RIEMANN SURFACES

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1. Motivation

Riemann surfaces are *domains* of most general type which can be used to replace the complex plane in studying holomorphic functions of one complex variable.

To illustrate this, let us consider the problem of defining the square-root function $\sqrt{z}$ for $z \in \mathbb{C}$. In other words, we would like to solve the equation

$$w^2 = z.$$

There is no single function $z \mapsto w(z)$ that is defined in the whole complex plane $\mathbb{C}$. However, if we cut along the axis $[0, \infty)$, we can define two holomorphic functions $\pm\sqrt{z}$ defined on $\mathbb{C} \setminus [0, \infty)$ by

$$z = re^{i\theta} \mapsto +\sqrt{z} = \sqrt{r}e^{i\frac{\theta}{2}}$$
$$z = re^{i\theta} \mapsto -\sqrt{z} = -\sqrt{r}e^{i\frac{\theta}{2}}$$

Now, observe that as $z = re^{i\theta}$ goes to r as θ goes to 0, we get that $+\sqrt{z} \to +\sqrt{r}$ and $-\sqrt{z} \to -\sqrt{r}$, whereas if we let θ go to 2π, then $+\sqrt{z}$ goes to $-\sqrt{r}$, and $-\sqrt{z}$ goes to $+\sqrt{r}$.

There are two issues here. First, the function $\sqrt{z}$ is not defined on all of $\mathbb{C}$, and second if we remove $[0, \infty)$, there are two values of $\sqrt{z}$, i.e. it is a multi-valued function.

Let us write D for the bordification of the domain $\mathbb{C} \setminus [0, \infty)$ where we regard distinct sides of the cut as distinct edges of D. We can extend $\pm\sqrt{z}$ to D by continuity. To distinguish the two extension, we write $D_\pm$ for the domain of the extension of $\pm\sqrt{z}$.

Now, we construct a "Riemann surface" S by gluing D_+ and D_- in a way that the upper side of the cut in D_+ gets identified with lower side of the cut in D_-, and the lower side of the cut in D_+ gets identified with the uppser side of the cut in D_-. Let $\phi : S \to \mathbb{C}$ be the natural map taking points in $D_\pm$ to the "same" points in $\mathbb{C}$. The functions $\pm\sqrt{\phi(z)}$ on $D_\pm$ taken together give a single-valued function $w : S \to \mathbb{C}$ such that

$$w(\phi(z))^2 = \phi(z)$$

Thus, the basic idea of Riemann surface theory is to construct a domain S on which the branches of the multi-valued functions on $\mathbb{C}$ fit together to define a single-valued function.

Similarly, we can associate a "domain" to any multi-valued holomorphic function $w(z)$ satisfying

$$P(z, w(z)) = 0$$

where $P(z,w)$ is an irreducible polynomial. The resulting Riemann surface is an "algebraic curve"

$$C = \{(z,w) \in \mathbb{C}^2 : P(z,w) = 0, (\partial_w P)(z,w) \neq 0\}$$

The points where $(\partial_w P)(z,w) = 0$ are called ramification points. They could be filled in, and the function $w : C \to \mathbb{C}$ can be extended but its derivative will vanish at these points, so these are the "true" singularities of the original multi-valued function.

2. Basic definitions

Definition 1. *A topological surface is a Hausdorff topological space S which is locally homeomorphic to $\mathbb{C}$.*

Here, locally homeomorphic means that and $p \in S$ has an open neighborhood U in S which is homeomorphic to an open subset V of $\mathbb{C}$.

Definition 2. *A Riemann surface is given by the following:*

- *A Hausdorff topological space S*
- *A collection of open sets $U_\alpha \subset S$ such that $\bigcup_\alpha U_\alpha = S$,*
- *For each α we have a homeomorphism $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) = V_\alpha \subset \mathbb{C}$ such that whenever $U_\alpha \circ U_\beta \neq 0$, the map*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

  is **holomorphic***.*

We recall that a function $f : U \to V$ with $U, V \subset \mathbb{C}$ is holomorphic if the limit

$$f'(z_0) = \lim_{h \in \mathbb{C}^*, h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exits for all $z_0 \in U$. Writing this in real coordinates, $z = x + iy$, $f(z) = u(x,y) + iv(x,y)$, this condition is equivalent to u and v being continuously differentiable satisfying the Cauchy-Riemann equations:

$$\partial_x u = \partial_y v$$
$$\partial_x v = -\partial_y u$$

and it is also equivalent to the existence of a power series expansion, for each $z_0 \in U$:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

absolutely convergent in an open set (and uniformly convergent on every compact).

A single (ϕ_α, U_α) are referred to as a "chart" or a "local co-ordinate", the maps $\phi_\beta \circ \phi_\alpha^{-1}$ relating different local co-ordinates are called transition functions, the whole collection of charts $\{(\phi_\alpha, U_\alpha)\}$ is called an "atlas".

Informally, one can think of a Riemann surface as open subsets of $\mathbb{C}$ "glued together" via holomorphic maps.

Definition 3. *A map $f : S \to T$ between Riemann surfaces is called holomorphic if for every choice of co-ordinate ϕ in S and ψ is T, the composition*

$$\psi \circ f \circ \phi^{-1}$$

is a holomorphic function on its domain of definition.

In particular, a function $f : S \to \mathbb{C}$ is holomorphic if $f \circ \phi^{-1}$ is holomorphic wherever it is defined.

We shall identify two Riemann surfaces whenever there is a holomorphic bijection $f : S \to T$. This allows us to remove the dependence of the definition of a Riemann surface on a particular atlas.

2.1. **First Examples.**

- Any open subset of $\mathbb{C}$ is a Riemann surface by using a single chart. Some important examples of this are the whole complex plane $\mathbb{C}$, the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and the upper halfplane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$.
  Exercise: i) Show that $\mathbb{H}$ and $\mathbb{D}$ are isomorphic Riemann surfaces via the map:
  $$\mathbb{H} \to \mathbb{D}$$
  $$z \mapsto \frac{z-i}{z+i}$$
  ii) Show that $\mathbb{C}$ and $\mathbb{D}$ are not isomorphic Riemann surfaces, by using Liouville's theorem (a *bounded* entire function is necessarily constant).
- The Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is topologically given as a one-point compactification of $\mathbb{C}$ - a collection of basic open sets of ∞ is given by the family of sets $B(\infty, r) = \{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}$. Next, the following two charts makes S^2 into a Riemann surface:
  $$U_0 = \mathbb{C}, \quad \phi_0(z) = z$$
  $$U_1 = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}, \quad \phi_1(z) = \begin{cases} 1/z & \text{if } z \neq \infty \\ 0 & \text{if } z = \infty \end{cases}$$
  Exercise: Show that the projective line
  $$\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times,$$
  the space of lines through origin in $\mathbb{C}^2$ is a Riemann surface which is isomorphic to the Riemann sphere.
- The cylinder, $\mathbb{C}/\mathbb{Z}$, is the quotient space of $\mathbb{C}$ where two points are identified if they differ by an integer. Consider the two open sets $V_0 = (0,1) \times \mathbb{R} \subset \mathbb{C}$ and $V_{1/2} = (1/2, 3/2) \times \mathbb{R} \subset \mathbb{C}$. When restricted to V_0 and $V_{1/2}$, the projection map $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ is a homeomorphism, hence, $\mathbb{C}/\mathbb{Z}$ is a topological surface and we can define charts by letting $U_0 = \pi(V_0)$, $\phi_0 = \pi^{-1}_{|V_0}$ and $U_{1/2} = \pi(V_{1/2})$, $\phi_{1/2} = \pi^{-1}_{|V_{1/2}}$. Furthermore, the transition function
  $$\phi_{1/2} \circ \phi_0^{-1} : ((0,1/2) \cup (1/2,1)) \times \mathbb{R} \to ((1/2,1) \cup (1,3/2)) \times \mathbb{R}$$

given by

$$z \mapsto z+1, \text{ if } \mathrm{Re} z \in (0, 1/2)$$
$$z \mapsto z, \text{ if } \mathrm{Re} z \in (1/2, 3/2)$$

is a holomorphic function. Similarly, one can check that the transition function $\phi_0 \circ \phi_{1/2}^{-1}$ is a holomorphic function.

Exercise: Show that $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ are isomorphic Riemann surfaces.

- The complex torus associated to a lattice Λ is the quotient space $\mathbb{C}/\Lambda$, where Λ is a discrete subgroups of $\mathbb{C}$ generated by two non-zero complex numbers ω_1 and ω_2 that are linearly independent over $\mathbb{R}$, i.e. $\omega_1/\omega_2 \notin \mathbb{R}$. One can show that these are Riemann surfaces via an argument similar to the one given for the cylinder.

  Exercise: (i) Show that $\mathbb{C}/\Lambda$ is a Riemann surface for any lattice Λ. (ii) Show that any such Riemann surface $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$.
- Hyperbolic surfaces. Recall the group $SL_2(\mathbb{R})$ is given by

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

  $SL_2(\mathbb{R})$ acts on the upper half-plane $\mathbb{H}$ by holomorphic isomorphisms via

$$z \mapsto \frac{az+b}{cz+d}$$

  This action actually descends to an action of $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$.

  Now, let $\Gamma \subset SL_2(\mathbb{R})$ be a torsion-free discrete subgroup of $SL_2(\mathbb{R})$, then $\mathbb{H}/\Gamma$ is a Riemann surface.

  Exercise: Consider the subgroup of $SL_2(\mathbb{R})$ isomorphic to $\mathbb{Z}$ given by

$$\left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, n \in \mathbb{Z} \right\}$$

  Show that $\mathbb{H}/\mathbb{Z}$ is isomorphic to $\mathbb{D}^\times = \mathbb{D} \setminus \{0\}$ as a Riemann surface.

  More generally, for $\lambda > 1$, let $\lambda^{\mathbb{Z}}$ be the subgroup of $\mathbb{H}$ given by

$$\left\{ \begin{bmatrix} 1 & n\lambda \\ 0 & 1 \end{bmatrix}, n \in \mathbb{Z} \right\}$$

  Then $\mathbb{H}/\lambda^{\mathbb{Z}}$ is isomorphic to the annulus $A(r) = \{z : r < |z| < 1\}$ with

$$r = \exp(-2\pi^2/\log\lambda).$$

In fact, above we have listed all the Riemann surfaces.

Theorem 4. *Every Riemann surface S is isomorphic to one of*

$$\mathbb{C}, S^2, \mathbb{C}/\mathbb{Z}, \mathbb{C}/\Lambda, \mathbb{H}/\Gamma$$

where $\Lambda \subset \mathbb{C}$ is some lattice, and $\Gamma \subset SL_2(\mathbb{R})$ is a torsion-free discrete subgroup.

This is a deep theorem called the uniformization theorem for Riemann surfaces with a long history. The first proofs are due to Poincaré and Koebe around 1907. There are also several modern proofs. We will not cover any of the proofs.

3. Algebraic Curves

3.1. **Affine curves.** A large class of examples of Riemann surfaces are obtained via polynomials in two variables.

Definition 5. *An affine algebraic curve in* $\mathbb{C}^2$ *is a subset of the form*

$$C = \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0\}$$

for some polynomial $f(z, w)$ *of two complex variables without any repeated factors.*

Here, we say that $f(z, w)$ has no repeated factors (or is square-free, or reduced), if we cannot write it as

$$f(z, w) = h(z, w)^2 g(z, w)$$

for some polynomial $g(z, w)$ and some non-constant polynomial $h(z, w)$.

The reason we restrict to this class of polynomials is because a polynomial $f(z, w)$ with no repeated factors is determined up to a non-zero scalar by the curve C that it defines. To prove this, we need to recall some elementary algebra.

Recall that a ring R is a *unique factorization domain*, UFD, if every non-zero element of R can be factored uniquely, up to units and the ordering of the factors, into irreducible ones. Some examples of UFDs are $R = \mathbb{Z}$ or R is a field. A non-example is the ring $R = \mathbb{C}[x, y, z]/(z^2 - xy)$. The element z^2 has two essentially different decompositions into irreducible elements $z^2 = z \cdot z = x \cdot y$.

Lemma 6. *The polynomial ring* $\mathbb{K}[X]$ *over a field* $\mathbb{K}$ *is a UFD.*

Proof. This is proven just as one proves $\mathbb{Z}$ is a UFD, namely using the fact that both of these rings are Euclidean i.e there is a division with an Euclidean algorithm. In other words, given any $f, g \in \mathbb{K}[X]$ we can find two polynomials q,r that satisfy

$$f = qg + r, \quad deg(r) \leq deg(g).$$

□

The following is a strengthening of the above by Gauss.

Proposition 7. *If* R *is a UFD, then* $R[X]$ *is a UFD.*

Proof. The crucial ingredient is the Gauss Lemma which is easy to check directly: If R is a UFD, then the product of two primitive polynomials (ones with relatively prime coefficients) is again primitive. This implies that any irreducible element $f \in R[X]$ remains irreducible in $K[X]$ where K is the quotient field of R, and irreducible elements in $R[X]$ are irreducible elements in $K[X]$ which are primitive in $R[X]$.

In order to decompose a polynomial in $f \in R[X]$, first divide by greatest common divisor of all of its coefficients. So, $f = cf_1$ where f_1 is a primitive polynomial. Now decompose c using the fact R is a UFD, and decompose f_1 in $K[X]$ using the fact that $K[X]$ is a UFD (which we know by Lemma 6). Use Gauss Lemma to deduce that the resulting decomposition of f is given by irreducibles in $R[X]$. □

Corollary 8. *For any field* $\mathbb{K}$*, the polynomial ring* $K[X_1, \ldots, X_n]$ *is a UFD.*

A field $\mathbb{K}$ is algebraically closed if any non-constant polynomial $f \in \mathbb{K}[X]$ has a root. It follows that f can factored as:

$$f(x) = c \prod (x - r_i)^{e_i}, \quad , c, r_i \in \mathbb{K}$$

where r_i are distinct roots of f. A polynomial of degree d has d roots counted with multiplicity. Some examples of algebraically closed fields are $\mathbb{C}$ and $\overline{\mathbb{Q}}$.

Proposition 9. *Let $\mathbb{K}$ be an algebraically closed field and $f(z,w), g(z,w) \in \mathbb{K}[z,w]$ be two polynomials.*

(i) (Weak form of Bezout's theorem) If f and g are relatively prime, then the curves $f(z,w) = 0$ and $g(z,w) = 0$ intersect only at finitely many points.

(ii) (Weak form of Hilbert's Nullstellensatz) If f is irreducible and g vanishes at all points of the curve $f(z,w) = 0$ then f divides g.

Proof. (i) We may regard the polynomials f, g as elements of $\mathbb{K}[z][w]$. By Gauss lemma, f and g are relatively prime in $\mathbb{K}(z)[w]$. Applying Euclid's algorithm in $\mathbb{K}(z)[w]$, we can find polynomials $a, b \in \mathbb{K}(z)[w]$, such that

$$af + bg = 1$$

in $\mathbb{K}(z)[w]$. Clearing out the denominators, by multiplying with a polynomial $q \in \mathbb{K}[z]$, we get that

$$(aq)(z,w)f(z,w) + (bq)(z,w)g(z,w) = q(z)$$

in $\mathbb{K}[z,w]$. Now, if f and g vanish along (z_i, w_i) for $i = 1, \ldots \infty$, we get that $q(z_i) = 0$ for $i = 1, \ldots, \infty$. But, q is a polynomial hence has only finitely many roots. Thus, the set $\{z_i\}_{i=1}^{\infty}$ for $i = 1, \ldots \infty$ is finite. Applying the same argument with the role of z and w exchanged, gives that the set $\{w_i\}_{i=1}^{\infty}$ must be finite.

(ii) By (i), f and g cannot be relatively prime, but since f is irreducible, it must be that f divides g.

□

Corollary 10. *If $f(z,w)$ and $g(z,w)$ are polynomials with no repeated factors such that*

$$\{(z,w) : f(z,w) = 0\} = \{(z,w) : g(z,w) = 0\}$$

then there exists a $\lambda \in \mathbb{C}^{\times}$ such that

$$f(z,w) = \lambda g(z,w)$$

Proof. It suffices to show that both f and g have the same irreducible factors. Suppose h is an irreducible factor f. Then g vanishes at all points of the curve $h(z,w) = 0$. By Prop. 9 (ii), it follows that h divides g. □

In view of the above result, we can define properties of a curve C via its defining square-free polynomial. Note that given a polynomial $f(z,w)$, by throwing away extra factors, you can always find a reduced polynomial $f_{red}(z,w)$ such that

$$\{(z,w) \in \mathbb{C}^2 : f(z,w) = 0\} = \{(z,w) \in \mathbb{C}^2 : f_{red}(z,w) = 0\}$$

Throughout, when we study algebraic curves associated with polynomials, we will almost always implicitly assume that the polynomials are reduced.

Definition 11. *The degree d of a curve C is the degree of its defining polynomial f, i.e. $d = max(r+s, c_{r,s} \neq 0)$ where $f(z,w) = \sum_{r,s} c_{r,s} z^r w^s$.*

Definition 12. *Let C be an algebraic curve defined by a polynomial $f(z,w)$. A point (z_0, w_0) is called singular point of S if*

$$f(z_0, w_0) = f_z(z_0, w_0) = f_w(z_0, w_0) = 0.$$

We say that C is non-singular or smooth if C has no singular points.

Exercise. i) The algebraic curve defined by $f(z,w) = w^2 - p(z)$ where $p \in \mathbb{C}[z]$ a polynomial, is smooth if and only if $p(z)$ has no multiple roots. ii) Show that the curve $f(z,w) = w^2 + wz - z^3$ is not smooth (this singulariy is called a *node*).

We will mostly avoid studying curves with singularities in this class even though this is in itself a very interesting topic. Let us nonetheless give the following definition that captures first order information about singularities of algebraic curves:

Definition 13. *The multiplicity of the curve C defined by $f(z,w)$ at a point $(z_0, w_0) \in C$ is the smallest positive integer m such that*

$$\frac{\partial^m f}{\partial z^i \partial w^j}(z_0, w_0) \neq 0$$

for some $i, j \geq 0$ with $i + j = m$.

In other words, the Taylor expansion of f at (z_0, w_0) looks like

$$f(z,w) = \sum_{k \geq m} \sum_{i+j=k} \frac{\partial^k f}{\partial z^i \partial w^j}(z_0, w_0) \frac{(z-z_0)^i (w-w_0)^j}{i!j!}$$

and the multiplicity m at (z_0, w_0) is the degree of the first non-zero term in the Taylor expansion at (z_0, w_0). Note that since f is a polynomial, the Taylor expansion is finite. Thus using a point of multiplicity m, we can write f as

$$f = f_m + f_{m+1} + \ldots + f_d$$

where f_i are polynomials in the variables $z - z_0$, $w - w_0$ consisting of monomials of the form

$$(z-z_0)^j (w-w_0)^{i-j} \text{ , for } 0 \leq j \leq i$$

By definition, a point (z_0, w_0) is singular if and only if the multiplicity at that point is greater than 1. In either case, we can consider the degree m part of the Taylor expansion

$$TC(z,w) = f_m(z,w) = \sum_{i+j=m} \frac{\partial^m f}{\partial z^i \partial w^j}(z_0, w_0) \frac{(z-z_0)^i (w-w_0)^j}{i!j!}$$

If (z_0, w_0) is smooth then

$$TC(z,w) = f_1(z,w) = \frac{\partial f}{\partial z}(z_0, w_0)(z - z_0) + \frac{\partial f}{\partial w}(w - w_0)$$

is called the *tangent line* at (z_0, w_0). At a singular point, by the fundamental theorem of algebra, we can factorize T into linear factors:

$$TC(z,w) = \prod_{i=1}^{m} \alpha_i(z - z_0) + \beta_i(w - w_0)$$

The curve

$$\{(z,w) \in \mathbb{C}^2 : TC(z,w) = 0\}$$

is called the *tangent cone* at (z_0, w_0). It is a generalization of the tangent line in the sense that the factors of $TC(z,w)$ give different tangent directions at (z_0, w_0). If there are no repeated factors in $TC(z,w)$, the singularity is called ordinary. In other words, at an ordinary singularity, there must be m distinct tangent lines.

Exercise: Examine the tangent directions at $(0,0)$ of the curves: $y^2 = x^3 + x^2$, $y^2 = x^3$, $(x^4 + y^4)^2 = x^2y^2$, $(x^4 + y^4 - x^2 - y^2)^2 = 9x^2y^2$. What are the multiplicities at $(0,0)$? Which singularities are ordinary?

Definition 14. *A curve is irreducible if its defined by a polynomial $f(z,w)$ which has no nonconstant polynomial factors other than scalar multiples of itself.*

Recall that since $\mathbb{C}[z,w]$ is a UFD, every polynomial can be factored into irreducible factors. Thus, given a curve defined by a polynomial $f(z,w)$ without any repeated factors, we can write

$$f = f_1 f_2 \dots f_k$$

for irreducible polynomials f_i, which are relatively prime. Then the curves $C_i = \{f_i = 0\}$ are called the components of the curve $C = \{f = 0\}$. The notion of irreducibilily really depends on the coefficient field. For example, the polynomial $f(z,w) = z^2 + w^2$ is irreducible over $\mathbb{R}[z,w]$ (or $\mathbb{Q}[z,w]$) but not over $\mathbb{C}[z,w]$. On the other hand, clearly if f is reducible in $\mathbb{K}[z,w]$ for $\mathbb{K} \subset \mathbb{C}$ a subfield of $\mathbb{C}$, then f is reducible in $\mathbb{C}[z,w]$.

Exercise. Show that a smooth connected curve is irreducible (Hint. Show that the points at which two or more components of a curve intersect are singular.)

However, an irreducible curve need not be smooth. In general, it is not so simple to check whether a singular curve is irreducible or not. We will next discuss a nice criteria that ensures irreducibility.

Definition 15. *Given a polynomial $f(z,w) = \sum_{r,s} c_{r,s} z^r w^s$, the Newton polytope Δ_f is the convex hull in $\mathbb{R}^2$ of the exponent vectors (r,s) of all monomials appearing with $c_{r,s} \neq 0$.*

Recall that convex hull $conv(S)$ of a set S in $\mathbb{R}^2$ is the smallest convex set in $\mathbb{R}^2$ that contains S. It is straightforward to check that

$$conv(S) = \left\{ \sum_{i=1}^{|S|} t_i x_i | x_i \in S, t_i \geq 0, \sum_{i=1}^{|S|} t_i = 1 \right\}$$

Recall also that the Minkowski sum of two convex polytopes Δ_1, Δ_2 is

$$\Delta_1 + \Delta_2 = \{x + y \in \mathbb{R}^2 : x \in \Delta_1, y \in \Delta_2\}$$

Proposition 16. *Let $\mathbb{K}$ be any field, if $f, g, h \in \mathbb{K}[z, w]$ such that $f = gh$, then $\Delta_f = \Delta_g + \Delta_h$.*

Proof. By multiplying g and h, we can see immediately that $\Delta_f \subset \Delta_g + \Delta_h$. To prove $\Delta_g + \Delta_h \subset \Delta_f$, let v be any vertex of $\Delta_g + \Delta_h$. We can write $v = v_1 + v_2$ for $v_1 \in \Delta_g$ and Δ_h. We show that in fact v_1 and v_2 are unique. Suppose we have $v_1' \in \Delta_g$ and $v_2' \in \Delta_h$ such that

$$v = v_1 + v_2 = v_1' + v_2'.$$

Then we have

$$v = \frac{1}{2}(v_1 + v_2') + \frac{1}{2}(v_1' + v_2)$$

Since $v_1 + v_2', v_1' + v_2 \in \Delta_g + \Delta_h$ and v is a vertex of $\Delta_g + \Delta_h$, it follows that

$$v = v_1 + v_2' = v_1' + v_2.$$

which in turn is equal to $v_1 + v_2 = v_1' + v_2'$ by our assumption. From these, it follows immediately that $v_1 = v_1'$ and $v_2 = v_2'$. Since v is a vertex of $\Delta_g + \Delta_h$, it must be that v_1 and v_2 are vertices of Δ_g and Δ_h respectively. Thus, there is a unique monomial in the expansion of gh that has v as an exponent vector. Hence, $v \in \Delta_f$. As a result, we see that all vertices of $\Delta_g + \Delta_h$ are in Δ_f, hence $\Delta_g + \Delta_h \subset \Delta_f$ as claimed. □

We call a convex polytope $\Delta \subset \mathbb{R}^2$ integral if all of its vertices have integer co-ordinates. We say that Δ is integrally decomposable if $\Delta = \Delta' + \Delta''$ for some integral convex polytopes Δ' and Δ'' (each of which is not just a point).

Corollary 17. *A polynomial $f \in \mathbb{K}[z, w]$ not divisible by z or w is irreducible (for any field $\mathbb{K}$) if Δ_f is not integrally decomposable.*

Exercise: Let S_1 and S_2 be subsets of $\mathbb{R}^2$, show that

$$conv(S_1 \cup S_2) = conv(S_1) \cup conv(S_2).$$

Exercise: Show that all the polynomials $f(z, w) = a + bz + cz^2 + dzw$ are irreducible for any non-zero $a, b, c, d \in \mathbb{C}$.

Exercise: Show that the Newton polytope of $f(z, w) = z^6 + w^6 + 1$ is integrally decomposable, but f is irreducible over $\mathbb{C}$.

We next return to smooth curves. Smooth algebraic curves give Riemann surfaces. To prove this, we will need to appeal to the holomorphic implicit function theorem.

Theorem 18. *Suppose (z_0, w_0) is a point of C and $(\partial_w f)(z_0, w_0) \neq 0$, then there are disks $D(z_0, r)$ and $D(w_0, r')$ centred around z_0 and w_0 in $\mathbb{C}$ respectively, and a holomorphic map $\phi : D(z_0, r) \to D(w_0, r')$ such that*

$$C \cap ((D(z_0, r) \times D(w_0, r')) = \text{graph}(\phi) := \{(z, \phi(z)) : z \in D(z_0, r)\}$$

Proof. Recall that for g a holomorphic function defined on a neighbourhood of a disk $D \subset \mathbb{C}$. Suppose g does not vanish on ∂D, then the number of zeros of the g in D is given by the contour integral:

$$\frac{1}{2\pi i}\int_{\partial D}\frac{g'(w)}{g(w)}dw$$

This is the Argument principle and proved by applying Cauchy integral formula. Now, fix z, and consider the function $(f_z)(w) := f(z,w)$. We have that $f_{z_0}(w_0) = 0$, and for sufficiently small disk $D(w_0, r')$, $w = w_0$ is the unique zero of f_{z_0}. (Otherwise f_{z_0} would have to be a constant function but $(\partial_w)(f)(z_0, w_0) \neq 0$). In particular, f_{z_0} does not vanish along $\partial D(w_0, r')$. By continuity, for $z \in D(z_0, r)$ and sufficiently small r, f_z also does not vanish along $\partial D(w_0, r')$. Now, consider the integral

$$\frac{1}{2\pi i}\int_{\partial D(w_0,r')}\frac{(f_z)'(w)}{f_z(w)}dw.$$

This integral is continuous as a function of z, takes integer values and is equal to 1 at z_0. So, it has to be 1 for all $z \in D(z_0, r)$. Thus, there is also a unique zero, call it $\phi(z)$, of f_z in $D(w_0, r')$. This, can be computed by another application of the Cauchy integral formula as

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial D(w_0,r')} w\frac{(\partial_w f)(z,w)}{f(z,w)}dw$$

Finally, note that $\phi(z)$ is clearly holomorphic. □

Proposition 19. *Let C be an algebraic curve defined by a polynomial f, and let* $\mathrm{Sing}(C)$ *be the singular points of C. Then, $C \setminus \mathrm{Sing}(C)$ is a Riemann surface.*

Proof. First, note that $\mathrm{Sing}(C)$ is a closed subset of C since it is defined by vanishing of the derivatives. Now, suppose $(z,w) \in C \setminus \mathrm{Sing}(C)$, then either $(\partial_w f)(z,w) \neq 0$ or $\partial_z f(z,w) \neq 0$. In the former case, the implicit function theorem gives that an open neighbourhood of (z,w) is the graph of a holomorphic function g defined on the z-plane, and in the latter case an open neighbourhood of (z,w) is a graph of a function h defined on the w-plane. So, any point of $(z,w) \in C \setminus \mathrm{Sing}(C)$ has a neighborhood U, such that either

$$U = \{(z, g(z)) : z \in V, \text{ for some } g \text{ defined on an open set } V \subset \mathbb{C}\}$$

or

$$U = \{(h(w), w) : w \in V, \text{ for some } h \text{ defined on open set } V \subset \mathbb{C}\}$$

The local-coordinates are given by projecting to z in the former case and projecting w in the latter case. The transition functions are $z \to g(z)$ or $w \to h(w)$ which are holomorphic, hence we have constructed an atlas on $C \setminus \mathrm{Sing}(C)$. □

3.2. **Projective curves.** So far, we have studied curves in $\mathbb{C}^2$. These are necessarily non-compact.

Theorem 20. *Every holomorphic function defined on a compact connected Riemann surface S is constant.*

Proof. The proof is a consequence of the maximum modulus principle (which can be proved, for example by Cauchy integral formula) for holomorphic functions on $\mathbb{C}$ which states that if $f : U \to \mathbb{C}$ is a holomorphic function defined on an open set $U \subset \mathbb{C}$ which has a local maximum then it is constant. In other words, if $z_0 \in U$ is a point such that $|f(z_0)| > |f(z)|$ for all z in a neighborhood of z_0, then f is constant.

Now, suppose $f : S \to \mathbb{C}$ is a holomorphic function defined on a compact Rieamnn surface S. Then, by compactness of S, $|f|$ attains its maximum at some point $p_0 \in S$. Let $\phi : U \to \mathbb{C}$ be chart for S such that $p_0 \in U$. $z_0 = \phi(p_0)$. Then the function $f \circ \phi^{-1} : U \to \mathbb{C}$ has a local maximum at z_0, hence it has to be constant. Thus, the set of points of S on which $|f|$ takes its maximum value is open, but since this set is the preimage of the maximum value of $|f|$, it is also closed. By connectedness of S, this set has to be all of S. □

On an algebraic curve $C \subset \mathbb{C}^2$, the co-ordinate projections $z, w : C \to \mathbb{C}$ give non-constant holomorphic functions, hence algebraic curves in $\mathbb{C}^2$ cannot be compact. In the case of algebraic curves, we can give another proof using algebraic methods as in the following exercise.

Exercise. Given a non-constant polynomial $f(z, w)$, show that for all but finitely many values of z, there exists $w \in \mathbb{C}$, such that $f(z, w) = 0$. Deduce that algebraic curves in $\mathbb{C}^2$ are non-compact.

Exercise. Use the removable singularity theorem to deduce that any *bounded* holomorphic function from $\mathbb{C}$ to $\mathbb{C}$ extends to a holomorphic function from $\mathbb{P}^1 \to \mathbb{C}$, hence is constant. (You may recall this as Liouville's theorem.)

For many purposes it is useful to compactify algebraic curves in $\mathbb{C}^2$. Let us give one example. Suppose that we have an affine C defined by a (square-free) polynomial f of degree d, and we wish to compute its intersection points with a straight line L. By a suitable change of co-ordinates, we can assume that L passes through the origin in $\mathbb{C}^2$, so its points are parametrized as:

$$w = at, z = bt, \text{ for some } a, b \in \mathbb{C}^\times$$

Substituting this into f gives:

$$f_d(a, b)t^d + f_{d-1}(a, b)t^{d-1} + \ldots + f_1(a, b)t + f_0 = 0 \tag{1}$$

where $f_i(z, w)$ are homogeneous polynomials of degree i such that $f(z, w) = \sum_{i=0}^{d} f_i(z, w)$. As long as $f_d(a, b) \neq 0$, there are d solutions to this equation. So, we have that C and L intersect at d points (counted with multiplicity taking into account multiple roots of the Equation 1). On the other hand, it is possible that $f_d(a, b) = f_{d-1}(a, b) = \ldots = f_{k+1}(a, b) = 0$, and $f_k(a, b) \neq 0$ for some $d > k \geq 0$. In that case, the intersection of C and L is only k

points. In this case, it turns out that the remaining $d-k$ points are intersctions at infinity. Namely, substituting $1/s$ for t in Equation 1 and multiplying it with s^d, we arrive at the equation:

$$f_d(a,b) + f_{d-1}(a,b)s + \ldots + f_1(a,b)s^{d-1} + f_0 s^d = 0$$

which has a root of multiplicity $d-k$ at $s=0$ if $f_d(a,b) = f_{d-1}(a,b) = \ldots f_{k+1}(a,b) = 0$. Therefore, to get a uniform answer that C and L intersects at d points without any further condition we must add points at infinity to $\mathbb{C}^2$.

The idea is to identify $(x,y) \in \mathbb{C}^2$ with the one dimensional complex linear subspace of $\mathbb{C}^3$ spnned by $(x,y,1)$. Every one-dimensional linear space in $\mathbb{C}^3$ which is not on the plane $\{(x,y,z) \in \mathbb{C}^3 : z = 0\}$ contains a unique point of the form $(x,y,1)$. Thus the one dimensional subspaces of $\{(x,y,z) \in \mathbb{C}^3 : z = 0\}$ can be thought as "points at infinity".

Definition 21. *The n-dimensional complex projective space $\mathbb{P}^n$ is the set of complex one-dimensional subspaces of $\mathbb{C}^{n+1}$.*

Thus, $\mathbb{P}^n$ is given by the quotient space

$$\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts by

$$\lambda : (X_0, \ldots, X_n) \mapsto (\lambda X_0, \ldots, \lambda X_n)$$

We write $[X_0, \ldots, X_n] \in \mathbb{P}^n$ for the one-dimensional subspace of $\mathbb{C}^{n+1}$ containing $(X_0, \ldots, X_n)$.

The topology on $\mathbb{P}^n$ is induced by the quotient topology with respect to the map

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$$
$$(X_0, X_1, \ldots, X_n) \mapsto [X_0, X_1, \ldots, X_n]$$

Exercise. Show that $\mathbb{P}^n$ is compact and Hausdorff as a topological space. Hint: Consider the projection $\pi : S^{2n+1} = \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} : |x_0|^2 + \ldots + |x_n|^2 = 1\} \to \mathbb{P}^n$.

We can identify a large chunk of $\mathbb{P}^n$ with $\mathbb{C}^n$ via

$$(x_0, x_1, \ldots, x_{n-1}) \in \mathbb{C}^n \mapsto [x_0, x_1, \ldots, x_{n-1}, 1] \in \mathbb{P}^n.$$

This is a well-defined map with inverse:

$$[X_0, X_1, \ldots, X_n] \in \mathbb{P}^n \setminus \{X_n = 0\} \mapsto \left(\frac{X_0}{X_n}, \ldots, \frac{X_{n-1}}{X_n}\right) \in \mathbb{C}^n$$

The complement of this $\mathbb{C}^n$ in $\mathbb{P}^n$ is given by elements of the form $[X_0, X_1, \ldots, X_{n-1}, 0]$, which can be identified with $\mathbb{P}^{n-1}$. Hence, have a decomposition:

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$$

In particular, $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, and $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1$.

This decomposition is not unique. Let us write $U_i \subset \mathbb{P}^n$ to be the open set consisting of points $[X_0, X_1, \ldots, X_n]$ where $X_i \neq 0$. Then U_i is a copy of $\mathbb{C}^n$, and we have

$$\mathbb{P}^n = U_0 \cup U_1 \cdots \cup U_n$$

To define complex projective curves in $\mathbb{P}^2$, we need the notion of a homogeneous polynomial:

Definition 22. *A polynomial* $F(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ *is homogeneous of degree* d *if*

$$F(\lambda X, \lambda Y, \lambda Z) = \lambda^d F(X,Y,Z)$$

for all $\lambda \in \mathbb{C}$.

In other words, a degree d homogeneous polynomial is a sum of monomials of the form

$$X^i Y^j Z^k \ , \ i+j+k = d$$

The zero-sets of homogeneous polynomials are well-defined as a set in $\mathbb{P}^2$, since for a homogeneous polynomial $F(X,Y,Z) = 0$ if and only if $F(\lambda X, \lambda Y, \lambda Z) = 0$ for all $\lambda \in \mathbb{C}^\times$.

Definition 23. *Let* $F(X,Y,Z)$ *be a non-constant polynomial with no repeated factors, the projective curve defined by* p *is*

$$C = \{[X,Y,Z] \in \mathbb{P}^2 : F(X,Y,Z) = 0\}$$

We will try to stick with the convention that polynomials in $\mathbb{C}[x,y]$ defining (affine) curves in $\mathbb{C}^2$ are named with lower-case letters, and *homogeneous* polynomials in $\mathbb{C}[X,Y,Z]$ defining projective curves in $\mathbb{P}^2$ are names with captital letters.

Projective curves are given as closed subset of the compact space $\mathbb{P}^2$, therefore they are also compact. Just as for curves in $\mathbb{C}^2$, two homogeneous polynomials with no repeated factors define the same projective curves $\mathbb{P}^2$ if and only if they are scalar multiples of each other.

Definition 24. *The degree of a projective curve* C *in* $\mathbb{P}^2$ *defined by a homogeneous polynomial* F *is the degree of* F.

Definition 25. *A point* (X_0, Y_0, Z_0) *of a projective curve* C *in* $\mathbb{P}^2$ *defined by a homogeneous polynomial* F *is called singular if*

$$F_X(X_0,Y_0,Z_0) = F_Y(X_0,Y_0,Z_0) = F_Z(X_0,Y_0,Z_0) = 0$$

A curve is non-singular if it has no singular points.

Exercise: Show that the projective curve $X^2 + Y^2 = Z^2$ is non-singular and in fact isomorphic to $\mathbb{P}^1$ as a Riemann surface.

Exercise: For which values of $\lambda \in \mathbb{C}$, are the curves defined by

$$X^3 + Y^3 + Z^3 + \lambda XYZ = 0$$

nonsingular?

To distinguish curves in $\mathbb{C}^2$ and $\mathbb{P}^2$, we call curves in $\mathbb{C}^2$ affine, and curves in $\mathbb{P}^2$ projective. Although, these are really different notions, they are closely related. From an affine curve C, one can always obtain a projective curve $\overline{C}$ by adding "points at infinity" as we next explain.

Let us now consider a homogeneous polynomial $F(X,Y,Z)$ of degree d and the corresponding projective curve $\overline{C} \subset \mathbb{P}^2$. Recall that we have a decomposition:

$$\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1$$

where $\mathbb{C}^2 = U \subset \mathbb{P}^2$ is the open set in $\mathbb{P}^2$ given by $Z \neq 0$, and $\mathbb{P}^1 = L \subset \mathbb{P}^2$ is the line at infinity given by the points $\{[X, Y, 0] \in \mathbb{P}^2\}$.

Let us consider the affine curve

$$C = \overline{C} \cap U \subset \mathbb{C}^2$$

By definition, C is given by the zero set of the polynomial

$$f(x, y) = F(x, y, 1)$$

If Z is not a factor of $F(X, Y, Z)$, then $f(x, y)$ is also a polynomial of degree d, and by Proposition 9 (i), we have that $(\overline{C} \cap L)$ is given by finitely many points. (If Z is a factor of $F(X, Y, Z)$, then $L \subset \overline{C}$). Thus, we have

$$\overline{C} = C \cup (\overline{C} \cap L)$$

$(\overline{C} \cap L)$ is given by the zeros of $F(X, Y, 0)$ in $\mathbb{P}^1$, a finite set and we see that $\overline{C}$ is a compactification of C obtained by adding to C a finite set of points.

Conversely, suppose $f(x, y)$ is a polynomial of degree d defining a curve C in $\mathbb{C}^2$, we can define a homogeneous polynomial by

$$F(X, Y, Z) = Z^d f(\frac{X}{Z}, \frac{Y}{Z})$$

such that $F(X, Y, 1) = f(x, y)$.

In this way, we get a bijective correspondence between affine curves in $\mathbb{C}^2$ and projective curves in $\mathbb{P}^2$ not containing the line at infinity given by $Z = 0$.

If $\overline{C}$ is non-singular, so is C but the converse is not true. The curve $\overline{C}$ may have singularities at infinity, even though C is smooth.

Exercise. (*Hyperelliptic curves*) Consider the affine curve C defined by

$$y^2 = (x - a_1)(x - a_2) \dots (x - a_{2g+1})$$

where $a_1, a_2, \dots, a_{2g+1}$ distinct points. Show that the affine curve C is non-singular. Show that the projectivization $\overline{C}$ intersects the line at infinity at a unique point. Show that for $g > 1$, this point is singular.

Now, recall $\mathbb{P}^2$ is covered by three copies of $\mathbb{C}^2$, thus $\mathbb{P}^2 = U_0 \cup U_1 \cup U_2$ where $U_0 = \{X \neq 0\}, U_1 = \{Y \neq 0\}, U_2 = \{Z \neq 0\}$.

Suppose that $F(X, Y, Z)$ defines a projective curve $\overline{C} \in \mathbb{P}^2$, if Z is not a factor of $F(X, Y, Z)$, then we can consider $\overline{C}$ as a compactification of the curve C defined by the polynomial $f(x, y) = F(X, Y, 1)$ in U_2. If C has no singular points, we can make it into a Riemann surface. We can repeat the discussion replacing Z by X and Y. Thus, we can also consider the affine curves defined by $F(1, Y, Z) = 0$ and $F(X, 1, Z) = 0$ in U_0 and U_1 respectively. If these do not have singular points, we can again make them into a Riemann surface. It is easy to see that the three Riemann surface structures are equivalent on their overlaps $\overline{C} \cap U_i \cap U_j$.

Exercise. Show that the three Riemann surface structures are equivalent on the overlaps $\overline{C} \cap U_i \cap U_j$.

Thus this makes $\overline{C}$ into a Riemann surface. We state this as follows:

Proposition 26. *Suppose $F(X,Y,Z)$ is a homogeneous polynomial of degree $d \geq 1$, and the only solution of the equations:*

$$\partial_X F = \partial_Y F = \partial_Z F = 0$$

is given by $X = Y = Z = 0$, i.e. the curve $\overline{C}$ defined by F is nonsingular, then $\overline{C}$ is a compact Riemann surface.

Proof. We have seen above how to define a Riemann surface structure on $\overline{C}$ as long as F is not divisible by X, Y or Z, and each affine curve defined by F is non-singular.

Now, if F is divisible by X, then let us write

$$F(X,Y,Z) = XG(X,Y,Z)$$

Then, we have:

$$\partial_X F = G + X\partial_X G,\ \partial_Y F = X\partial_Y G,\quad \partial_Z F = X\partial_Z G$$

Hence, if $(0, Y_0, Z_0)$ is a point where G vanishes, we see that the hypothesis on derivatives of F is not satisfied. Similarly, we can see that F is not divisible by Y or Z.

Next, let us consider the affine curve $f(x,y) = F(x,y,1)$. We want to show that this curve is non-singular. To see that this holds under our assumption, we will use Euler's identity for homogeneous polynomials:

$$X\partial_X F + Y\partial_Y F + Z\partial_Z F = dF$$

This can be seen by differentiating the equality

$$F(\lambda X, \lambda Y, \lambda Z) = \lambda^d F(X,Y,Z)$$

valid for $\lambda \in \mathbb{C}^\times$ with respect to λ, and setting $\lambda = 1$. Suppose that there exists a point (x_0, y_0) such that

$$f(x_0,y_0) = (\partial_x f)(x_0,y_0) = (\partial_y f)(x_0,y_0) = 0$$

Translating this to homogeneous polynomial, we get that

$$F(x_0,y_0,1) = 0, (\partial_X F)(x_0,y_0,1) = (\partial_Y F)(x_0,y_0,1) = 0$$

Now, it follows by the Euler's identity that $(\partial_Z F)(x_0,y_0,1) = 0$ as well, hence $[x_0,y_0,1]$ is a singular point of the curve defined by F, which contradicts the hypothesis. Other affine curves defined by F are treated the same way. □

Exercise. Let A be a matrix in $GL(3,\mathbb{C})$. Show that A induces map from $\mathbb{P}^2$ to itself. Let F be homogeneous polynomial defining a projective curve $C \subset \mathbb{P}^2$. Show that A sends C to another projective curve $A(C) \subset \mathbb{P}^2$ defined by another homogeneous polynomial F_A. Show that if C is non-singular, so is $A(C)$ and the map $A : C \to A(C)$ gives an isomorphism of Riemann surfaces.

4. Meromorphic functions

Definition 27. *A meromorphic function on a Riemann surface S is a function*

$$f : S \to \mathbb{C} \cup \{\infty\}$$

and such that in every co-ordinate patch, it can be represented as a quotient g/h where g, h are holomorphic functions and h is not identically zero.

Equivalently, a meromorphic function is a function $f : S \to \mathbb{C} \cup \{\infty\}$ such that $f : S \setminus f^{-1}(\infty) \to \mathbb{C}$ is a holomorphic function and near a point $a \in f^{-1}(\infty)$, f can be represented as

$$f(z) = \frac{g(z)}{(z-a)^m}$$

for some $m > 0$ and $g(z)$ holomorphic near a and $g(a) \neq 0$. At such a point a, f is said to have a *pole of order* m. A meromorphic function f has a Laurent series expansion at a pole of order m at pole a:

$$f(z) = \sum_{n \geq -m} c_n (z-a)^n$$

The coefficient c_{-1} is called the *residue* of f at a, written $\mathrm{Res}_a f$.

From the above definition, it is easy to verify that the set of meromorphic functions form a field via the usual multiplication and addition of functions. We denote this field by $K(S)$ and call it the *function field* of S.

Proposition 28. *Let* $\mathrm{Mor}(S, \mathbb{P}^1)$ *denote the set of holomorphic functions from $S \to \mathbb{P}^1$, and c_∞ the holomorphic function on S that is constant with value equal to ∞. Then,*

$$K(S) \equiv \mathrm{Mor}(S, \mathbb{P}^1) \setminus \{c_\infty\}$$

Proof. Recall that we cover $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ with two charts. $U_0 = \mathbb{P}^1 \setminus \{\infty\}$ and $\phi_0 : U_0 \to \mathbb{C}$ is the identity map on $U_0 = \mathbb{C}$, and $U_1 = \mathbb{P}^1 \setminus \{0\}$ and $\phi_1 : U_1 \to \mathbb{C}$ is the map $z \to 1/z$ on U_1.

For $f : S \to \mathbb{P}^1$ holomorphic, take $p \in S$. If $p \in S \setminus f^{-1}(\infty)$, then take a local co-ordinate (V, ψ) near p, and $\phi_0 \circ f \circ \psi^{-1}(z)$ restricts to this chart as a holomorphic function, hence f is a holomorphic function outside $f^{-1}(\infty)$. If $f(p) = \infty$, then $\phi_1 \circ f \circ \psi^{-1}(z) = 1/(f \circ \psi^{-1})(z) = h(z)$ is a holomorphic function. Hence, $f \circ \psi^{-1}(z) = 1/h(z)$, thus f is meromorphic.

Conversely, a meromorphic function f on S defines a continuous mapping $f : S \to \mathbb{P}^1$ by simply sending poles to ∞. Again, it is clear that f is holomorphic near a point $p \in S \setminus f^{-1}(\infty)$. If $p \in f^{-1}(\infty)$, then in a local-coordinate on S, $f(z) = g(z)/(z-p)^m$ with $g(z)$ holomorphic and $g(p) \neq 0$, thus $\phi_2 \circ f(z) = (z-p)^m / g(z)$ is a holomorphic function. Hence, f defines a morphism $S \to \mathbb{P}^1$. □

Example: Suppose $C \subset \mathbb{C}^2$ is a smooth affine algebraic curve defined by an irreducible polynomial $f(z, w) \in \mathbb{C}[z, w]$. The coordinate functions $z, w : C \to \mathbb{C}$ define holmorphic functions. Any rational function of the form

$$r(z,w) = \frac{h(z,w)}{g(z,w)}$$

defines a meromorphic function $C \to \mathbb{C}$ as long as g does not vanish on C identically. By, Prop. 9, this happens if and only if f divides g.

Similarly, if F defines a smooth projective curve in $\mathbb{P}^2$, and G, H are homogeneous polynomial of the *same* degree such that F does not divide H. Then, the ratio

$$R(X,Y,Z) = \frac{G(X,Y,Z)}{H(X,Y,Z)}$$

defines a meromorphic function on the projective curve defined by F.

Thus, on Riemann surfaces that are defined algebraically, there are plenty of meromorphic functions. As we have seen above, meromorphic functions on S are essentially just the holomorphic maps from S to $\mathbb{P}^1$. Before we go on, let us give the following theorem which describes the local behaviour of maps between Riemann surfaces:

Theorem 29. *Let S and T be Riemann surfaces, and $f : S \to T$ be a holomorphic map with $f(s) = t$, for some $s \in S$ and $t \in T$, and f not constant near s. Then, there exists a unique $n \geq 1$, a local co-ordinate around s, $\psi : U \to \mathbb{D}$ with $\psi(s) = 0$, and a local co-ordinate around t, $\phi : V \to \mathbb{D}$ with $\phi(t) = 0$ such that $\phi \circ f \circ \psi^{-1} : \mathbb{D} \to \mathbb{D}$ is the map $z \mapsto z^n$. In other words, the following diagram commutes:*

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & V \\ \psi \downarrow & & \downarrow \phi \\ \mathbb{D} & \xrightarrow[z \mapsto z^n]{} & \mathbb{D} \end{array}$$

Proof. First, choose arbitrary chart $U \to \tilde{U} \subset \mathbb{C}$. By translating, we may assume that our chart is centred at 0. Thus, in local charts f is a holomorphic function defined in a neighborhood of 0 in $\mathbb{C}$ and $f(0) = 0$. The Taylor expansion of f is of the form:

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots = a_n z^n \left(1 + \frac{a_{n+1}}{a_n} z + \frac{a_{n+2}}{a_n} z^2 + \ldots\right),$$

where $a_n \neq 0$ is the first non-zero coefficient. We can pick b such that $b^n = a_n$ and an n^{th} root of the function $1 + \frac{a_{n+1}}{a_n} z + \frac{a_{n+2}}{a_n} z^2 + \ldots$ for z sufficiently small. Thus, by shrinking $\tilde{U}$, we have:

$$f(z) = b^n z^n (1 + b_1 z + b_2 z^2 + \ldots)^n = (bz + bb_1 z^2 + \ldots)^n = g(z)^n$$

for some holomorphic function $g(z)$ with $g'(0) = b \neq 0$. Now, inverse function theorem says that g is a holomorphic function defined near 0, with $g(0) = 0$ and $g'(0) \neq 0$, then g has a holomorphic inverse defined near 0. (This is a special case of Theorem 18 applied to

$f(z,w) = g(z) - w$ where we interchanged roles of z and w). Using the inverse of g^{-1} as a coordinate change, we see that

$$f \circ g^{-1}(z) = z^n$$

which holds in a sufficiently small neighborhood of 0. Rescaling appropriately, we get the result stated. □

Exercise. Use Theorem 18 to give a proof of Inverse function theorem.

Exercise. Show that a holomorphic function $f : S \to T$ is an open mapping i.e. it sends open sets to open sets.

Definition 30. *Given a non-constant holomorphic function $f : S \to T$, and points $s \in S$ and $t = f(s) \in T$, we can express f in local co-ordinates (centered at s and t) as $z \to z^n$ for some $n \geq 1$. If $n > 1$, we say that s is a ramification point and t is a branch point. We also define $v_f(s) = n$ to be the ramification index.*

Note that $s \in S$ is a ramification point if and only if $f'(s) = 0$ (in local coordinates). Thus, the set of ramification (and branch) points are isolated.

Suppose $f : S \to T$ is a holomorphic map between compact connected surfaces. Given a point $t \in T$, define the integer valued function:

$$d_f(t) = \sum_{s \in f^{-1}(t)} v_f(s)$$

Theorem 31. *$d_f : T \to \mathbb{Z}$ is constant.*

Proof. Let $t \in T$ and $\{s_1, \dots, s_k\} = f^{-1}(t)$. For each s_i, there exists $n_i \geq 1$, co-ordinate chart U_i centred at s_i and co-ordinate chart V_i centred at t such that f looks like $z \to z^{n_i}$ in these co-ordinates. Thus, by using these local models, we can directly see that for t in the open set $\bigcap_i V_i$,

$$d_f(t) = \sum_{i=1}^{k} n_i$$

is constant. Hence, d_f is a locally constant function on T. Since T is connected, it follows that d_f is constant.

□

This theorem allows us to make the following definition:

Definition 32. *Let $f : S \to T$ be a holomorphic map between compact connected surfaces. We define* $\deg(f) = d_f(t)$ *where t is any fixed point in T.*

More generally, one can extend the above definitions to the case *proper* holomorphic maps between connected Riemann surfaces S and T. (Recall that a map $f : S \to T$ is proper, if the preimage of every compact subset of T is compact in S).

Exercise. Show that if $f : S \to T$ is proper holomorphic map between connected Riemann surfaces then $f^{-1}(t)$ is a finite set for all $t \in T$.

Exercise. View a polynomial $f(z) \in \mathbb{C}[z]$ as a holomorphic function from $\mathbb{P}^1 \to \mathbb{P}^1$ by sending $\infty \to \infty$. Show directly that $v_f(\infty)$ is the same as the degree of the polynomial.

Next, let us consider some special cases of function fields on Riemann surfaces.

Proposition 33. *$K(\mathbb{P}^1)$ is isomorphic to $\mathbb{C}(z)$, the field of fractions of the polynomial ring in one variable, $\mathbb{C}[z]$.*

Proof. It is clear that every element of $\mathbb{C}(z)$ defines a meromorphic function on $\mathbb{P}^1$. Conversely, suppose f is a meromorphic function on $\mathbb{P}^1$. Since $\mathbb{P}^1$ is compact, the set of poles $f^{-1}(\infty)$ is a finite set. Let us restrict f to $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$. In this local chart, f can be written as

$$f(z) = \sum_{i=1}^{n} p_i(z) + q(z)$$

where

$$p_i(z) = \frac{b_{im_i}}{(z-a_i)^{m_i}} + \ldots + \frac{b_{i1}}{z-a_i}$$

is the principal part of the Laurent expansion of f around a_i and $q(z)$ is some holomorphic function. If ∞ is not a pole of f, then $q(z)$ is a bounded holomorphic function, therefore constant. Hence, $f(z) = \sum_{i=1}^{n} p_i(z) + c$ is a rational function. If ∞ is a pole of f, then the expansion of $q(1/z)$ can only have finitely many number of negative powers, hence $q(z)$ has only finitely many positive powers, therefore is a polynomial. As a result,

$$f(z) = \sum_{i=1}^{n} p_i(z) + q(z)$$

is a rational function. □

Exercise: Give another proof of the last proposition as follows: Given $f \in K(\mathbb{P}^1)$, let z_i for $i = 1, \ldots, n$ be the zeros of f and p_j for $j = 1, \ldots m$ be the poles of f (repeated as necessary). Consider the function:

$$g(z) = \frac{\Pi_{i=1}^{n}(z - z_i)}{\Pi_{j=1}^{m}(z - p_j)}$$

Show that $f(z)/g(z)$ is a meromorphic function with no zeroes and poles. Hence, it is constant.

Exercise: Let S be a compact connected Riemann surface. Suppose $f : S \to \mathbb{P}^1$ is a meromorphic function having exactly one pole at $s \in S$ and $v_f(s) = 1$, then f is an isomorphism of Riemann surfaces between S and $\mathbb{P}^1$.

4.1. **Elliptic functions.**

Exercise. Recall that if $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$ with $\mathrm{Im}\tau > 0$, is a lattice, we have a Riemann surface structure on $\mathbb{C}/\Lambda$. Show that $K(\mathbb{C}/\Lambda)$ is isomorphic to the field of doubly periodic meromorphic functions on $\mathbb{C}$ with period $(1, \tau)$.

By definition, an *elliptic function* is a doubly periodic meromorphic function on $\mathbb{C}$. Consider the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$ with $\mathrm{Im}\tau > 0$. A $(1, \tau)$-periodic function on $\mathbb{C}$ is a map $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi(z+1) = \phi(z)$ and $\phi(z+\tau) = \phi(z)$. Such a holomorphic map would give a holomorphic map on $\mathbb{C}/\Lambda$ but since $\mathbb{C}/\Lambda$ is a compact Riemann surface. All the holomorphic maps on it are constant.

Let us next turn to doubly periodic meromorphic functions. A fundamental parallelogram P for the lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ is a subset of $\mathbb{C}$ of the form

$$z_0 + t_1 + t_2\tau, \quad 0 \leq t_i < 1$$

Clearly, a doubly periodic function on $\mathbb{C}$ is determined by its values on a fundamental parallelogram P.

Proposition 34. *Let P be a fundamental parallelogram for Λ, and f is an elliptic function which has no poles on its boundary $\partial\overline{P}$. Then the sum of the residues of f in P is 0.*

Proof. Let $a_1, \ldots a_m$ be the poles of f inside P. By the Cauchy integral formula and the periodicity of f, we have

$$2\pi i \sum_{i=1}^{m} \mathrm{Res}_{a_i} f = \int_{\partial\overline{P}} f(z)dz = 0$$

□

As a corollary we see that an elliptic function must have at least two poles (counting multiplicity) on the torus $\mathbb{C}/\Lambda$.

We note that this is different than the case of $\mathbb{P}^1$, where residues of a meromorphic function on $\mathbb{P}^1$ can be arbitrary.

The simplest doubly periodic functions must have two poles, either a double pole or two simple poles with opposite residues. We next study a classical example of Weierstrass, who chose a function with a double pole at the origin.

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(z-(m+n\tau))^2} - \frac{1}{(m+n\tau)^2} \right)$$

This function is called the Weierstrass $\wp$ function.

Lemma 35. *Given a lattice $\Lambda \subset \mathbb{C}$ and $r > 2$, the sum $\sum_{\lambda\in\Lambda\setminus\{(0,0)\}} \frac{1}{|\lambda|^r}$ converges.*

Proof. There exists a constant $c > 0$, such that

$$|m + n\tau| \geq c(|m| + |n|) \text{ for all } \ (m,n) \in \mathbb{Z}^2$$

It is easy to see this by noticing that the function $(m,n) \mapsto |m+n\tau|/(|m|+|n|)$ is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ and is invariant under scaling $(m,n) \to (tm,tn)$, hence its values are determined by its values on the unit circle, which is a compact set.

Next, let us consider the $4k$ elements $m + n\tau$ of Λ such that $|m| + |n| = k$. For such (m,n), we have $|m + n\tau| \geq ck$, thus we have:

$$\sum_{\lambda\in\Lambda\setminus\{(0,0)\}} \frac{1}{|\lambda|^r} \leq \sum_{k=1}^{\infty} \frac{4k}{(ck)^r} = 4c^{-r} \sum_{k=1}^{\infty} \frac{1}{k^{r-1}} < \infty$$

□

Note from the form of the construction that $\wp(z) = \wp(-z)$, hence $\wp(z)$ is an even meromorphic function. we can get an odd meromorphic function by differentiation

$$\wp'(z) = -2 \sum_{(m,n)\in\mathbb{Z}^2} \frac{1}{(z-(m+n\tau))^3}.$$

$\wp'(z)$ is a doubly periodic meromorphic function with a pole of order 3 at 0. Moreover,

$$\wp'(-z) = -\wp'(z)$$

In other words, $\wp'(z)$ is an odd function.

Corollary 36. *$\wp(z)$ defines a meromorphic function on $\mathbb{C}/\Lambda$ with a double pole at each lattice point and no other pole.*

Proof. We have for $|\lambda| > 2|z|$,

$$\left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| = \left|\frac{(z-2\lambda)z}{(z-\lambda)^2\lambda^2}\right| \leq \frac{10|z|}{|\lambda|^3}$$

Hence, by Lemma 35, the series defining $\wp(z)$ converges for all z.

Thus, $\wp'(z)$ is also well-defined. It is obvious that $\wp'(z)$ is $(1,\tau)$-periodic. To see that $\wp(z)$ is $(1,\tau)$ periodic, consider the functions:

$$\wp(z+1) - \wp(z) \text{ and } \wp(z+\tau) - \wp(z)$$

Differentiating these, by double periodicity of $\wp'(z)$, we see that both of these functions have to be constant. To determine the constant, we evaluate these functions at $z = -1/2$ and $z = -\tau/2$, respectively and use the fact that $\wp(z)$ is even. □

The function $\wp(z)$ when viewed as a holomorphic function from $\mathbb{C}/\Lambda \to \mathbb{P}^1$ has degree 2, therefore it must have two zeros, and similarly $\wp'(z)$ must have three zeros. It is easy to find the zeros of $\wp'(z)$. Namely,

$$\wp'(1/2) = \wp'(\tau/2) = \wp'((1+\tau)/2) = 0$$

This follows from periodicity of $\wp'(z)$ and that it is an odd function.

Since $\wp(z)$ is an even function, the zeros of $\wp(z)$ must be given by $\pm z_0$ for some z_0. It is difficult to give a formula for z_0. (If you want to read about it, there is a paper by Eichler and Zagier).

Exercise. Let e_1, e_2, e_3 be the values of $\wp(1/2)$, $\wp(\tau/2)$, $\wp((1+\tau)/2)$ respectively. (i) Show that e_1, e_2, e_3 are all distinct. (ii) Show that for any $a \in \mathbb{C} \setminus \{e_1, e_2, e_3\}$, the equation $\wp(z) = a$ has exactly two distinct solutions.

We are now ready to compute the function field of $\mathbb{C}/\Lambda$.

Theorem 37. *The field of meromorphic functions on $\mathbb{C}/\Lambda$ is generated by $\wp(z)$ and $\wp'(z)$ over $\mathbb{C}$. In other words, every elliptic function is a rational function of $\wp(z)$ and $\wp'(z)$.*

Proof. Suppose f is an elliptic function. We can write it as a sum of even and odd functions:

$$f(z) = \frac{f(z)+f(-z)}{2} + \frac{f(z)-f(-z)}{2}$$

Furthermore, if f is odd then the product $f\wp'$ is even, so it suffices to prove that every even function is a rational function of $\wp$.

Thus, let us assume that f is an even elliptic function. f can be viewed as a holomorphic function from $\mathbb{C}/\Lambda \to \mathbb{P}^1$. Let $p_1, \dots p_n$ and $q_1, \dots q_n$ be the zeros and poles of f, repeated as neccesary. Note that their numbers is the same n since this is the degree of f. Given a zero a and a pole b of f, we define:

$$g(z) = \begin{cases} \frac{f(z)(\wp(z)-\wp(b))}{\wp(z)-\wp(a)}, & a \notin \Lambda, b \notin \Lambda \\ f(z)(\wp(z)-\wp(b)), & a \in \Lambda, b \notin \Lambda \\ \frac{f(z)}{\wp(z)-\wp(a)}, & a \notin \Lambda, b \in \Lambda \end{cases}$$

Now, $g(z)$ is even and doubly periodic meromorphic function, and g is either constant or else

$$\#\{g^{-1}(0)\} = \#\{f^{-1}(0)\} - 2$$

where # denotes the count of zeros in the quotient $\mathbb{C}/\Lambda$. This equality can be verified by using the fact that for $c \notin \Lambda$, the function

$$\wp(z) - \wp(c)$$

has exactly two zeros at $\pm c$, and has exactly two poles at 0. Hence, after repeating this construction finitely many times, we obtain that $f(z)$ is a rational function of $\wp(z)$. □

Corollary 38. *The field $K(\mathbb{C}/\Lambda)$ of meromorphic functions on $\mathbb{C}/\Lambda$ is isomorphic to the field extension $\mathbb{C}(\wp, \wp')$.*

Now, since $(\wp'(z))^2$ is an even elliptic function, it should be expressible as a rational function of $\wp(z)$. In fact, we have:

Theorem 39. $(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ *where* $g_2 = 60\sum_{\lambda\in\Lambda\setminus\{(0,0)\}} \lambda^{-4}$ *and* $g_3 = 140\sum_{\lambda\in\Lambda\setminus\{(0,0)\}} \lambda^{-6}$.

Proof. The even function $\wp$ has Laurent series expansion:

$$\wp(z) = \frac{1}{z^2} + 0 + az^2 + bz^4 + \dots$$

Then,

$$\wp'(z) = -2\frac{1}{z^3} + 2az + 4bz^3 + \dots$$

Consider the function

$$k(z) = \wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$$

where $g_2 = 20a$ and $g_3 = 28b$. Using the above formulae, we can check that $k(z)$ restricts to a holomorphic function near 0 and it vanishes at 0. Furthermore, since $\wp, \wp'$ are doubly periodic, we conclude that $k(z)$ is also doubly periodic, hence a constant and is identically zero. Thus, we conclude that

$$\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3 = 0$$

The required series expansions of g_2 and g_3 can be obtained by computing the power series expansion of

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{(0,0)\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

at $z = 0$. Namely,

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{(0,0)\}} \left(\frac{1}{\lambda^2}(1 + \frac{z}{\lambda} + (\frac{z}{\lambda})^2 + \dots)^2 - \frac{1}{\lambda^2}\right) \\ &= \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{(0,0)\}} \sum_{m=1}^{\infty} (m+1)(\frac{z}{\lambda})^m \frac{1}{\lambda^2} \\ &= \frac{1}{z^2} + \sum_{m=1}^{\infty} c_m z^m\end{aligned}$$

where

$$c_m = \sum_{\lambda\in\Lambda\setminus\{(0,0)\}} \frac{m+1}{\lambda^{m+2}}$$

□

Corollary 40. *The field of meromorphic functions on $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}(z)[w]/(w^2 - 4z^3 + g_2 z + g_3)$, the degree 2 extension of the field of rational functions $\mathbb{C}(z)$ obtained by adjoining roots of $w^2 = 4z^3 - g_2 z - g_3$.*

Exercise. Given a lattice $\Lambda \subset \mathbb{C}$. Let $C_\Lambda \subset \mathbb{P}^2$ be the projective curve defined by

$$Y^2 Z - 4X^3 + g_2 X Z^2 + g_3 Z^3 = 0$$

Show that there is a well-defined map $\mathbb{C}/\Lambda \to C_\Lambda$ given by

$$[z + \Lambda] \mapsto \begin{cases} [\wp(z), \wp'(z), 1] \text{ if } z \notin \Lambda \\ [0, 1, 0] \text{ if } z \in \Lambda \end{cases}$$

which is an isomorphism of Riemann surfaces.

Exercise (moduli of elliptic curves). Given a curve C_Λ by the equation

$$Y^2 Z - 4X^3 + g_2 X Z^2 + g_3 Z^3 = 0$$

we define

$$J(\Lambda) = \frac{g_2^3}{g_2^3 - 27 g_3^2}$$

(i) Show that $g_2^3 - 27g_3^2 \neq 0$ so that $J(\Lambda)$ is well-defined.

(ii) Given two lattice $\Lambda, \tilde{\Lambda} \subset \mathbb{C}$. Show that if $J(\Lambda) = J(\tilde{\Lambda})$ then the corresponding projective curves

$$Y^2 Z - 4X^3 + g_2(\Lambda) X Z^2 + g_3(\Lambda) Z^3 = 0 \text{ and } Y^2 Z - 4X^3 + g_2(\tilde{\Lambda}) X Z^2 + g_3(\tilde{\Lambda}) Z^3 = 0$$

are projectively equivalent in $\mathbb{P}^2$.

(iii) Given two lattices Λ, $\tilde{\Lambda} \subset \mathbb{C}$. Show that the following are equivalent:

- $\mathbb{C}/\Lambda$ is biholomorphic to $\mathbb{C}/\tilde{\Lambda}$.
- $\Lambda = c\tilde{\Lambda}$ for some $c \in \mathbb{C}^\times$.
- $J(\Lambda) = J(\tilde{\Lambda})$.

It is a non-trivial theorem that every Riemann surface admits non-constant meromorphic functions. The proof of this result requires analysis of the Laplace operator. In fact, meromorphic functions can be used to separate points.

Theorem 41. *Given two points p and q of a compact Riemann surface S, there exists a meromorphic function $f \in K(S)$ such that $f(p) = 0$ and $f(q) = \infty$.*

This theorem is closely related to the uniformization theorem, and can be proved assuming the uniformization theorem and we might see a proof along those lines later on. Also, the above theorem implies that every *compact* Riemann surface can be defined algebraically. (This is not the case for non-compact Riemann surfaces.)

5. Topology of Riemanns surfaces and Normalization

5.1. **Riemann-Hurwitz formula.** Recall that every compact connected orientable topological surface is homeomorphic to a sphere with g-handles attached. g is a topological invariant of the surface called the genus.

We will next prove a formula that allows us to compute the genus of the underlying toopological surface of a Riemann surface.

Definition 42. *Let $f : S \to T$ be a holomorphic map between compact Riemann surfaces. We define the total branching index of f to be:*

$$b_f = \sum_{t \in T} \sum_{s \in f^{-1}(t)} (v_f(s) - 1) = \sum_{t \in T} (deg(f) - |f^{-1}(t)|)$$

Theorem 43. *(Riemann-Hurwitz) Suppose $f : S \to T$ be a holomorphic map between compact Riemann surfaces then,*

$$2g(S) - 2 = (deg f)(2g(T) - 2) + b_f$$

(In particular, b_f must be even.)

A topological space is second countable, or has a countable base, if it contains a countable family of open subsets U_n such that every open set is union of some of U_n. For example, a countable base for the topology on $\mathbb{R}$ is open intervals with endpoints in $\mathbb{Q}$. A topological surface has a countable base if and only if it can be covered by a countably many disks. In particular, it follows that every compact surface has a countable base. In contrast, Prüfer has given an example of a connected surface which has no countable base. Such a surface is necessarily pathological and we shall not discuss it.

By a theorem of Radó a surface has a countable base if and only if it is triangulable. Furthermore, every Riemann surface has a countable base. We shall not give a proof of Radó's theorem as it would take us too far a field but from now on we will assume that our Riemann surfaces will be triangulable. To be precise, this means:

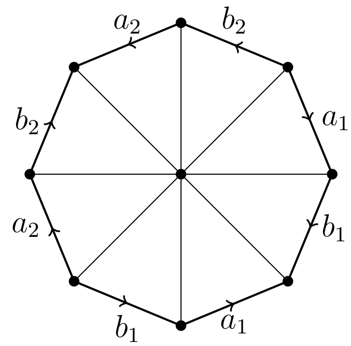


FIGURE 1. A triangulation of a genus 2 surface

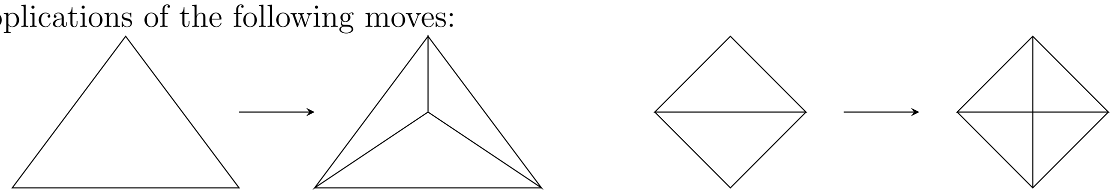
Definition 44. *A triangulation of a surface S is a triple (V, E, F), where V is a set of points on S, E is a set of edges i.e. a collection of homeomorphic images of the closed interval with end points on V. The edges are not allowed to intersect other than at the end points, and only finitely many edges may meet at a vertex. Finally, F is the union of complement of the edges on S. The connected components of F are called faces and their closure must be homeomorphic to a triangle.*

Definition 45. *Let S be a triangulated surface with finitely many faces. Then the Euler characteric of S is defined to be:*

$$\chi(S) = \#V - \#E + \#F$$

Theorem 46. *The Euler characteristic of S is independent of triangulation. For a compact orientable surface S, the Euler characteristic is $2 - 2g$.*

Proof. First, one sees by direct calculation that a refinement of a triangulation does not change the Euler characteristic. A refinement of a triangulation is given by repeated applications of the following moves:



Next, given two different triangulations τ, τ' of S, one shows that there is a common refinement. This is achieved by superimposing τ and τ' and adding vertices and edges to arrive at a triangulation that contains both τ and τ'.

To prove that Euler characteristic of a compact orientable surface S is $2 - 2g$, you can use any triangulation. For example, genus g surface can be obtained as taking a $4g$-gon and identifying the edges as indicated below for $g = 2$, where also a triangulation is given.

One computes the Euler characteristic as $V = 2 - 6g + 4g = 2 - 2g$. □

Exercise. Show that the surface given by the identification indicated in Figure 1 is homeomorphic to a compact surface of genus 2.

Proof of Riemann-Hurwitz: Pick a triangulation $\tau = (V, E, F)$ for T so that the branch points belong to the vertices of the triangulation. We can do this by refining any triangulation of T. Now, we can cover S and T by local charts U_i and V_i for some $i = 1, \dots N$, such that in each local chart $U_i \to V_i$, the map $f : S \to T$ has the standard local form $z \to z^{n_i}$ for some $n_i \geq 1$. If necessary, refine the triangulation τ so that each triangle of τ lies inside some V_i. It is then clear that the inverse images of the triangulation τ gives a triangulation $f^{-1}(\tau)$ of S. Write $f^{-1}(\tau) = (V', E', F')$. Then, we have:

$$\#V' = (degf)(\#V) - b_f \quad , \quad \#E' = (degf)(\#E), \quad (\#F') = (degf)(\#F)$$

Therefore, we conclude that

$$\chi(S) = \#V' - \#E' + \#F' = (degf)(\#V - \#E + \#F) - b = (degf)\chi(T) - b$$

□

5.2. **Normalization.** As we have seen before, one way to compactify affine algebraic curves given by a polynomial $f(z, w)$ is to consider their compactification in $\mathbb{P}^2$. However, this usually gives singular projective curves. The following theorem gives another way to compactify affine curves such that the compactification is smooth. This is a desingularization procedure called the normalization of the singular projective curve and in fact applies more generally when the affine curve is defined by an irreducible polynomial (so the affine curve may also have singularities). Note that if the affine curve is reducible, one can just consider each component separately.

Theorem 47. *Let*

$$\begin{aligned} f(z, w) &= p_0(z)w^n + p_1(z)w^{n-1} + \dots + p_n(z) \\ &= q_0(w)z^m + q_1(w)z^{m-1} + \dots + q_m(w) \end{aligned}$$

be an irreducible polynomial. If $n \geq 1$, define

$$S_f^z = \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0, f_w(z, w) \neq 0, p_0(z) \neq 0\}$$

and, similarly, if $m \geq 1$, define

$$S_f^w = \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0, f_z(z, w) \neq 0, q_0(w) \neq 0\}$$

Then, S_f^z and S_f^w are connected Riemann surfaces and there exists a unique compact connected Riemann surface $S = S_f$ that contains S_f^z and S_f^w. Furthermore, the coordinate functions z and w extend to meromorphic functions on S with branching points in $S \setminus S_f^z$ and $S \setminus S_f^w$ respectively.

We will prove this theorem in stages. Intuitively, S is obtained as follows: First, we define a Riemann surface structure on the parts S_f^z and S_f^w, as here the implicit function theorem applies. Then, we show that we can compactify this in unique way to a compact Riemann surface by "filling in the punctures" in a canonical way.

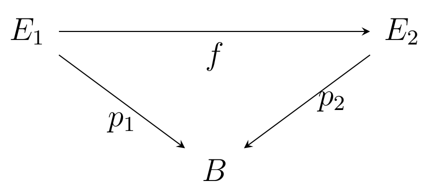
First, we will take the opportunity to review the theory of covering spaces from topology.

Definition 48. *A continuous map $p : E \to B$ between topological surfaces E and B is a covering map if for every $b \in B$, there is a neighborhood V such that $p^{-1}(V) = \bigsqcup U_i$; where U_i are pairwise disjoint and the restriction $p_{|U_i}$ is a homeomorphism.*

A covering map is a local homeomorphism, but the converse is not always true. However, a proper local homeomorphism is a covering map.

Exercise. Give an example of a local homeomorphism that is not a covering map.

An isomorphism of covering maps $p_i : E_i \to B$, $i = 1, 2$ is given by a homeomorphism $f : E_1 \to E_2$ such that the following diagram commutes:



Lemma 49. *Let $p : E \to B$ be a covering. Given a path $f : [0, 1] \to B$ and a lift of $f(0) = b_0$, that is a point $e_0 \in E$ that satisfies $b_0 = p(e_0)$, then there exists a unique path $\tilde{f} : [0, 1] \to E$ such that $\tilde{f}(0) = e_0$ satisfying $f = p \circ \tilde{f}$.*

Proof. By definition of a covering space, we can cover B with a collection of open sets U_α such that $p^{-1}(U_\alpha) = \sqcup_{i \in I_\alpha} U_{\alpha,i}$ such that $p_{|U_{\alpha,i}} : U_{\alpha,i} \to U_\alpha$ is a homeomorphism.

Since I is compact, we can find a subdivision of I to $I_i = [t_k, t_{k+1}]$ for $0 = t_0 < t_1 < \ldots < t_n = 1$ such that for each i there exists a U_α such that $f(I_k)$ lies entirely in some U_α. Assume inductively that we have defined a lift $\tilde{f}$ at $\{t_k\}$. Now, in the preimage $p^{-1}(U_\alpha)$ we also get a distinguished open set U_{α,i_k} which contains $\tilde{f}(t_k)$. We can then extend, $\tilde{f}$ to all of $[t_k, t_{k+1}]$ by composing $f : [t_k, t_{k+1}] \to B$ with the homeomorphism $p^{-1} : U_\alpha \to U_{\alpha,i_k}$. Thus by, induction we get a lift $\tilde{f} : I \to E$. We note that we did not have any choice in constructing the lift except the lift e_0 of the first point $f(0) = b_0$. □

We shall briefly recall the notion of a fundamental group and its role in the classification of coverings of a surface. Details will not be given. You can consult any basic introduction to algebraic topology for details.

Given a topological space B and a base-point $b_0 \in B$, the fundamental group is $\pi_1(B, b_0)$ consists of homotopy classes of loops based at b_0. Here by a "loop" we mean a continuous map $\gamma : [0, 1] \to B$ such that $\gamma(0) = \gamma(1) = b_0$, and by a homotopy between two loops γ_0 and γ_1, we mean a continuous 1-parameter family $\gamma_t : [0, 1] \to B$ for $t \in [0, 1]$. By concatenation of loops, $\pi_1(B, b_0)$ becomes a group. More precisely, define the composition law of loops α, β by

$$[\alpha] \circ [\beta] = [\alpha\beta], \text{ where } \alpha\beta(s) = \begin{cases} \alpha(2s) \text{ if } 0 \le s \le 1/2 \\ \beta(2s - 1) \text{ if } 1/2 \le s \le 1 \end{cases}$$

Identity element is given by the constant loop. Inverse of a loop γ is given by $\gamma^{-1}(s) = \gamma(1-s)$.

The isomorphism class of the fundamental group does not depend on the basepoint, hence the basepoint is sometimes dropped from the notation.

Examples. 1) $\pi_1(B) = 0$ if $B = \mathbb{C}$ or $B = \mathbb{D}$, or more generally any contractible topological space.

2) If $B = \mathbb{C}^\times$ is the punctured plane, $\pi_1(B) = \mathbb{Z}$ generated by a loop that goes around once.

3) IF $B = T^2$ torus, then $\pi_1(B) = \mathbb{Z}^2$.

4) If $B = \Sigma_g$ the compact surface of genus g, then there are $2g$ generators $a_1, b_1, \ldots, a_g, b_g$ of $\pi_1(B)$ subject to the relation:

$$[a_1, b_1][a_2, b_2] \ldots [a_g, b_g] = 1$$

where $[a, b] = aba^{-1}b^{-1}$.

The following result states the correspondence between the fundamental group and coverings in the case of surfaces. (This is a topological statement and holds in a much more general setting.)

Proposition 50. *Let B be a connected surface and b_0 a basepoint in B. There is a one-to-one correspondence between:*

- *equivalence classes of coverings $p : E \to B$ with E connected.*
- *conjugacy classes of subgroups in $\pi_1(B, b_0)$.*

To give an indication of how this correspondence goes: Given a covering $p : E \to B$, choose a point $e_0 \in p^{-1}(b_0)$, then the map p induces an injective homomorphism of groups:

$$p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$$

and one considers the subgroup $p_*(\pi_1(E, e_0))$. Different choices of e_0 leads to conjugate subgroups.

To go the other way, one first constructs a "universal cover", corresponding to the trivial subgroup. The universal cover $\tilde{B}$ can be realized as the set of pairs (b, γ) where $b \in B$ and $\gamma : [0, 1] \to B$ is a homotopy class of paths in B with $\gamma(0) = b_0$ and $\gamma_1(b)$, where the covering map $\tilde{B} \to B$ is given by projection $(b, \gamma) \to b$.

By the path lifting property proved above, $\pi_1(B, b_0)$ acts on $\tilde{B}$ such that

$$B = \tilde{B}/\pi_1(B, b_0)$$

The universal cover is unique and it is characterised by the property that it is *simply-connected*, i.e. its fundamental group is trivial.

Now, any subgroup $H \subset \pi_1(B, b_0)$ acts on $\tilde{B}$, and the corresponding covering space is obtained by $\tilde{B}/H$. Furthermore, the degree of the covering map is given by the index of the subgroup H in $\pi_1(B)$. A covering map is called a finite covering if this index is finite.

Example. The universal cover of $\mathbb{C}^\times$ is $\mathbb{C}$ with the covering map given by: $z \mapsto e^{2\pi i z}$.

where the action of $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$ on $\mathbb{C}$ is given by $k : z \to z + k$. The connected covering $\mathbb{C}^\times \to \mathbb{C}^\times$ given by $z \mapsto z^n$ corresponds to the subgroup $n\mathbb{Z} \subset \mathbb{Z} = \pi_1(\mathbb{C}^\times)$.

Exercise. Show that the universal covering of $\mathbb{D}^\times$ is $\mathbb{H}$ where the covering map $\mathbb{H} \to \mathbb{D}^\times$ is given by $z \to e^{2\pi i z}$.

Lemma 51. *Let $f : S \to \mathbb{D}^\times = \mathbb{D} \setminus \{0\}$ is a proper holomorphic map of degree n without any branched points. If S is connected, then S is isomorphic to $\mathbb{D}^\times$ in such a way that f becomes the map $z \mapsto z^n$.*

Proof. A proper holomorphic map of degree n without branched points is a covering map $f : S \to \mathbb{D}^\times$. But, by the correspondence with the subgroups of fundamental groups, we know that all the finite degree connected covering spaces of $\mathbb{D}^\times$ is given by the maps $\mathbb{D}^\times \to \mathbb{D}^\times$ sending $z \to z^n$ for some n since the subgroups $n\mathbb{Z} \subset \mathbb{Z}$ are all finite index subgroups. □

Lemma 52. *Let T be a compact Riemann surface and $\Sigma \subset T$ be a finite subset, $T^* = T \setminus \Sigma$, and S^* is a Riemann surface. Assume that $f^* : S^* \to T^*$ is a holomorphic map of finite degree without any branched points. Then there exists a unique compact Riemann surface $S \supset S^*$ such that f^* extends to a unique morphism $f : S \to T$. Moreover, $S \setminus S^*$ is a finite set.*

Proof. Let $t \in T \setminus T^*$ and (V, ϕ) be a co-ordinate disk for T centred at t. By rescaling, we may assume that V is isomorphic to $\mathbb{D}$. Let $\mathbb{D}^\times = V \setminus \{t\}$. Let $U_1^* \cup U_2^* \cup \dots U_r^* = f^{-1}(\mathbb{D}^\times)$ be decomposition of $f^{-1}(\mathbb{D}^\times)$ into connected components. Restriction of f to each U_i^* gives a covering map $f_{|U_i^*} : U_i^* \to \mathbb{D}^\times$, hence by Lemma 51 each U_i^* is isomorphic to $\mathbb{D}^\times$ in such a way that f becomes the map $z \mapsto z^{m_i}$ for some m_i. This leads us to add an extra point p_i which plays the role of the centre by forming $U_i = U_i^* \cup \{p_i\}$ and extending f by sending p_i to t. It is easy to check that the resulting surface S has the structure of a Riemann surface by using the inverse of the obvious map from $\mathbb{D}$ to S as a new chart. Compactness of S follows by checking that $f : S \to T$ is a proper map. The uniqueness of S and the extension of f is proved by the following lemma. □

Lemma 53. *Let S and T be compact Riemann surfaces, and $\sigma \subset S$ and $\tau \subset T$ be collections of finite sets. Assume that $S^* = S \setminus \sigma$ and $T^* = T \setminus \tau$ are isomorphic Riemann surfaces, then S and T must also be isomorphic.*

Proof. Let $f : S^* \to T^*$ be an isomorphism of Riemann surfaces. Given $x \in \sigma$, take an arbitrary sequence (x_i) of points in S^* converging to x. Since T is compact the sequence $f(x_i)$ has a limit point y. We have $y \in \tau$ since otherwise, x_i would need to converge to $f^{-1}(y) \in S^*$ which is a contradiction. Say $|\tau| = k$ and let $V_1, V_2, \dots, V_k$ be disjoint disk neighborhoods of the different points in τ. Let $x \in \sigma$, and U be a neighborhood of x such that $U \cap S^* = U \setminus \{x\}$. By shrinking U if necessary, we can arrange that $f(U \setminus \{x\}) \subset V_i$ for the i such that $y \in V_i$. By removable of singularity, we extend f to U by setting $f(x) = y$. Thus, we get a holomorphic map $f : S \to T$ of degree 1, hence is an isomorphism of Riemann surfaces. □

We are finally ready to prove the normalization theorem.
Proof of Theorem 47: As we have seen before, the implicit function theorem makes S_f^z and S_f^w into Riemann surfaces such that the projections to z and w are holomorphic functions.

Moreover, since $f_w(z,w) \neq 0$ and $p_0(z) \neq 0$ in S_f^z (resp. $f_z(z,w) \neq 0$, and $q_0(w) \neq 0$ in S_f^w), the projection $z : S_f^z \to \mathbb{C}$ (resp. $w : S_f^w \to \mathbb{C}$) is a covering map of degree n (resp. m).

Moreover, since f and f_w can have only finitely many common zeros (by Hilbert's Nullstellensatz), we see that $z : S_f^z \to \mathbb{P}^1$ misses only finitely many points $\{a_1, a_2, \dots, a_r, \infty\}$.

Now, let W be a connected component of S_f^z, then $z : W \to \mathbb{P}^1 \setminus \{a_1, a_2, \dots, a_r, \infty\}$ is a covering map of degree $d \leq n$. We can then apply Lemma 52 to obtain a compact Riemann surface $\widehat{W}$ and a map $z : \widehat{W} \to \mathbb{P}^1$.

It remains to prove that S_f^z is connected. Consider the symmetric functions:

$$s_1(z) = \sum w_i(z), s_2(z) = \sum w_i(z)w_j(z), \dots, s_d(z) = \prod w_i(z)$$

where $(z, w_1(z)), \dots, (z, w_d(z)) \in S_f^z$ are the preimages of $z \in \mathbb{P}^1 \setminus \{a_1, a_2, \dots, a_r, \infty\}$ by the function $z : W \to \mathbb{C}$. In other words, $w_i(z)$ are the roots of the polynomial $f(z,T)$ when considered as a polynomial in the variable $T = w$. (Using the covering map property, check that $s_i(z)$ are well-defined holomorphic functions on all of $\mathbb{P}^1 \setminus \{a_1, a_2, \dots, a_r, \infty\}$). The argument principle implies that near a_i, the functions $w_i(z)$ are bounded in terms of the coefficients of the polynomial $f(z,T) \in \mathbb{C}[T]$ (z fixed). (Alternatively, see the exercise below.) Similarly, $1/w_i(z)$ are bounded near ∞. Therefore, $s_i(z)$ extend to rational functions on all of $\mathbb{P}^1$. We now consider the polynomial

$$G(z,w) = s(z)(w^d - s_1(z)w^{d-1} + s_2(z)w^{d-2} - \dots \pm s_d(z))$$

where $s(z)$ is the least common multiple of the denominators of the rational functions $s_i(z)$. Now, by construction we have that $G(z,w)$ vanishes identically on W and so does the irreducible polynomial $F(z,w)$. Thus, by Nullstellensatz, it follows that G has to be a multiple of F. Hence $deg_w(G) = d \geq deg_w(F) = n$. But, we had $d \leq n$ by definition. It follows that $n = d$, $F = G$ and $S_f^z = W$, as required.

The same process can be done for S_f^w, and that since S_f^z and S_f^w differ by a finite number of points, by lemma 53 the constructed compact connected Riemann surfaces are isomorphic.

□

Exercise. If $x^n + c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n = 0$, then $|x| < 2\text{max}|c_k|^{1/k}$.

Exercise. Let S be the compact Riemann surface associated to the equation $z^{2a} - 2w^b z^a + 1 = 0$, for fixed positive integers a, b. Identify the branch points of the covering of the Riemann sphere defined by the projection to the z co-ordinate, and hence show that the genus of S is $ab - a$.

Thus, we proved that every irreducible curve defines a compact connected Riemann surface in a unique way. As was mentioned before, it is also true that every compact connected Riemann surface is obtained this way. Assuming Theorem 41, we will next work towards a proof of this.

5.3. **Function field of a Riemann surface.** The next result will make use of the Primitive Element Theorem from field theory which states that:

Theorem 54. *Any finite extension $k \subset K = k(\alpha_1, \ldots, \alpha_n)$ between fields of characteristic 0 is generated by a single primitive element $\beta \in K$, which can be chosen of the form $\beta = k_1\alpha_1 + \ldots + k_n\alpha_n$, $k_i \in k$.*

Proposition 55. *Suppose $g \in K(S)$ be a meromorphic function on S of degree n. Then the field extension $\mathbb{C}(g) \subset K(S)$ has degree $\leq n$. (In fact, the degree equals exactly n).*

Proof. It suffices to show that every element $h \in K(S)$ satisfies a polynomial of degree $\leq n$ with coefficients in $\mathbb{C}(g)$. Then the result follows the primitive element theorem.

Let $w_1(z), w_2(z), \ldots, w_n(z)$ be the preimages of g by z, counted according to the multiplicities. Consider the symmetric expressions:

$$b_1(z) = \sum h(w_i(z)), b_2(z) = \sum h(w_i(z))h(w_j(z)), \ldots, b_n(z) = \prod h(w_i(z))$$

As in the proof of Theorem 47, these symmetric functions $b_i(z)$ define rational functions on all of $\mathbb{P}^1$. Set

$$p(w) = \prod(h(w) - h(w_i(g(w)))) = \sum(-1)^k b_k(g(w))h(w)^{n-k}$$

which has to vanish identically since one of the $w_i(g(w))$ has to be w. Therefore, we see that the polynomial

$$P(X) = X^n - b_1(g)X^{n-1} + b_2(g)X^{n-2} - \ldots \pm b_n(g)$$

has h as one of its roots. Thus, $K(S)$ is a degree $\leq n$ extension of $\mathbb{C}(g)$. By the primitive element theorem, we can find $h \in K(S)$ such that $K(S) = \mathbb{C}(g, h)$.

□

The next theorem shows that any Riemann surface is of the form S_f for some irreducible polynomial f of two variables. Its proof will use Theorem 41 which we assume without proof.

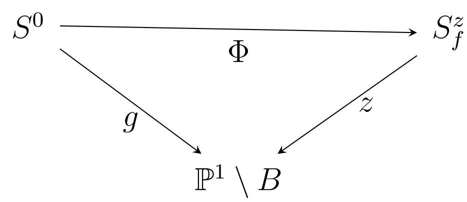
Theorem 56. *Let $K(S) = \mathbb{C}(g, h)$ and let $f(z, w)$ be an irreducible polynomial such that $f(g, h) = 0$. Then the map*

$$S \xrightarrow{\Phi} S_f$$
$$p \mapsto (g(p), h(p))$$

defines an isomorphism of Riemann surfaces.

Let (f) denote the ideal of $\mathbb{C}[z, w]$ generated by f. Then, the quotient field of $\mathbb{C}[z, w]/(f)$ is isomorphic to $K(S)$ via the map $z \mapsto g, w \mapsto h$, and deg(g) is equal to the degree of the extension $\mathbb{C}(g) \subset K(S)$.

Proof. We will first show that Φ is well-defined. Recall that S_f constructed as the unique compact connected Riemann surface containing S_f^z which is a covering space for $\mathbb{P}^1 \setminus \{a_1, a_2, \ldots, a_r, \infty\}$ via the projection to z. Let $B = \{a_1, a_2, \ldots, a_r, \infty\}$ and $S^0 = S \setminus g^{-1}(B)$. We have that the following diagram commutes:



Observe that if $g(p) = a \in \mathbb{P}^1 \setminus B$, then the value of h at p must be one of the n distinct roots of the polynomial $f(a, T)$, hence the function $\Phi(p)$ gives a well-defined point of S_f^z for every $p \in S^0$. To be able to extend Φ to all of S, we need to show that $\Phi : S^0 \to S_f^z$ is a covering map, but this follows because both g and z are covering maps.

Finally, we have to show that the map Φ has degree 1. Suppose that this is not the case. Then the fibres of all but finitely many points $p = (a, b) \in S_f^z$ would contain at least two points p_1 and p_2. Let ϕ be an arbitrary meromorphic function. As $K(S)$ is generated by g and h, it follows that

$$\phi = \frac{\sum a_{ij} g^i h^j}{\sum b_{ij} g^i h^j}$$

hence $\phi(p_1) = \frac{\sum a_{ij} a^i b^j}{\sum b_{ij} a^i b^j} = \phi(p_2)$. This means that for all these pairs of points any meromorphic function ϕ takes the same values. Thus, no meromorphic function can have 0 at p_1 and a pole at p_2, contradicting Theorem 41.

Next, note that the assignment $z \to g$ and $w \to h$ defines a homomorphism of $\mathbb{C}$-algebras:

$$\mathbb{C}[z, w]/(f) \to K(S)$$

If $r \in \mathbb{C}[z, w]$ is a polynomial which is sent to 0 under this homomorphism, then $r(g, h) = 0$ but this means that $r(z, w)$ vanishes identically on the curve defined by $f(z, w) = 0$. Since f is irreducible, by the Nullstellensatz this means that r is in the ideal (f).

Finally, the degree of the field extension $\mathbb{C}(g) \subset K(S)$ is equal to the degree of the minimal polynomial of h over $\mathbb{C}(g)$, namely the polynomial $f(g, T) \in \mathbb{C}(g)[T]$. This degree is equal to the degree of f as a polynomial in w when we write

$$f(z, w) = p_0(z) w^n + p_1(z) w^{n-1} + \ldots + p_n(z),$$

which is also equal to the degree of the projection $z : S \to \mathbb{P}^1$, and this is also the degree of g (since Φ is of degree 1). □

Thus, we have shown the equivalence between the following objects:

- compact connected Riemann surfaces
- function fields of one variable (i.e. finite extensions of $\mathbb{C}(X)$).
- irreducible algebraic curves: $\{(z, w) \in \mathbb{C}^2 : f(z, w) = 0\}$.

This is actually part of a (contravariant) functorial equivalence of categories, that associates to each compact connected Riemann surface S its function field $K(S)$, and to every holomorphic map $f : S_1 \to S_2$, the $\mathbb{C}$-algebra homomorphism $f^* : K(S_2) \to K(S_1)$.

5.4. **Monodromy.** Recall that the conjugacy classes of subgroups of $\pi_1(B, b_0)$ correspond to equivalence classes of covering $p : E \to B$ with E connected. The subgroups of index d correspond to d-sheeted coverings. There is another way to interpret subgroups of index d, namely, they correspond to transitive permutation representations. More precisely, suppose we have a set F of d elements and a choice of one element $f_0 \in F$. Then given an action of $\pi = \pi_1(B, b_0)$ on F (which is the same as a homomorphism to the permutations of F), the stabilizer of f_0 is a subgroup of π. If the action is transitive, the stabilizer has index d. Conversely, if H is a subgroup of index d, then π acts on the set of cosets π/H, which has d elements, and H is the stabiliser of the coset of identity. Changing the choice of the preferred point $f_0 \in F$, just changes the stabilizer by conjugation. Thus, a connected covering space of B, is equivalent to a transitive representation

$$\pi_1(B, b_0) \to \mathfrak{S}_d$$

determined up to conjugacy, where $\mathfrak{S}_d$ is the symmetric group of d elements.

Let S be connected Riemann surface. Given a morphism $f : S \to T$ of degree d let $t_1, \dots t_n \in T$ branch points, $t \in T$ a regular value, then we can construct a homomorphism

$$M_f : \pi_1(T \setminus \{t_1, \dots, t_n\}, t) \to \mathfrak{S}(f^{-1}(t))$$

called the monodromy of f as follows, where $\mathfrak{S}(f^{-1}(t))$ is the group of permutation of the finite set $f^{-1}(t)$.

Firstly since,

$$f : S \setminus \{f^{-1}(\{t_1, \dots t_n\})\} \to T \setminus \{t_1, \dots, t_n\}$$

is a covering map, we get a subgroup $H \subset \pi_1(T \setminus \{t_1, t_2, \dots, t_n\})$, hence a monodoromy homomorphism to the symmetric group. In fact, we can give a more intuitive description. Namely, if γ is a loop in T with $\gamma(0) = \gamma(1) = t$, we can lift γ to a path $\tilde{\gamma} : [0, 1] \to S \setminus \{f^{-1}(\{t_1, \dots, t_n\})\}$, starting at any given point $s \in f^{-1}(t)$, and the end point $\tilde{\gamma}(1) \in f^{-1}(t)$, then we define $M_f(\gamma)$ by

$$M_f(\gamma)(\tilde{\gamma}(1)) = \tilde{\gamma}(0)$$

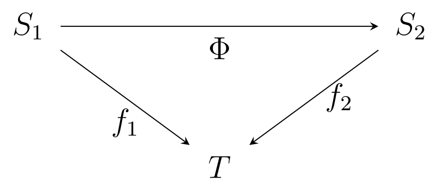
Exercise. Check that M_f is a homomorphism into $\mathfrak{S}(f^{-1}(t))$. Show that M_f is transitive (since S is connected.)

Definition 57. *Given $f : S \to T$ of degree d with S connected, and branched points $t_1, \dots, t_n$, the monodromy group of f is the*

$$\mathrm{Mon}(f) = \{M_f(\gamma) : [\gamma] \in \pi_1(T \setminus \{t_1, \dots, t_n\})\} \subset \mathfrak{S}_d$$

The classification result of coverings, together with the uniqueness of the filling in punctures result we saw above implies the following theorem.

Theorem 58. *Let $f_i : S_i \to T$ for $i = 1, 2$ be two morphisms of degree d with the same branching points $t_1, \dots, t_n$. Then f_1 and f_2 have conjugate monodromies if and only if they are isomorphic (branched) coverings, i.e. there exits an isomorphism of Riemann surfaces $\Phi : S_1 \to S_2$ such that*



An interesting special case of this result when $T = \mathbb{P}^1$ is known as the *Riemann's existence theorem*. Namely, given a topological map $f : S \to \mathbb{P}^1$ which restricts to a covering map $f : S \setminus \{f^{-1}(\{t_1, \dots t_n\})\} \to \mathbb{P}^1 \setminus \{t_1, \dots t_n\}$, which in turn is determined by the monodromy, we can construct a unique Riemann surface structure on S such that $f : S \to \mathbb{P}^1$ is holomorphic map, and any holomorphic map from a Riemann surface $S \to \mathbb{P}^1$ is given by such data.

6. Belyi's theorem

Definition 59. *We say that a Riemann surface S is defined over a field $K \subset \mathbb{C}$ if $S \cong S_f$ for some irreducible polynomial $f = \sum a_{ij} z^i w^j$ with coefficients $a_{ij} \in K$.*

For example, if S has genus 0, then S is isomorphic to $\mathbb{P}^1$, and therefore it is defined over $\mathbb{Q}$ (take $f(z, w) = z$). The Riemann surface S corresponding to the curve $z^2 = w^3 - \pi^3$ is defined over the transcendental extension $\mathbb{Q}(\pi)$ but, in fact, it is also defined over $\mathbb{Q}$ since it is isomorphic to the surface $z^2 = w^3 - 1$ via $(z, w) \mapsto (\frac{z}{\pi}, \frac{w}{\pi\sqrt{\pi}})$.

One problem of interest is to decide when is S defined over some *number field* (a finite extension of $\mathbb{Q}$). Since the number of coefficients involved in a polynomial $f(z, w)$ is finite, this question is equivalent to asking when S is defined over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$.

The following theorem of Belyi gives a very interesting answer to this problem:

Theorem 60. *Let S be a compact Riemann surface. The following statements are equivalent:*

(i) *S is defined over $\overline{\mathbb{Q}}$.*
(ii) *S admits a morphism $f : S \to \mathbb{P}^1$ with at most three branching values (which can be taken to be a subset of $\{0, 1, \infty\}$).*

We say that a meromorphic function on S is a *Belyi function* if its branching values is a subset of $\{0, 1, \infty\}$. Note that if such a function has 1 or 2 branching values then S is isomorphic to $\mathbb{P}^1$. Indeed, if there is 0 branching points, then $f : S \to \mathbb{P}^1$ is a covering map, but since $\mathbb{P}^1$ is simply-connected (i.e $\pi_1(\mathbb{P}^1) = 0$), there are no non-trivial coverings of $\mathbb{P}^1$, so f is an isomorphism.

Similarly, if ∞ is the only branching point of f, then $f : S \setminus f^{-1}(\infty) \to \mathbb{P}^1 \setminus \{\infty\}$ is a covering map and hence an isomorphism, since $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ is simply-connected. It follows that from Lemma 53 that S is isomorphic to $\mathbb{P}^1$.

Finally, if $\{0, \infty\}$ is the branching values, then, $f : S \setminus f^{-1}(\{0, \infty\}) \to \mathbb{C}^\times$ is a covering map. We have seen that all these coverings are isomorphic to $\mathbb{C}^\times \to \mathbb{C}^\times$, sending $z \to z^k$ for some k. It follows again that S is isomorphic to $\mathbb{P}^1$.

We will prove (i) implies (ii) part of Theorem 60. To warm up, we study now a special case. Consider the Riemann surface defined by the equation

$$z^2 = w(w-1)(w-\lambda), \lambda = \frac{m}{m+n}, \quad m, n \in \mathbb{N}$$

This is clearly defined over $\overline{\mathbb{Q}}$. Thus, we should be able to find a function $f : S_\lambda \to \mathbb{P}^1$ with branching points $\{0, 1, \infty\}$. The function $w : S \to \mathbb{P}^1$ defined by $(z, w) \mapsto w$ is a meromorphic function of degree 2, which has ramification over $\{0, 1, \lambda, \infty\}$. Belyi considers the polynomials

$$p_{m,n}(w) = p_\lambda(w) = \frac{(m+n)^{m+n}}{m^m n^n} w^m (1-w)^n$$

Proposition 61. *As a function $p_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$, Belyi's polynomial satisfies the following properties:*

(i) *p_λ ramifies only at the points $w = 0, 1, \infty, \lambda$.*
(ii) *$p_\lambda(0) = 0$, $p_\lambda(1) = 0$, $p_\lambda(\infty) = \infty$, and $p_\lambda(\lambda) = 1$.*

Proof. The zeroes of the derivative of p_λ are solutions to the equation:

$$w^{m-1}(1-w)^{n-1}((m+n)w - m) = 0$$

From this, the assertions above follow easily. □

We now see that the composition of the functions $w : S_\lambda \to \mathbb{P}^1$ with $p_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ produces a function

$$f = p_\lambda \circ w : S_\lambda \to \mathbb{P}^1$$

that ramifies at the points $(0, 0), (1, 0), (\lambda, 0), \infty \in S$ with branching values $0, 1, \infty \in \mathbb{P}^1$ respectively, so f is a Belyi function.

Proof of Theorem 60 (i) implies (ii):

Let S be Riemann surface that is defined over $\overline{\mathbb{Q}}$. Thus, there exists a polynomial

$$f(z, w) = p_0(z)w^n + p_1(z)w^{n-1} + \ldots + p_n(z) \in \overline{\mathbb{Q}}[z, w]$$

such that $S = S_f$.

We want to construct a function $g : S \to \mathbb{P}^1$ with ramification in $\{0, 1, \infty\}$.

First, we show that it suffices to find a function $g : S \to \mathbb{P}^1$ which is ramified over a set of rational values $\{0, 1, \infty, \lambda_1, \ldots, \lambda_n\} \subset \mathbb{Q} \cup \{\infty\}$. To see this, we observe that after composing g with Möbius transformations $T(x) = 1 - x$ and $M(x) = 1/x$ if necessary, we can assume that $0 < \lambda_1 < 1$. Therefore, λ_1 can be written in the desired form $\lambda_1 = \frac{m}{m+n}$. Composing g with the rational function p_{λ_1}, we would get a morphism $p_{\lambda_1} \circ g$ with strictly smaller number of branching values, namely $\{0, 1, \infty, p_{\lambda_1}(\lambda_2), \ldots, p_{\lambda_1}(\lambda_n)\} \subset \mathbb{Q} \cup \{\infty\}$. By iterating this argument, we arrive at a function with branching set at $\{0, 1, \infty\}$.

In order to show the existence of such a function $g : S_f \to \mathbb{P}^1$ consider the morphism $z : S_f \to \mathbb{P}^1$ given by $(z, w) \mapsto z$.

Let $B_0 = \{\mu_1, \ldots, \mu_s\}$ be the set of branching values of z. Thus μ_i is either the zero of the function $p_0(z)$ or the point $\infty \in \mathbb{P}^1$, or the first coordinate of a common zero of the polynomials $f(z, w)$ and $f_w(z, w)$ in $\overline{\mathbb{Q}}[z, w]$. The roots of $p_0(z)$ are in $\overline{\mathbb{Q}} \cup \{\infty\}$. From the proof of Proposition 9(i), we also see that the z coordinates of the common zeros of $f(z,w)$ and $f_w(z,w)$ are also in $\overline{\mathbb{Q}} \cup \{\infty\}$. Thus, $B_0 \subset \overline{\mathbb{Q}} \cup \{\infty\}$. Now, if $B_0 \subset \mathbb{Q} \cup \{\infty\}$ we are done, otherwise we use the following inductive argument.

Let $m_1(T) \in \mathbb{Q}[T]$ be the minimal polynomial of $\{\mu_1, \mu_2, \ldots, \mu_s\}$ (i.e. the monic polynomial of lowest degree that vanishes at μ_i for $i = 1, \ldots, s$ (or at μ_i for $i = 1, \ldots, (s-1)$, if $\mu_s = \infty$). Let $\{\beta_1, \beta_2, \ldots, \beta_d\}$ be the roots of $m_1'(t)$ and $p(t)$ their minimal polynomial. By definition $\deg(p(t)) \leq \deg(m_1'(t))$.

Consider the composition

$$S_f \xrightarrow{z} \mathbb{P}^1 \xrightarrow{m_1} \mathbb{P}^1$$
$$(z,w) \mapsto z \mapsto m_1(z)$$

The branching values of this composition is given by:

$$B_1 = m_1(\{\text{roots of } \ m_1'\}) \cup \{0, \infty\}$$

(This follows because $\text{Branch}(u \circ v) = \text{Branch}(u) \cup u(\text{Branch}(v))$.)

Now, if $B_1 \subset \mathbb{Q} \cup \{\infty\}$ we are done. If not, we denote by $m_2(T) \in \mathbb{Q}[T]$ the minimal polynomial of the branch value set of m_1, that is $m_1(\{\text{roots of } \ m_1'\}) = \{m_1(\beta_1), m_1(\beta_2), \ldots, m_1(\beta_d)\}$.

Clearly, $[\mathbb{Q}(m_1(\beta_i)) : \mathbb{Q}] \leq [\mathbb{Q}(\beta_i) : \mathbb{Q}]$, which means that the degree of the minimal polynomial of $m_1(\beta_i)$ is lower than or equal to the degree of the minimal polynomial of β_i. Moreover, by elementary Galois theory, two algebraic numbers β_i and β_j have the same minimal polynomial if and only if there is some field embedding $\sigma : \mathbb{Q}(\beta_i) \to \overline{\mathbb{Q}}$ such that $\sigma(\beta_i) = \beta_j$. But in that case, $\sigma(m_1(\beta_i)) = m_1(\beta_j)$, and so $m_1(\beta_i)$ and $m_1(\beta_j)$ have the same minimal polynomial. It follows that

$$\deg(m_2(T)) \leq \deg(p(T)) \leq \deg(m_1'(T)) < \deg(m_1(T))$$

Next, we consider the composition $m_2 \circ m_1 \circ z : S \to \mathbb{P}^1$. Its branching value set is

$$B_2 = m_2(\{\text{roots of } \ m_2'\}) \cup m_2(B_1)$$

By construction, $m_2(B_1)$ is $0, \infty$ and $m_2(0)$ hence $m_2(B_1) \in \mathbb{Q} \cup \{\infty\}$. If $B_2 \subset \mathbb{Q} \cup \{\infty\}$ we are done, otherwise, we continue the process by $m_3(T) \in \mathbb{Q}[T]$ the minimal polynomial of $m_2(\{\text{roots of } \ m_2'\})$. The branching values of $m_3 \circ m_2 \circ m_1 \circ z$ is given by

$$B_3 = m_3(\{\text{roots of } \ m_3'\}) \cup (m_3 \circ m_2)(B_1)$$

Again $m_3 \circ m_2(B_1) \subset \mathbb{Q} \cup \{\infty\}$ and we are done if $B_3 \subset \mathbb{Q} \cup \{\infty\}$. This process ends in finitely many steps, since each $\deg(m_{i+1}(T)) < \deg(m_i(T))$. □

Exercise. Let S_f be the compact Riemann surface defined by the irreducible polynomial

$$f(z,w) = z^2 - w(w-1)(w-\sqrt{2})$$

Use the above prescription to construct a Belyi function on S_f.

6.1. **Galois action.** Let us denote by $\text{Gal}(\mathbb{C}) = \text{Gal}(\mathbb{C}/\mathbb{Q})$ the group of all field automorphisms of $\mathbb{C}$. It is easy to see that other than the identity, the only other *continuous* field automorphism of $\mathbb{C}$ is given by complex conjugation. Using Axiom of Choice, one can show that the complex numbers have crazy automorphisms: any automorphism of any subfield of $\mathbb{C}$ can be extended to an automorphism of all of $\mathbb{C}$. (In fact, such an automorphism leaves a dense subset of $\mathbb{R}$ pointwise fixed but maps the real line onto a dense subset of the plane! [1])

For a given $\sigma \in \text{Gal}(\mathbb{C})$, we write a^σ instead of $\sigma(a)$. Given a polynomial $f(z, w) = \sum_{i,j} a_{ij} z^i w^j$, we write $f^\sigma(z, w) := \sum_{i,j} a_{ij}^\sigma z^i w^j$. In this way, σ induces automorphisms of the polynomial ring $\mathbb{C}[z, w]$ and the function field $\mathbb{C}(z, w)$.

Given a compact Riemann surface S defined by an irreducible polynomial $f \in \mathbb{C}[z, w]$ we set S^σ for the compact Riemann surface defined by the irreducible polynomial f^σ.

Given two compact Riemann surfaces S_f and S_g defined by irreducible polynomials, defining a morphism $S_f \to S_g$ is equivalent to specifying a pair of rational functions (r_1, r_2) where

$$r_i(z, w) = p_i(z, w)/q_i(z, w)$$

with $p_i, q_i \in \mathbb{C}[z, w]$ and $q_i(z, w)$ not divisible by f such that

$$q_1(z, w)^m q_2(z, w)^n g(r_1, r_2) = h(z, w) f(z, w)$$

for some $h \in \mathbb{C}[z, w]$, where n and m are defined by writing

$$g(z, w) = a_0(z)w^n + a_1(z)w^{n-1} + \ldots + a_n(z) = b_0(w)z^m + b_1(w)z^{m-1} + \ldots + b_m(w)$$

Thus, by sending (r_1, r_2) to (r_1^σ, r_2^σ), we also get a map sending $\Phi : S_f \to S_g$ to $\Phi^\sigma : S_{f^\sigma} \to S_{g^\sigma}$. In particular, meromorphic functions $K(S_f)$ can be described as $r(z, w) = p(z, w)/q(z, w)$, and we put $r^\sigma(z, w) = p^\sigma(z, w)/q^\sigma(z, w)$ for the corresponding meromorphic function in $K(S_{f^\sigma})$.

Next, recall that S_f is constructed from a non-compact algebraic curve $S_f^z \subset \mathbb{C}^2$ by adding finitely many points. Given a point $(a, b) \in S_f^z$, we get a point $(a^\sigma, b^\sigma) \in S_{f^\sigma}^z$. It is not too hard to show that this extends to S_f to give a bijection between points of S_f and S_{f^σ}. (This requires a digression to theory of valuations, which we will skip. If you are curious, you can read it in Section 3.4 of the book of Girondo-Gonzalez-Diez.)

Note that this bijection S_f to S_{f^σ} is not continuous (except if σ is the complex conjugation.)

We will be concerned with action of $\text{Gal}(\mathbb{C})$ on pairs (S, f) where S is a compact Riemann surface and $f : S \to \mathbb{P}^1$ is a meromorphic function. We summarize the elementary properties of this action below without proof.

Theorem 62. *Let S be a compact Riemann surface, $f \in K(S)$, and $\sigma \in \text{Gal}(\mathbb{C})$*

- $deg(f^\sigma) = deg(f)$
- $f(p)^\sigma = f^\sigma(p^\sigma)$
- $v_{f^\sigma}(p^\sigma) = v_f(p)$
- $a \in \mathbb{P}^1$ *is a branching point of f if and only if a^σ is a branching point of f^σ*

[1] Paul B. Yale. Automorphisms of the complex numbers, Math. Mag. 39 (1966), 135-141

- *The genus of S^σ is the same as the genus of S, i.e. S and S^σ are homeomorphic.*
- *$Aut(S,f)$ and $Aut(S^\sigma, f^\sigma)$ are isomorphic.*
- *$Mon(S,f)$ and $Mon(S^\sigma, f^\sigma)$ are isomorphic.*

Proof of Theorem 60 (ii) implies (i): To prove this, we will need the following theorem which we accept without proof (see Section 3.7 of Girondo-Gonzalez-Diez for a proof).

Theorem 63. *For a compact Riemann surface S the following conditions are equivalent:*

- *S is defined over $\overline{\mathbb{Q}}$.*
- *The family $\{S^\sigma\}_{\sigma \in \mathrm{Gal}(\mathbb{C})}$ contains only finitely many isomorphism classes of Riemann surfaces.*

With this criterion at hand, we complete the proof of Belyi's theorem as follows. If $f : S \to \mathbb{P}^1$ is a morphism of degree d whose only branching values are $0, 1, \infty$, then for any $\sigma \in \mathrm{Gal}(\mathbb{C})$, the conjugated morphism $f^\sigma : S^\sigma \to \mathbb{P}^1$ is a morphism of the same degree d having the same branching values $\sigma(0) = 0$, $\sigma(1) = 1$, $\sigma(\infty) = \infty$. In particular, this family $\{f^\sigma\}$ of morphisms gives rise to only finitely many different monodromy homomorphisms $M_{f^\sigma} : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\} \to \mathfrak{S}_d$.(note that the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the free group in two generators). Therefore, by Theorem 58 it follows that among all the Riemann surfaces $\{S^\sigma\}_{\sigma \in \mathbb{C}}$, there are only finitely many equivalence classes.

7. Dessins d'enfants

Definition 64. *A dessin d'enfant, or simply a dessin, is a pair $(X, \mathcal{D})$ where X is an oriented compact topological surface, and $\mathcal{D} \subset X$ is a finite graph such that:*

(i) *$\mathcal{D}$ is connected.*
(ii) *$\mathcal{D}$ is bicoloured, i.e. the vertices are coloured either black or white, and the vertices connected by an edge have different colours.*
(iii) *$X \setminus \mathcal{D}$ is the union of finitely many topological disks, which we call facets of $\mathcal{D}$.*

(In fact, the condition (i) is a consequence of condition (iii)).

It is important to keep the data of the embedding of the dessin $\mathcal{D}$ in the topological surface X.

The following are examples from Fig 4.1 of [GGD].

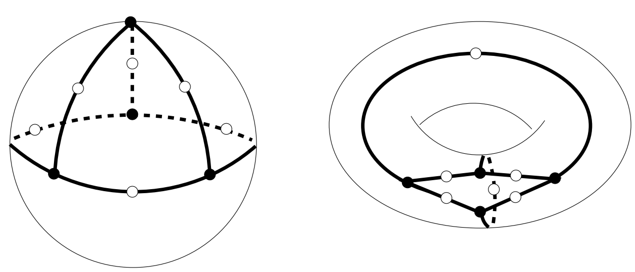


Figure 2. tetrahedral graph

Two dessins are said to be equivalent if there is an orientation preserving homeomorphism of the underlying surface whose restriction to the bicoloured graphs is an isomoprhism of coloured graphs.

7.1. **Representation of dessins by a pair of permutations.** Suppose $(X, \mathcal{D})$ is a dessin with N edges, label them with integer numbers from 1 to N. Consider two permutations $\sigma_0, \sigma_1 \in \mathfrak{S}_N$ defined as follows.

Around each vertex of the dessin, we can draw a little disk which has an induced orientation from the orientation of the surface, hence it makes sense to talk about counter-clockwise rotation in this disk. The edge labelled i is incident to a white vertex, at such a vertex, we set $\sigma_0(i) = j$, where j is the edge that follows i in the counter-clockwise direction (among edges incident to that particular white vertex). We define σ_1 using black vertices in a similar fashion.

Fig. 4.2 of [GGD] reproduced below is helpful in understanding the permutation representation of a dessin.

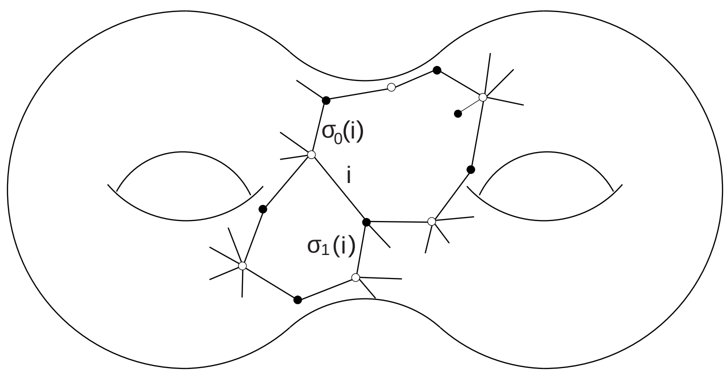


FIGURE 3. Permutation representation of dessins

Definition 65. *The pair (σ_0, σ_1) is called the permutation representation of the dessin.*

Relabelling the edges of the dessin gives a new permutation (σ_0', σ_1') that is related to (σ_0, σ_1) by conjugation with $(\tau\sigma_0\tau^{-1}, \tau\sigma_1\tau^{-1})$ for some τ.

The cycles of σ_0 are in one-to-one correspondence with white vertices of $\mathcal{D}$. The cycles of σ_1 are in one-to-one correspondence with black vertices of $\mathcal{D}$. The length of each cycle being the degree of the corresponding vvertex.

The cycles of $\sigma_1\sigma_0$ (or equivalently, those of $\sigma_0\sigma_1$ are in one-to-one correspondence with faces of $\mathcal{D}$.

These observations allow us to compute the Euler characteristic as follows.

Proposition 66. *Let g be the genus of X. Then the following formula holds:*

$$2 - 2g = \{\#\{cycles\ of\ \sigma_0\} + \#\{cycles\ of\ \sigma_1\}\} - N + \#\{cycles\ of\ \sigma_1\sigma_0\}$$

Example. It is possible to choose a label in for the dessins in Figure 2 such that in the first case we have $\sigma_0 = (1,10)(2,4)(3,9)(5,12)(6,7)(8,11)$, $\sigma_1 = (1,2,3)(4,5,6)(7,8,9)(10,11,12)$. The cycle decomposition of $\sigma_1\sigma_0$ is given as $(1,11,9)(2,5,10)(3,7,4)(6,8,12)$.

In the second case, we have $\sigma_0 = (1,10)(2,4)(3,8)(5,12)(6,7)(9,11)$, and $\sigma_1 = (1,2,3)(4,5,6)(7,8,9)(10,11,12)$, $\sigma_1\sigma_0 = (1,11,7,4,3,9,12,6,8)(5,10,2)$.

One can compute from this that in the first case we have $g = 0$, and in the second we have $g = 1$ in accordance with Figure 2.

Proposition 67. *Let σ_0 and σ_1 be two permutations in $\mathfrak{S}_N$ such that $\langle \sigma_0, \sigma_1 \rangle$ is a transitive group. There exists a dessin d'enfant $(X, \mathcal{D})$ such that its permutation representation pair is precisely (σ_0, σ_1)*

Proof. First we write cycle decomposition of $\sigma_1\sigma_0$ as

$$\sigma_1\sigma_0 = \tau_1 \dots \tau_k$$

where τ_j has order n_j and $\sum_{j=1}^k n_j = N$. Accordingly, we consider k faces (for the moment, disjoin) bounded by $2n_1$, $2n_2, \dots, 2n_k$ edges respectively. After assigning white and black colours to the vertices of each face, we label half of the edges as prescribed by the cycles of $\sigma_1\sigma_0$ and then we use σ_0 to label the remaining edges. It suffices now to glue together these pieces along edges with the same lable in order to form a connected surface. Note that transitivity of $\langle \sigma_0, \sigma_1 \rangle$ ensures that no face can remain disconnected from the rest. □

Definition 68. *The subgroup $\langle \sigma_0, \sigma_1 \rangle$ of $\mathfrak{S}_N$ generated by σ_0 and σ_1 is called the monodromy group of the dessin, and we will denote it by* $\mathrm{Mon}(\mathcal{D})$.

Exercise. Let $\sigma_0 = (1,5,4)(2,6,3)$, $\sigma_1 = (1,2)(3,4)(5,6)$. Construct the corresponding surface and the dessin d'enfant on it.

7.2. **The Belyi function associated to a dessin.** The goal of this section to prove the following theorem

Theorem 69. *Given a dessin $(X, \mathcal{D})$ there exists a Riemann surface $S_{\mathcal{D}}$ and a Belyi function $f_{\mathcal{D}} : S_{\mathcal{D}} \to \mathbb{P}^1$ such that X is homeomorphic to $S_{\mathcal{D}}$ and $\mathcal{D} = f^{-1}[0,1]$.*

Furthermore, the association $(X, \mathcal{D}) \to (S_{\mathcal{D}}, f_{\mathcal{D}})$ sends equivalent dessins to equivalent Belyi pairs inducing a bijection of equivalence classes.

For example, a Belyi function for the tetrahedron is given by

$$f(z) = -64\frac{(z^3+1)^3}{(z^3-8)^3 z^3}$$

and a Belyi function for the cube is given by

$$f(z) = -108\frac{(z^4+1)^4 z^4}{(z^8-14z^4+1)^3}$$

(See "Magot-Zvonkin Belyi functions for Archimedean solids" for these and other examples.)

Proof. To prove this theorem, we will first construct a *triangle decomposition* of X, by which we mean a collection of triangles that cover the whole of X, and such that the intersection of two triangles consists of a union of edges or vertices. Note that the triangle decompositions need not be triangulations, as triangles are allowed to meet at more than one edge. See the following figure taken from [GGD]:

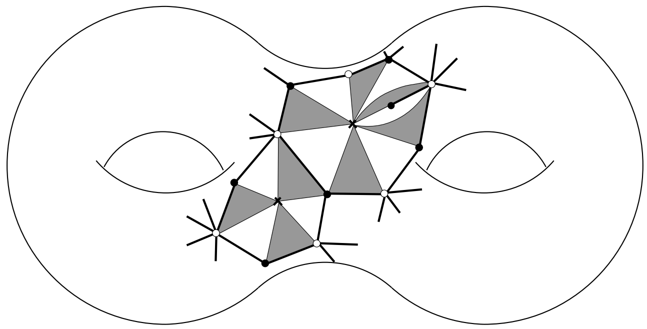


FIGURE 4. Triangle decomposition

To construct the triangle decomposition, we choose a centre labelled by $\times$ at each face of $X \setminus \mathcal{D}$. Now, for a vertex v, let $F_{v,1}, \ldots, F_{v,d}$ be the set of faces which have a vertex in v. For each certex v, choose topological paths $\gamma_{v,i}$, for $i = 1, \ldots d$, such that $\gamma_{v,i}$ stays inside the face $F_{v,i}$ and connects the vertex v to the centre $\times$ of $F_{v,i}$ by a topological path.

This gives a triangle decompostion $\mathcal{T}$ of X, such that each triangle contains one vertex of each type: $\circ$, $\bullet$, $\times$.

Using the orientation of X we distinguis two types of triangles. We call a triangle T positive if the circuite $\circ \to \bullet \to \times \to \circ$ follows the positive orientation of ∂T. We call T negative otherwise. The positive triangles are white and the negative ones are shaded in Figure 5.

Note that each edge of $\mathcal{D}$ belongs to exactly two triangles (of different type).

The most simple dessin on the sphere has a triangle decomposition given as follows:

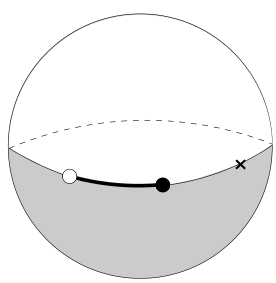


FIGURE 5. Simplest dessin on $\mathbb{P}^1$.

Let us write $T_0^+ \cup T_0^- = \mathbb{P}^1$ for the triangle decomposition of $\mathbb{P}^1$ given as above, where T_0^+ and T_0^- correspond to northern and southern hemispheres, respectively. Note that these meet along their boundary given by $\mathbb{R} \cup \{\infty\}$.

Now, given $(X, \mathcal{D})$, define a function $f : X \to \mathbb{P}^1$ as follows: For all positive/negative triangles $T^{\pm}$ choose a homeomorphism $f^{\pm} : T^{\pm} \to T_0^+$

$$f^{\pm} : \begin{cases} \partial T^{\pm} & \to \mathbb{R} \cup \{\infty\} \\ \circ & \to 0 \\ \bullet & \to 1 \\ \times & \to \infty \end{cases}$$

such that if T_j^+ and T_j^- are triangles adjacent to the edge lablelled j, then the chosen homeomorphisms f_j^+ and f_j^- agree along the edge j.

These homeomorphisms glue together to a topological map $f : X \to \mathbb{P}^1$ whose restriction $f : X \setminus f^{-1}(\{0, 1, \infty\}) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is a covering map. This allows us to equip $X \setminus f^{-1}(\{0, 1, \infty\})$ with the unique Riemann surface structure that makes f holomorphic. We can then extend this Riemann surface structure to uniquely to X using Lemma 52.

Choosing different centres of the faces or different paths connecting the centres to the vertices results in a different triangle decomposition of X. However, it is easy to see that there is a homeomorphism of X preserving the dessin $\mathcal{D}$ that sends the two triangle decompositions and commuting with the associated maps from X to $\mathbb{P}^1$. Moreover, if $F : X \to X$ is any orientation-preserving homeomorphism, a triangle decomposition associated to $\mathcal{D}$ is sent to a triangle decomposition associated to $F(\mathcal{D})$ and the resulting Belyi pairs $(S_{\mathcal{D}}, f_{\mathcal{D}})$ and $(S_{F(\mathcal{D})}, f_{F(\mathcal{D})})$ are equivalent.

Note that in the above construction $\mathcal{D}$ is recovered as $f^{-1}[0, 1]$ with $f^{-1}(0)$ the set of white vertices and $f^{-1}(1)$ the set of black vertices. The triangle decomposition of X has edges given by the union of the sets $f^{-1}[0, 1]$, $f^{-1}[1, \infty]$ and $f^{-1}[-\infty, 0]$.

Thus, given a holomorphic function $f : S \to \mathbb{P}^1$ which ramifies only over $\{0, 1, \infty\}$. We let X to be the underlying topological space of S and $\mathcal{D} = f^{-1}([0, 1])$. Some simple topological considerations show that this gives the inverse to map $(X, \mathcal{D}) \to (S_{\mathcal{D}}, f_{\mathcal{D}})$ up to equivalence.

□

Given $(X, \mathcal{D})$, the function $f_{\mathcal{D}}$ that we constructed has the following properties.

- $f_{\mathcal{D}}$ only ramifies along points labelled $\circ, \bullet, \times$, and sends those points to $0, 1, \infty$ respectively.
- $deg(f_{\mathcal{D}})$ agree with the number of edges of $\mathcal{D}$ as can be seen by counting the number of preimages of $1/2$.
- The multiplicity at a vertex v of $\mathcal{D}$ is half the number of triangles surrounding v.
- The multiplicity at a vertex $\times$ of a face of $\mathcal{D}$ equals half the number of edges of that face (an edge that belongs at both sides of the same face is counted twice).
- $\mathrm{Mon}(\mathcal{D}) = \mathrm{Mon}(f_{\mathcal{D}})$.

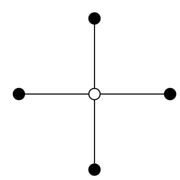
7.3. **Plane trees and Shabat polynomials.** We shall give an exposition of the general theory above in the case $X = \mathbb{P}^1$. Recall that all maps $\mathbb{P}^1 \to \mathbb{P}^1$ are given by $\mathbb{C}(z)$, the field of rational functions. We will restrict our attention to just polynomials $\mathbb{C}[z]$. Recall that a polynomial $p(z)$ of degree n has a unique pole of multiplicity n at ∞.

Definition 70. *A polynomial $p(z) \in \mathbb{C}[z]$ with at most two critical values is called a Shabat polynomial.*

Any Shabat polynomial $p(z)$ with critical values at y_1 and y_2 maybe normalized by considering $p(z) - y_1/(y_2 - y_1)$ so that the critical values are at 0 and 1.

A Shabat polynomial $p : \mathbb{P}^1 \to \mathbb{P}^1$ defines a Belyi function, and since the only pole is at ∞, the associated dessin is a *bicolored plane tree.*

Examples. 1) The simplest Shabat polynomials are $p(z) = z^n$ for $n > 0$. This has only one critical point of order n at 0 with one critical value at 0. The associated dessin is the star-shaped tree given by the following picture for $n = 4$:



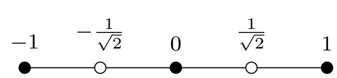
The permutation pair associated to an n-star is $\sigma_1 = (1, 2, \ldots, n)$, $\sigma_2 = \mathrm{Id}$.

2) Let us consider the Tchebychev polynomial T_n characterized by the equality $T_n(\cos z) = \cos(nz)$. $T_0(z) = 1$, $T_1(z) = z$, $T_2(z) = 2z^2 - 1$, $T_3(z) = 4z^3 - 3z$, $T_4(z) = 8z^4 - 8z^2 + 1$. In general, the explicit formula is

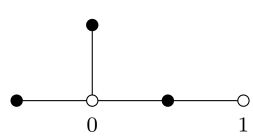
$$T_n(z) = \sum_{k \text{ even}} (-1)^{k/2} \binom{n}{n-k} z^{n-k} (1 - z^2)^{k/2}$$

as can be seen from De Moivre's identity $(\cos z + i \sin z)^n = \cos(nz) + \sin(nz)$.

Now, since $T'_n(z) = 0$ if and only if $z = \cos \frac{d\pi}{n}$ for $d = 1, \ldots, (n-1)$, and $T_n(\cos \frac{d\pi}{n}) = \pm 1$, we see that T_n has three ramification values $\{1, -1, \infty\}$, that can be send to $\{0, 1, \infty\}$ by considering instead $\frac{1+T_n(z)}{2}$. A straightforward computation shows that all finite ramification points of $T_n(z)$ have branching order 2, therefore the dessin corresponding the Belyi function $\frac{1+T_n(z)}{2}$ is a linear graph in n edges, as depicted below for $n = 4$,



3) Let us consider the following tree with 4-edges:



We placed the white vertices to be at 0 and 1, thus the corresponding polynomial of degree 4 has to be of the form:

$$p(z) = cz^3(z - 1)$$

for some constant $c \neq 0$. To determine the constant, observe that we know a priori that f has to have exactly one point α of multiplicity 2 corresponding to the black vertex between the two white vertices. We must have that $p'(z)$ vanishes to order 1 at α and $p(\alpha) = 1$. Computing the derivative of p, we see

$$p'(z) = cz^2(4z-3)$$

Thus $\alpha = 3/4$, and $p(3/4) = -\frac{27c}{256}$, hence the Shabat polynomial is

$$p(z) = -\frac{256}{27}z^3(z-1)$$

4) Consider the elliptic curves of the form

$$y^2 = x(x-1)(x-\lambda)$$

for some $\lambda \neq 0, 1$. By applying the change of variable $x \mapsto x + \frac{\lambda+1}{3}$ and $y \mapsto \frac{y}{2}$, we see that these curves are equivalent to:

$$y^2 = 4(x+\frac{\lambda+1}{3})(x+\frac{\lambda-2}{3})(x+\frac{1-2\lambda}{3}) = 4x^3+4\frac{-\lambda^2+\lambda-1}{3}x+4\frac{-2\lambda^3+3\lambda^2+3\lambda-2}{27}$$

We have seen (as part of the homework) that the j-invariant of such a curve is given by:

$$j(\lambda) = (4\frac{\lambda^2-\lambda+1}{3})^3 / \left((4\frac{\lambda^2-\lambda+1}{3})^3 - 27(4\frac{2\lambda^3-3\lambda^2-3\lambda+2}{27})^2\right)$$
$$= \frac{4(\lambda^2-\lambda+1)^3}{4(\lambda^2-\lambda+1)^3-(2\lambda^3-3\lambda^2-3\lambda+2)^2} = \frac{4}{27}\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$

Two such curves C_λ and C_μ are isomorphic if and only if $j(\lambda) = j(\mu)$.

The j-function viewed as a rational function on $\mathbb{P}^1$ is a Belyi function. To see this, note first that the two obvious ramification values of j are at ∞ (which is attained at each of the points $0, 1$ and ∞ with multiplicity 2), and 0 (attained at the two roots of the polynomial $\lambda^2 - \lambda + 1$ with multiplicity 3). Morever, the computation of the derivative gives:

$$j'(\lambda) = \frac{4}{27}\frac{(\lambda^2-\lambda+1)^3}{\lambda^3(\lambda-1)^3}(2\lambda-1)(\lambda^2-\lambda-2)$$

Thus, j has three more branching points, located at $\lambda = 1/2$ and the two roots of $\lambda^2-\lambda-2$. A simple calculation shows that j maps these three points to 1. Thus, the bracnhing values of j are $\{0, 1, \infty\}$. Hence, j is a Belyi function.

Putting together all the relevant information about zeroes, poles, multiplicities and so on we find that the corresponding dessin $\mathcal{D}$ has six edges, two white vertices of degree 3, three black vertices of degree 2 and three faces, each of them with four edges on their boundary. One can easily get convinced that the only possible graph fulfilling all these requirements is the one shown in Figure 6

From Figure 6 one sees that the permutation representation pair of $\mathcal{D}$ is

$$\sigma_0 = (1,2,3)(4,5,6), \sigma_1 = (1,4)(2,6)(3,5).$$

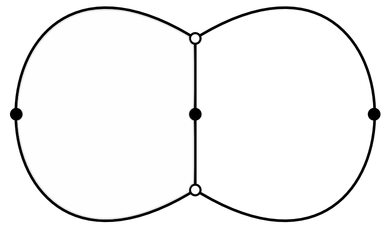
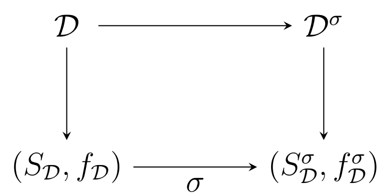


FIGURE 6. j-function

7.4. **The action of** $\mathrm{Gal}(\overline{\mathbb{Q}})$ **on dessins.** Previously, we have seen how $Gal(\mathbb{C})$ acts on compact Riemann surfaces and morphisms by conjugation on the coefficients of the polynomials describing these objects. Since Belyi pairs $(X, \mathcal{D})$ are defined over $\mathbb{Q}$ the absolute Galois group $\mathrm{Gal}(\mathbb{Q})$ acts on them in the same way, and therefore we can make $\mathrm{Gal}(\mathbb{Q})$ act on the dessins themselves. The action of an element $\sigma \in \mathrm{Gal}(\mathbb{Q})$ is defined via the following diagram:

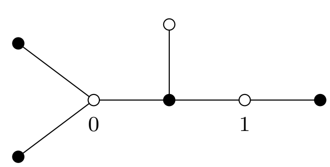


The following properties of this action are consequences of Theorem 62:

Theorem 71. *The following properties of the dessin $\mathcal{D}$ remains invariant under the action of* $\mathrm{Gal}(\overline{\mathbb{Q}})$*:*

- *The number of edges.*
- *The number of white vertices, black vertices and faces.*
- *The degree of the white vertices, black vertices and faces.*
- *The genus.*
- *The monodromy group.*
- *The automorphism group.*

Example. Let $\mathbb{D}$ be the tree given below.



Let us assume that the white vertices labelled by 0 and 1 are at 0 and 1, and the remaining white vertex is at a. The corresponding Belyi function is of the form

$$f(z) = cz^3(z-1)^2(z-a)$$

Computing the derivative we get:

$$f'(z) = cz^2(z-1)(6z^2 + (-5a-4)z + 3a)$$

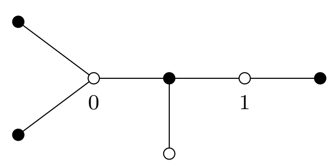
As $\mathcal{D}$ has a black vertex of degree 3, f must have a branch point α of order 3, distinct from 0 and 1, that occurs as a double root of f'. Thus, the discriminant of $6z^2 + (-5a-a)z + 3a$ must vanish, thus

$$25a^2 - 32a + 16 = 0$$

Hence a can be either $a_1 = \frac{4}{25}(4+3i)$ or $a_2 = \frac{4}{25}(4-3i)$. Correspondingly α will be either $\alpha_1 = \frac{3+i}{5}$ or $\alpha_2 = \frac{3-i}{5}$. These in turn determined the value of the constant c. Using the condition that $f(\alpha_1) = 1$, we get $c = \frac{3+i}{5}$. Using the condition $f(\alpha_2) = 1$, we get $c = \frac{3-i}{5}$. As a result, we get the following two possible candidates:

$$f_1(z) = \frac{3+i}{5} z^3 (z-1)^2 \left(z - \frac{4}{25}(4+3i) \right)$$
$$f_2(z) = \frac{3-i}{5} z^3 (z-1)^2 \left(z - \frac{4}{25}(4-3i) \right)$$

Now, we observe that the dessin $\overline{\mathcal{D}}$ given in the following figure also has the properties that would lead us to conclude that its Belyi function is either f_1 or f_2:



It is easy to see that there is no orientation-preserving homeomorphism of the sphere sending $\mathcal{D}$ to $\overline{\mathcal{D}}$ we may conclude that, as claimed, f_1 and f_2 correspond to the two non-equivalent dessins $\mathcal{D}$ and $\overline{\mathcal{D}}$.

Furthermore, they are related by the action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ as we have $f_2 = f_1^\sigma$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ with $\sigma(i) = -i$. Furthermore, since these kind of elements are the only elements that act non-trivially on f_1, it follows that $\{\mathcal{D}, \overline{\mathcal{D}}\}$ is the complete $\mathrm{Gal}(\mathbb{Q})$ orbit.

By computing the inverse image of the segment $[0,1]$, it can be directly seeen that $f_1 = f_{\mathcal{D}}$ and $f_2 = f_{\overline{\mathcal{D}}}$.

We now prove the faithfulness of the Galois action on dessin d'enfants of genus 0.

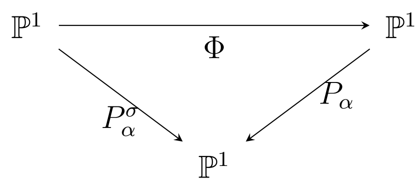
Theorem 72. *The action of* $\mathrm{Gal}(\overline{\mathbb{Q}})$ *on Shabat polynomials is faithful. In particular,* $\mathrm{Gal}(\overline{\mathbb{Q}})$ *acts faitfhully on dessins of genus 0.*

Proof. Let $\alpha \in \overline{\mathbb{Q}}$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ such that $\sigma(\alpha) \neq \alpha$. We need to construct a Shabat polynomial which is not preserved by σ. Let us consider the polynomial given by the indefinite integral

$$p_\alpha(z) = \int z(z-1)^2(z-\alpha)^3 dz \in \mathbb{Q}(\alpha)[z]$$

The branching values of p_α are $\{p_\alpha(0), p_\alpha(1), p_\alpha(\alpha), \infty\} \subset \overline{\mathbb{Q}} \cup \{\infty\}$. Applying the algorithm given in the proof of Theorem 60, we can find a polynomial $q_\alpha \in \mathbb{Q}[z]$ such that $P_\alpha = q_\alpha \circ p_\alpha$ is a Belyi function, i.e. a Shabat polynomial.

The conjugation action of σ on P_α is given by $P_\alpha^\sigma = q_\alpha^\sigma \circ p_\alpha^\sigma = q_\alpha \circ p_{\sigma(\alpha)}$. We want to show that the Shabat poynomial P_α^σ is not equivalent to the Shabat polynomial P_α. Suppose for contradiction that there is such an equivalence. This means that there is an automorphism Φ of $\mathbb{P}^1$ such that the following diagram commutes:



Recall all automorphisms of $\mathbb{P}^1$ are given by Möbius transformations. In our situation, we also have $\Phi(\infty) = \infty$, therefore, it follows that $\Phi(z) = az + b$ for some $a \neq 0$ and b. Then, we have:

$$q_\alpha(p_\alpha(az + b)) = P_\alpha(az + b) = P_\alpha^\sigma(z) = q_\alpha(p_{\sigma(\alpha)}(z))$$

Since $p_\alpha(az+b)$ and $p_{\sigma(\alpha)}(z)$ are of the same degree, an elementary term by term comparison using the above identity implies that

$$p_\alpha(az + b) = cp_{\sigma(\alpha)}(z) + d$$

(see Lemma 73 below.).

Now, the ramification points of p_α are 0,1 and α with multiplicities $2, 3$ and 4. This implies that $p_\alpha(az + b)$ has ramification points given by $-b/a$, $\frac{1-b}{a}$ and $\frac{\alpha-b}{a}$. On the other hand, the ramification points of $cp_{\sigma(\alpha)} + d$, agrees with the ramification points of $p_{\sigma(\alpha)}$ and these are $0, 1$ and $\sigma(\alpha)$ with multiplicity 2,3 and 4. It follows that $b = 0$, $a = 1$ and $a\sigma(\alpha) + b = \sigma(\alpha) = \alpha$, which is a contradiction. □

Lemma 73. *Let h_1, h_2 be two monic polynomials of the same degree such that $h_1(0) = h_2(0) = 0$. Assume that there exist polynomials g_1, g_2 such that $g_1 \circ h_1 = g_2 \circ h_2$, then $h_1 = h_2$. More generally, if h_1 and h_2 are two polynomials (not necessarily monic or vanishing at 0), under the same hypothesis, we get $h_2 = ch_1 + d$ for some constants c, d.*

Proof. Write

$$h_1(z) = z^m + \alpha_{m-1}z^{m-1} + \ldots + \alpha_1 z$$
$$h_2(z) = z^m + \beta_{m-1}z^{m-1} + \ldots + \beta_1 z$$
$$g_1(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$$
$$g_2(z) = b_n z^n + b_{n-1}z^{n-1} + \ldots + b_0$$

We have

$$g_1 \circ h_1(z) = a_n(z^m + \alpha_{m-1}z^{m-1} + \ldots + \alpha_1 z)^n + \ldots + a_0$$
$$g_2 \circ h_2(z) = b_n(z^m + \beta_{m-1}z^{m-1} + \ldots + \beta_1 z)^n + \ldots + b_0$$

Comparing the highest degree terms, we get $a_n = b_n$. Comparing terms of degree $nm-1$, we get $na_n\alpha_{m-1} = nb_n\beta_{m-1}$, hence $\alpha_{m-1} = \beta_{m-1}$.

In general, it is easy to see that terms of degree $nm-j$, gives an identity which implies $\alpha_{m-j} = \beta_{m-j}$.

For the second part, just consider $\tilde{h}_1 = \frac{h_1-\alpha_0}{\alpha_m}, \tilde{h}_2 = \frac{h_2-\beta_0}{\beta_m}, \tilde{g}_1 = g_1(\alpha_m z + \alpha_0), \tilde{g}_2 = g_2(\beta_m z + \beta_0)$. □

In the case of genus 1, the faithfulness of the action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ follows easily using the j-invariant as follows:

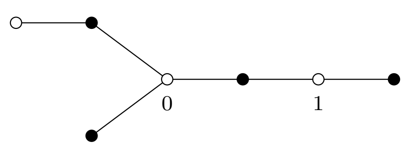
Proposition 74. $\mathrm{Gal}(\mathbb{C})$ *acts faithfully on the isomorphism classes of compact Riemann surfaces of genus 1.*

Proof. Let $\sigma \in \mathrm{Gal}(\mathbb{C})$ and $z \in \mathbb{C}$ such that $\sigma(z) \neq z$. Consider the curves C_λ defined by $y^2 = x(x-1)(x-\lambda)$. Take λ with $j(\lambda) = z$. Clearly, we have $C_\lambda^\sigma = C_{\lambda^\sigma}$ which has j-invariant equal to $j(\lambda^\sigma) = j(\lambda)^\sigma = \sigma(z) \neq z$, therefore it cannot be isomorphic to C_λ. □

As a corollary, we see that $\mathrm{Gal}(\overline{\mathbb{Q}})$ also acts faithfully on dessins of genus 1.

It is also true that $\mathrm{Gal}(\overline{\mathbb{Q}})$ acts faithfully on dessin of genus > 1. You can read a proof in Theorem 4.53 of [GGD].

We will end with an interesting example. Consider the tree $\mathcal{D}$ given by the following figure:



Let $z = a$ be the remaining white vertex. The Belyi function associated $\mathcal{D}$ is of the form:

$$f(z) = cz^3(z-1)^2(z-a)$$

Taking the derivative, we see

$$f'(z) = cz^2(z-1)(6z^2 - (5a+4)z + 3a)$$

Let $g(z) = 6z^2 - (5a+4)z + 3a$. Now, since there are two black vertices of ramification order 2, the last factor, $g(z)$ in $f'(z)$ should have non-vanishing discriminant, i.e.

$$25a^2 - 32a + 16 \neq 0$$

Let w_1, w_2 be the two distinct roots of $g(z)$. Consider the Euclidean division

$$f(z) = q(z)g(z) + r(z)$$

where $\deg(r(z)) \leq 1$, hence $r(z) = Az + B$ for some A, B. Now, if we evaluate the last equation at w_1 and w_2, we get:

$$Aw_1 + B = f(w_1) = 1 = f(w_2) = Aw_2 + B$$

but since $w_1 \neq w_2$, it follows that $A = 0$. By formally applying the Euclidean division of f by g, we get:

$$A = \frac{-c}{6^5}(25a^2 - 32a + 16)(25a^3 - 12a^2 - 24a - 16)$$

and

$$B = \frac{c}{2^5 3^4} a(5a - 8)(25a^3 - 6a^2 + 8)$$

The condition $A = 0$, together with the non-vanishing of $25a^2 - 32a + 16$ implies that

$$25a^3 - 12a^2 - 24a - 16 = 0$$

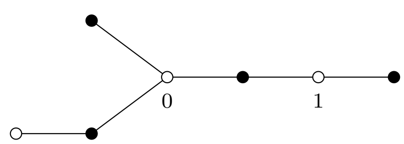
We obtain three possible values for a as the roots of this polynomial. Call them a_1, a_2 and a_3. For each such value we get a Belyi function.

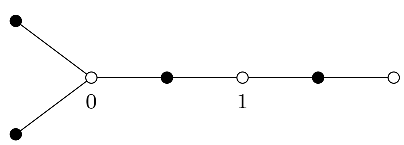
$$f_k(z) = c_k z^3 (z - 1)^2 (z - a_k)$$

Furthermore, the value of c_k can be determined by $1 = f_k(w_1) = B$, which yields

$$c_k = \frac{2^5 3^4}{a_k(5a_k - 8)(25a_k^3 - 6a_k^2 + 8)}$$

The following two trees have the same properties that we used above, so the above arguments apply to them.





Furthermore, it is easy to see that there is no orientation-preserving homeomorphisms between any two of the three graphs.

If σ is a Galois group element that permutes the roots of the polynomial $25a^3 - 12a^2 - 24a - 16$, then it will also permute the polynomials $\{f_1, f_2, f_3\}$, and this is the only kind of Galois conjugation that acts non-trivially on them.

We conclude that the three dessins drawn form a Galois orbit with three elements. This example illustrates how the Galois group action is far from being a continuous operation.

It remains an open problem to determine the orbit decomposition of the Galois group action on dessins...