Curves - Homework 3

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1) Given a lattice $\Lambda \subset \mathbb{C}$. Let $C_\Lambda \subset \mathbb{P}^2$ be the projective curve defined by

$$Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3 = 0$$

Show that there is a well-defined map $\mathbb{C}/\Lambda \to C_\Lambda$ given by

$$[z + \Lambda] \mapsto \begin{cases} [\wp(z), \wp'(z), 1] \text{ if } z \notin \Lambda \\ [0, 1, 0] \text{ if } z \in \Lambda \end{cases}$$

which is an isomorphism of Riemann surfaces.

2) Given a curve C_Λ by the equation

$$Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3 = 0$$

we define

$$J(\Lambda) = \frac{g_2^3}{g_2^3 - 27g_3^2}$$

(i) Show that $g_2^3 - 27g_3^2 \neq 0$ so that $J(\Lambda)$ is well-defined.

(ii) Given two lattice $\Lambda, \tilde{\Lambda} \subset \mathbb{C}$. Show that if $J(\Lambda) = J(\tilde{\Lambda})$ then the corresponding projective curves

$$Y^2Z - 4X^3 + g_2(\Lambda)XZ^2 + g_3(\Lambda)Z^3 = 0 \text{ and } Y^2Z - 4X^3 + g_2(\tilde{\Lambda})XZ^2 + g_3(\tilde{\Lambda})Z^3 = 0$$

are projectively equivalent in $\mathbb{P}^2$.

(iii) Given two lattices Λ, $\tilde{\Lambda} \subset \mathbb{C}$. Show that the following are equivalent:

- $\mathbb{C}/\Lambda$ is biholomorphic to $\mathbb{C}/\tilde{\Lambda}$.
- $\Lambda = c\tilde{\Lambda}$ for some $c \in \mathbb{C}^\times$.
- $J(\Lambda) = J(\tilde{\Lambda})$.

3) Compute the Euler characteristic of a compact orientable surface of genus g.

4) Consider the curve $C = \{[X,Y,Z] \in \mathbb{P}^2 : Z^2Y^2 = X^4 + Y^4 + Z^4\}$. Show that this curve is smooth. Consider the holomorphic map $f : C \to \mathbb{P}^1$ given by projection given by

$$[X : Y : Z] \to [X : Y].$$

Check that f gives a well-defined holomorphic map when restricted to C. What is the degree of f? What are the branch points of f? Compute the branching index b_f and use this together with the Riemann-Hurwitz formula to compute the genus of C_1.

4) Show that a proper, local homeomorphism between Riemann surfaces (or more generally between any topological manifolds) is a covering map. Give an example of a local homeomorphism that is not a covering map.

5) Let S be the compact Riemann surface associated to the equation $z^{2a} - 2w^b z^a + 1 = 0$, for fixed positive integers a, b with $(a, b) = 1$. Identify the branch points of the covering of the Riemann sphere defined by the projection to the z co-ordinate, and hence show that the genus of S is $ab - a$.

6) (i) Find the compact connected Riemann surface S associated to the affine curve $C \subset \mathbb{C}^2$ defined by the irreducible polynomial $f(x,y) = x^3 + y^3 - 1$ (Hint: Consider the projectivization and check that its smooth). How many points are there in $S \backslash C$? Show that projection x induces a map $S \to \mathbb{P}^1$ of degree 3. Using Riemann-Hurwitz formula for this projection show that the genus of S is 1.

(ii) Find the compact connected Riemann surface S associated to the affine curve $C \subset \mathbb{C}^2$ defined by the irreducible polynomial $f(x,y) = y^2 - x^6 + 1$. (Hint: Projectivization in $\mathbb{P}^2$ gives a singular curve. Find S by studying the map $C \to \mathbb{C}$ given by projection to x.) How many points are there in $S \setminus C$? Show that the projection to x induces a map $S \to \mathbb{P}^1$ of degree 2. Using the Riemann-Hurwitz formula for this projection show that the genus of S is 2.

7) Let $a_1, a_2, \ldots, a_r$ be distinct points in $\mathbb{C}$ and p a prime number. Let S be the compact connected Riemann surface associated to the curve

$$f(z,w) = z^p - (w - a_1)^{m_1}(w - a_2)^{m_2} \ldots (w - a_r)^{m_r}$$

$1 \leq m_i < p$.

Compute the branching points of the projection to w. How many points are there in $S \setminus S_f^w$? Use Riemann-Hurwitz formula show that

$$g = \begin{cases} \frac{(p-1)(r-1)}{2} & \text{if } \ gcd(\sum m_i, p) = 1 \\ \frac{(p-1)(r-2)}{2} & \text{otherwise} \end{cases}$$