Curves - Homework 1

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0) Read Chapter 1 of Kirwan's book.

1) i) Show that $\mathbb{H}$ and $\mathbb{D}$ are isomorphic Riemann surfaces via the map:

$$\mathbb{H} \to \mathbb{D}$$
$$z \mapsto \frac{z-i}{z+i}$$

ii) Show that $\mathbb{C}$ and $\mathbb{D}$ are not isomorphic Riemann surfaces, by using Liouville's theorem (a *bounded* entire function is necessarily constant).

2) Show that the projective line

$$\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times,$$

the space of lines through origin in $\mathbb{C}^2$ is a Riemann surface which is isomorphic to the Riemann sphere.

3) (i) Show that $\mathbb{C}/\Lambda$ is a Riemann surface for any lattice Λ.
(ii) Show that any such Riemann surface $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$.

4) Show that the subset of $\mathbb{C}^2$ consisting of points of the form

$$(t^2, t^3 + 1), \quad t \in \mathbb{C}$$

is a complex algebraic curve.

5) The algebraic curve defined by $f(z, w) = w^2 - p(z)$ where $p \in \mathbb{C}[z]$ a polynomial, is smooth if and only if $p(z)$ has no multiple roots. The smooths ones are called *hyperelliptic* curves if the degree of $p(z)$ is greater than 4.

6) Show that a smooth connected curve is irreducible (Hint. Show that the points at which two or more components of a curve intersect are singular.)

7) i) Show that the polynomial $f(z, w) = z^p w + w^q z - 1$ for $p, q \geq 1$ is irreducible.

ii) Show that the polynomial $f(z, w) = z^p + w^q$ for $p, q \geq 1$ is irreducible if and only if $gcd(p, q) = 1$.

iii) Show that all the polynomials $f(z,w) = a + bz + cz^2 + dzw$ is irreducible for any non-zero $a, b, c, d \in \mathbb{C}$[1].

iv) Show that the Newton polytope of $f(z,w) = z^6 + w^6 + 1$ is integrally decomposable, but f is irreducible over $\mathbb{C}$. Can you find a field over which f is reducible?

Computing project: There doesn't seem to be a way to decompose a polynomial $f \in \mathbb{C}[z,w]$ into irreducibles using Sage. I found a paper "Shugong Gao - Factoring multivariate polynomials via partial differential equations Math. Comp. 72 (2003), no. 242, 801–822" that gives an efficient algorithm for this purpose (see also "Hani Shaker-Topology and factorization of polynomials. Math. Scand. 104 (2009), no. 1, 51–5"). If you would like to implement this algorithm in Sage as a project, come talk to me and let's see if we can do it.

8) Examine the tangent directions at $(0,0)$ of the curves: $y^2 = x^3 + x^2$, $y^2 = x^3$, $(x^4+y^4)^2 = x^2y^2$, $(x^4 + y^4 - x^2 - y^2)^2 = 9x^2y^2$. What are the multiplicities at $(0,0)$? Which singularities are ordinary?

9) Using a modification of the map $t \mapsto (\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1})$, show directly that the curve C defined by $f(z,w) = z^2 + w^2 - 1$ is isomorphic to $\mathbb{C}^{\times}$.

10) Given a non-constant polynomial $f(z,w)$ that is not independent of w, show that for all but finitely many values of z, there exists $w \in \mathbb{C}$, such that $f(z,w) = 0$. Deduce that algebraic curves in $\mathbb{C}^2$ are non-compact.

[1] For other interesting examples see the paper Shugong Gao- Absolute irreducibility of polynomials via Newton Polytopes J. Algebra 237 (2001), no. 2, 501–520.