \{(x, y) \in \mathbb{C}^2 |\ f(x, y) = 0,\ f_y \neq 0\} \\ &= \{(x, y) \in \mathbb{C}^2 |\ f(x, y) = 0,\ y \neq 0\} \\ &= \{(x, y) \in \mathbb{C}^2 |\ f(x, y) = 0,\ x^6 \neq 1\}. \end{aligned}$$

f is a degree 2 polynomial in y, and so this map is of degree 2. It extends to a map $\hat{x} : S \to \mathbb{P}^1$ with branch points contained in the set $\{x_1, x_2, x_3, x_4, x_5, x_6, \infty\}$, where the x_i are the sixth roots of unity. To calculate $|\hat{x}^{-1}(x_i)|$, consider the circle of radius ε centred at x_i. $|\hat{x}^{-1}(x_i)|$ will be given by the number of disjoint circles in the preimage of $\hat{x}$ on this circle. Along the circle $|x - x_i| = \varepsilon$, we have

$$y^2 = \prod_{i=1}^{6} (x - x_i).$$

Since we only care about the terms on the right hand side containing $(x - x_i)$ (the others are approximately constant), we need to count solutions to

$$y^2 \approx x - x_i,$$

of which there will be $\gcd(2,1) = 1$. To compute $|\hat{x}^{-1}(\infty)|$, consider a circle $|x| = N > 0$ large enough. On this circle, we have that $x - x_i \approx x$, and so we must count the solutions to the equation

$$y^2 \approx x^6,$$

of which there will be $\gcd(2,6) = 2$. Therefore, we have that

$$b_{\hat{x}} = \sum_{x \in \{x_1, x_2, x_3, x_4, x_5, x_6, \infty\}} (\deg(\hat{x}) - |\hat{x}^{-1}(x)|) = 6$$

Since $g(\mathbb{P}^1) = 0$, we have by the Riemann-Hurwitz formula that

$$\begin{aligned} 2g(S) - 2 &= \deg(\hat{x})(2g(\mathbb{P}^1) - 2) + b_{\hat{x}} \\ &= 2(0-2) + 6 = 2, \\ g(S) &= 2. \end{aligned}$$

8. Let $a_1, a_2, \dots, a_r$ be distinct points in $\mathbb{C}$ and p a prime number. Let S be the compact connected Riemann surface associated to the curve

$$f(z,w) = z^p - (w-a_1)^{m_1}(w-a_2)^{m_2}\dots(w-a_r)^{m_r}$$

$1 \le m_i < p$.

Compute the branching points of the projection to w. How many points are there in $S \setminus S_f^w$? Use Riemann-Hurwitz formula to show that

$$g = \begin{cases} \frac{(p-1)(r-1)}{2} & \text{if } \gcd(\sum m_i, p) = 1 \\ \frac{(p-1)(r-2)}{2} & \text{otherwise} \end{cases}$$

Solution: Let $S = S_f$ be the compact Riemann surface associated to the irreducible polynomial $f(z,w) = z^p - (w-a_1)^{m_1}(w-a_2)^{m_2}\dots(w-a_r)^{m_r}$. We have the map $\hat{w} : S_f \to \mathbb{P}^1$, which is the extension of the projection map $w \colon S_f^w \to \mathbb{P}^1$, where S_f^w is given by

$$\begin{aligned} S_f^w &= \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0, f_z(z,w) \neq 0, \text{the coefficient of highest power of } z \text{ in } f \neq 0\} \\ &= \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0, pz^{p-1} \neq 0\} \\ &= \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0, z \neq 0\} \\ &= \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0, w \neq a_i \text{ for } i = 1, \dots, r\} \ . \end{aligned}$$

Remember for the branching values of $\hat{w}$ (i.e. $\mathrm{branch}(\hat{w})$), we have $\mathrm{branch}(\hat{w}) \subset \hat{w}(S_f \setminus S_f^w)$. It is clear that $\hat{w}(S_f \setminus S_f^w) = \{a_1, \dots, a_r, \infty\}$, hence we have

$$\mathrm{branch}(\hat{w}) \subset \{a_1, \dots, a_r, \infty\} \ .$$

To find the number of points in $S_f \setminus S_f^w$, observe that $|S_f \setminus S_f^w| = \sum_{w \in \{a_1, \dots, a_r, \infty\}} |\hat{w}^{-1}(w)|$. To compute $|\hat{w}^{-1}(a_i)|$ for a fixed $i \in \{1, \dots, r\}$, we will consider the circle $|w - a_i| = \varepsilon$ for a sufficiently small $\varepsilon > 0$. We can write $f(z,w) = 0$ as $z^p - (w-a_1)^{m_1}(w-a_2)^{m_2}\dots(w-a_r)^{m_r} = 0$ which gives

$$z^p = (w-a_1)^{m_1}(w-a_2)^{m_2}\dots(w-a_r)^{m_r} \ .$$

We only care about the terms with $(w - a_i)$ in the right hand side, hence we can consider

$$z^p = (w-a_i)^{m_i}$$

instead. If we draw the solutions of this equation in z for the circle $|w-a_i| = \varepsilon$, we will get $\gcd(p, m_i) = 1$ disjoint circle in the complex plane, where p is the power of z and m_i is the power of $(w - a_i)$ in the equation. Hence we get $|\hat{w}^{-1}(a_i)| = 1$.

To compute $|\hat{w}^{-1}(\infty)|$, we will consider the circle $|w| = N$ for a sufficiently large $N > 0$. We can write $f(z, w) = 0$ as $z^p = (w - a_1)^{m_1}(w - a_2)^{m_2} \dots (w - a_r)^{m_r}$ as shown above. Since w is very large, we have $w - a_i \approx w$ approximately for any $i \in \{1, \dots, r\}$, hence we approximately get

$$z^p \approx w^{\sum m_i} .$$

If we draw the solutions of this equation in z for the circle $|w| = N$, we will get $\gcd(p, \sum m_i)$ disjoint circle in the complex plane, where p is the power of z and $\sum m_i$ is the power of w in the equation. Hence we get $|\hat{w}^{-1}(\infty)| = \gcd(p, \sum m_i)$. Therefore, we get

$$|S_f \setminus S_f^w| = \sum_{w \in \{a_1, \dots, a_r, \infty\}} |\hat{w}^{-1}(w)| = r \times 1 + \gcd(p, \sum m_i) .$$

If $\gcd(p, \sum m_i) \neq 1$, then $\gcd(p, \sum m_i) = p$ since p is prime. Hence

$$|S_f \setminus S_f^w| = \begin{cases} r + 1 & \text{if } \gcd(p, \sum m_i) = 1 \\ r + p & \text{otherwise} \end{cases} .$$

To find the genus of S_f (i.e. $g(S_f)$), we will use the Riemann-Hurwitz formula for $\hat{w} \colon S_f \to \mathbb{P}^1$, which is

$$2g(S_f) - 2 = \deg(\hat{w})(2g(\mathbb{P}^1) - 2) + b_{\hat{w}}$$

where $b_{\hat{w}}$ is the branching index of $\hat{w}$. We know $g(\mathbb{P}^1) = 0$, and $\deg(\hat{w})$ is just the degree of f seen as a polynomial in z. Hence $\deg(\hat{w}) = p$. Also, we have

$$\begin{aligned} b_{\hat{w}} &= \sum_{w \in \text{branch}(\hat{w})} (\deg(\hat{w}) - |\hat{w}^{-1}(w)|) \\ &= \sum_{w \in \{a_1, \dots, a_r, \infty\}} (\deg(\hat{w}) - |\hat{w}^{-1}(w)|) \\ &= r(p-1) + (p - \gcd(p, \sum m_i)) \\ &= \begin{cases} (r+1)(p-1) & \text{if } \gcd(p, \sum m_i) = 1 \\ r(p-1) & \text{otherwise} \end{cases} . \end{aligned}$$

Finally, we get

$$\begin{aligned} 2g(S_f) - 2 &= \deg(\hat{w})(2g(\mathbb{P}^1) - 2) + b_{\hat{w}} \\ 2g(S_f) - 2 &= \begin{cases} p(2 \times 0 - 2) + (r+1)(p-1) & \text{if } \gcd(p, \sum m_i) = 1 \\ p(2 \times 0 - 2) + r(p-1) & \text{otherwise} \end{cases} \\ g(S_f) &= \begin{cases} \frac{(r-1)(p-1)}{2} & \text{if } \gcd(p, \sum m_i) = 1 \\ \frac{(r-2)(p-1)}{2} & \text{otherwise} \end{cases} . \end{aligned}$$