Curves - Homework 2

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1) Show that the projective curve $X^2 + Y^2 = Z^2$ is non-singular and in fact isomorphic to $\mathbb{P}^1$ as a Riemann surface.

2) For which values of $\lambda \in \mathbb{C}$, are the curves defined by

$$X^3 + Y^3 + Z^3 + \lambda XYZ = 0$$

non-singular?

3) Consider the affine curve C defined by

$$y^2 = (x - a_1)(x - a_2)\dots(x - a_{2g+1})$$

where $a_1, \dots, a_{2g+1}$ are distinct points. Show that the affine curve C is non-singular. Show that the projectivization $\overline{C}$ intersects the line at infinity at a unique point. Show that for $g > 1$, this point is singular.

4) Let A be a matrix in $GL(3, \mathbb{C})$. Show that A induces a map from $\mathbb{P}^2$ to itself. Let F be a homogeneous polynomial defining a projective curve $C \subset \mathbb{P}^2$. Show that A sends C to another projective curve $A(C) \subset \mathbb{P}^2$ defined by another polynomial F_A. Show that if C is non-singular, so is $A(C)$ and the map $A : C \to A(C)$ gives an isomorphism of Riemann surfaces.

5) (i) If a line with slope t intersects the circle $x^2 + y^2 = 1$ in the points $(-1, 0)$ and (x, y), show that x and y are both rational functions of t. (A rational function is one that can be written as the quotient of two polynomials.) By taking $t = p/q$ to be a rational number, construct the general solution of the equation $x^2 + y^2 = z^2$ for which x, y, z are coprime integers.
(ii) Given any smooth conic in $\mathbb{P}^2$ defined over rational numbers, i.e a smooth curve defined by an equation of the form

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0, \quad a, b, c, d, e, f \in \mathbb{Q}$$

find a change of co-ordinates (as in Exercise 4) over $\mathbb{Q}$ so that the resulting curve is defined by

$$\alpha x^2 + \beta y^2 = z^2, \quad \alpha, \beta \in \mathbb{Q}$$

6) (i) Show that if $f : S \to T$ is a non-constant, proper holomorphic map between connected Riemann surfaces then $f^{-1}(t)$ is a finite set for all $t \in T$.
(ii) Show that a non-constant holomorphic function $f : S \to T$ between connected Riemann surfaces is an open mapping i.e. it sends open sets to open sets.

7) View a non-constant polynomial $f(z) \in \mathbb{C}[z]$ as a holomorphic function from $f : \mathbb{P}^1 \to \mathbb{P}^1$ by sending ∞ to ∞. Show directly that $v_f(\infty)$ is the same as the degree of the polynomial. As a corollary, give a proof fundamental theorem of algebra.

8) Another proof of the fact $K(\mathbb{P}^1)$ is isomorphic to $\mathbb{C}(z)$ can be given as follows. Given $f \in K(\mathbb{P}^1)$, let z_i for $i = 1, \ldots, n$ be the zeros of f and p_j for $j = 1, \ldots m$ be the poles of f (repeated as necessary). Consider the function:

$$g(z) = \frac{\Pi_{i=1}^n (z - z_i)}{\Pi_{j=1}^m (z - p_j)}$$

Show that $f(z)/g(z)$ is a meromorphic function with no zeroes and poles. Hence, it is constant.

9) Recall that if $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$ with $\mathrm{Im}\tau > 0$, is a lattice, we have a Riemann surface structure on $\mathbb{C}/\Lambda$. Show that $K(\mathbb{C}/\Lambda)$ is isomorphic to the field of doubly periodic meromorphic functions on $\mathbb{C}$ with period $(1, \tau)$.

10) Let e_1, e_2, e_3 be the values of $\wp(1/2)$, $\wp(\tau/2)$, $\wp((1+\tau)/2)$ respectively. (i) Show that e_1, e_2, e_3 are all distinct. (ii) Show that for any $a \in \mathbb{C} \setminus \{e_1, e_2, e_3\}$, the equation $\wp(z) = a$ has exactly two distinct solutions.