Algebraic Curves – Exam 2019

1) (i) (10/25) Prove directly that the projective curve defined by $F(X, Y, Z) = XZ - Y^2$ is isomorphic to $\mathbb{P}^1$.

(ii) (10/25) Consider the family of elliptic curves in $\mathbb{P}^2$ given by

$$C_a = \{([X : Y : Z] \in \mathbb{P}^2 : X^3 + Y^3 + Z^3 = 3aXYZ\}, \quad a \in \mathbb{C}$$

Determine the values of a for which C_a is singular, and for those values of a for which C_a is singular, determine all the singular points of C_a.

(iii) (5/25) For singular C_a show that each irreducible component of C_a is isomorphic to $\mathbb{P}^1$. (HINT: Show that the lines in $\mathbb{C}P^2$ joining the singular points of C_a are components of C_a. It suffices to do this only for one singular curve C_a as the other cases are similar).

2) (i) (5/25) Let f be a holomorphic map $S_h \to S_g$ between compact Riemann surfaces of genus h and g respectively. State the definition of the branching index b_f. State the Riemann-Hurwitz formula.

(ii) (10/25) Show that if $h < g$ then the only holomorphic maps between S_h and S_g are the constant maps.

(iii) (10/25) Let X_F be the projective curve of degree d defined by the homogeneous polynomial $F(X, Y, Z) = X^d + Y^d + Z^d$, called the *Fermat curve*. Let $\pi : X_F \to \mathbb{P}^1$ be given by $\pi([X : Y : Z]) = [X : Y]$.

Check that Fermat curve is smooth. Show that π is a well-defined map of degree d. Use Riemann-Hurwitz formula to compute the genus of X to be

$$g(X_F) = \frac{(d-1)(d-2)}{2}.$$

3)(i) (10/25) Let $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. For fixed such τ, consider the following function

$$\vartheta(z; \tau) = \sum_{n \in \mathbb{Z}} e^{\pi\, in^2\tau + 2\pi inz}$$

Show that the series defining $\vartheta(z; \tau)$ converges for all $z \in \mathbb{C}$.

(ii) Fix τ with $\mathrm{Im}(\tau) > 0$ and write $\vartheta(z) = \vartheta(z; \tau)$. Verify the following identities.

a) (5/25) $\vartheta(z + 1) = \vartheta(z)$

b) (5/25) $\vartheta(z + \tau) = e^{-\pi\, i\tau - 2\pi\, iz}\vartheta(z)$.

c) (5/25) For $a, b \in \mathbb{Z}$, $\vartheta(z + a + b\tau) = e^{-\pi\, ib^2\tau - 2\pi\, ibz}\vartheta(z)$. (Hint: Use induction. Note that b can be negative.)

4)(i) (5/25) State the weak form of Bezout's theorem and weak form of Hilbert's Nullstellensatz concerning two curves in $\mathbb{C}^2$ defined by polynomials $f(z, w), g(z, w) \in$

$\mathbb{C}[z, w]$. Pay close attention to the hypotheses. (You are welcome to state the strong versions instead if you know them and you will be given full credit if you state them correctly.)

(ii) (10/25) Prove that an irreducible affine curve has only finitely many singular points.

(iii) (10/25) Find the singular points of the quartic curve in $\mathbb{C}P^2$ given by $F = (X^2 - Z^2)^2 - Y^2 Z(2Y + 3Z)$ and determine their multiplicities.

5) (i) (5/25) State Belyi's theorem.

(ii) (10/25) Suppose that f is Belyi function on a Riemann surface S. Consider the degree d map $g(z) = z^d$ viewed as a holomorphic map $\mathbb{P}^1 \to \mathbb{P}^1$. Show that $g \circ f : S \to \mathbb{P}^1$ is also a Belyi function.

(iii) (10/25) Let $f(z) = -\frac{4z^2(z-1)^2}{(2z-1)^2}$ define a holomorphic map from $\mathbb{P}^1$ to itself. Show that f is a degree 4 Belyi function. Sketch the dessin d'enfant associated to f.

Solutions:

1) (i) Let C be the curve defined by F. We have a regular map $\phi : \mathbb{P}^1 \to C$ defined by $[X : Y] \to [X^2 : XY : Y^2]$. Since $X^2Y^2 - (XY)^2 = 0$, we see that the image of F is in C.

Suppose $\phi([X_0 : Y_0]) = \phi([X_1 : Y_1])$. Then, we have

$$X_0^2 = \lambda X_1^2, X_0Y_0 = \lambda X_1Y_1, Y_0^2 = \lambda Y_1^2, \text{for some } \lambda \in \mathbb{C}^\times$$

Suppose $X_0 \neq 0$, then $X_1 \neq 0$ from the first equation. Dividing the second equation by the first one, we get

$$Y_0/X_0 = Y_1/X_1$$

Hence, $[X_0 : Y_0] = [X_1 : Y_1]$. If $X_0 = 0$, then we have $X_1 = 0$ from the first equation so again $[X_0 : Y_0] = [X_1 : Y_1]$. This proves that ϕ is injective. To prove surjectivity, note that given $[X_0 : Y_0 : Z_0]$ such that $X_0Z_0 = Y_0^2$, let $X, Y \in \mathbb{C}$ such that $X^2 = X_0$ and $Y^2 = Z_0$, then we have $(XY)^2 = Y_0^2$. If $XY = Y_0$, we have $\phi([X : Y]) = [X_0 : Y_0 : Z_0]$ otherwise, $\phi([X : -Y]) = [X_0 : Y_0 : Z_0]$.

(ii) Taking derivatives and setting them to zero, we see that

$$\partial_X = 3X^2 - 3aYZ = 0$$

$$\partial_Y = 3Y^2 - 3aXZ = 0$$

$$\partial_Z = 3Z^2 - 3aXY = 0$$

We note that $a \neq 0$ since otherwise the only solution is $X = Y = Z = 0$ which is not on $\mathbb{C}P^2$. We also note that if $X = 0$, then first equation implies either

$Y = 0$ or $Z = 0$, suppose $Y = 0$, then the last equation implies $Z = 0$. By symmetry, we conclude that the singular points of the singular curves satisfy $X \neq 0, Y \neq 0, Z \neq 0$. The above equations are equivalent to

$$X^2 = aYZ$$

$$Y^2 = aXZ$$

$$Z^2 = aXY$$

Multiplying both sides, we see

$$X^2Y^2Z^2 = a^3X^2Y^2Z^2$$

Since $X \neq 0, Y \neq 0, Z \neq 0$, we conclude $a^3 = 1$. By multiplying the first equation by X, the second by Y, the third by Z and comparing the right hand sides, we conclude that

$$X^3 = Y^3 = Z^3$$

Without loss of generality assume $Z = 1$. Let $(1, \xi, \xi^2)$ be the third roots of the unity. It is easy to verify that the following possibilities are the only solutions:

$$\begin{aligned}([X:Y:Z];a) = \{&([1:1:1];1),([\xi:\xi^2:1];1),([\xi^2:\xi:1];1),\\ &([\xi:\xi:1];\xi),([1:\xi^2:1];\xi),([\xi^2:1:1];\xi),\\ &([\xi:1:1];\xi^2),([1:\xi:1];\xi^2),([\xi^2:\xi^2:1],\xi^2)\}\end{aligned}$$

(iii) Following the hint, we consider the lines joining the singular points of C_a for $a = 1, \xi, \xi^2$.

Let $a = 1$. The singular points are $s_1 = [1:1:1], s_2 = [\xi:\xi^2:1], s_3 = [\xi^2:\xi:1]$. The lines through these are given by the following equations:

$$\begin{aligned}&l_{s_1s_2} : \xi^2X + Y + \xi Z = 0\\ &l_{s_2s_3} : X + Y + Z = 0\\ &l_{s_3s_1} : \xi X + Y + \xi^2 Z = 0\end{aligned}$$

where we use the well-known fact that $1 + \xi + \xi^2 = 0$. It remains to observe that

$$\begin{aligned}&(X + Y + Z)(\xi X + Y + \xi^2 Z)(\xi^2 X + Y + \xi Z)\\ &= (\xi X^2 + Y^2 + \xi^2 Z^2 + (1+\xi)XY + (\xi + \xi^2)XZ + (1+\xi^2)YZ)(\xi^2 X + Y + \xi Z)\\ &= X^3 + Y^3 + Z^3 - 3XYZ\end{aligned}$$

The other cases are similar. (Give full credit if the examiner works out one case and says the other cases are similar.)

2) (i) Branching index is defined by

$$b_f = \sum_{s \in S_2} (\deg(f) - |f^{-1}(s)|)$$

or it can also be defined by

$$b_f = \sum_{s \in S_2} \sum_{t \in f^{-1}(s)} (v_f(t) - 1)$$

where $v_f(t)$ is the ramification index of f at t (in local coordinates around t, f is given by $z \to z^{v_f(t)}$).

Riemann-Hurwitz states that

$$2h - 2 = \deg(f)(2g - 2) + b_f$$

(ii) For a non-constant holomorphic map Riemann-Hurwitz gives $2h-2 = d(2g-2) + b$ where $d = \deg(f) \geq 1$ and $b = b_f \geq 0$. If $g \geq h + 1$, we have that

$$d(2g - 2) + b \geq d\,2h + b \geq 2h$$

hence it cannot be equal to $2h - 2$.

(iii) To check smoothness, we observe that the only solutions to the system of equations $\partial_X F = dX^{d-1} = 0$, $\partial_Y F = dY^{d-1} = 0$, $\partial_Z F = dZ^{d-1} = 0$ is given by $X = Y = Z = 0$ which does not represent a point on $\mathbb{C}P^2$. Clearly π is well-defined as $[\lambda X : \lambda Y : \lambda Z] \to [\lambda X : \lambda Y] = [X : Y]$. For fixed $[X : Y]$, there are d solutions to $X^d + Y^d + Z^d = 0$ hence π has degree d. The only branching is when $X^d + Y^d = 0$. We may assume that $X = 1$, so there are d distinct branching points and the preimage above those points is geometrically a unique point, so the ramification index is $d - 1$. Thus, applying the Riemman-Hurwitz formula, we have:

$$2g(X_F) - 2 = -2d + d(d - 1)$$

In other words, $g(X_F) = \frac{(d-1)(d-2)}{2}$, as desired.

3) (i) Let $\tau = u + iv$ with $v > 0$ and $z = x + iy$, we observe that

$$\begin{aligned} |e^{\pi\, in^2\tau + 2\pi\, inz}| &= |e^{\pi\, iun^2 - \pi vn^2}||e^{2\pi inx - 2\pi ny}| \\ &= e^{-\pi vn^2 - 2\pi ny} \end{aligned}$$

For n large enough, $|n| < \pi n(vn + 2y)$ hence $|e^{\pi\, in^2\tau + 2\pi\, inz}| = e^{-\pi vn^2 - 2\pi ny} < e^{-|n|}$. As a result, the series converges absolutely, and very rapidly so.

(ii) a) This is clear since $e^{2\pi\, in(z+1)} = e^{2pi\, inz}$. b) $\vartheta(z+\tau) = \sum_{n\in\mathbb{Z}} e^{\pi\, in^2\tau + 2\pi\, in(z+\tau)}$ Re-indexing $n \to (n - 1)$, we get

$$\begin{aligned} \sum_{n\in\mathbb{Z}} e^{\pi\, i(n-1)^2\tau + 2\pi\, i(n-1)(z+\tau)} &= \sum_{n\in\mathbb{Z}} e^{\pi\, in^2\tau + 2\pi\, inz} e^{\pi\, i\tau} e^{-2\pi\, i(z+\tau)} \\ &= \left(\sum_{n\in\mathbb{Z}} e^{\pi\, in^2\tau + 2\pi\, inz} \right) e^{-\pi\, i\tau - 2\pi\, iz} \end{aligned}$$

c) The independence of a follows from part a). Suppose $b > 0$, we apply part b) to deduce

$$\begin{aligned}\vartheta(z+b\tau) &= \vartheta((z+(b-1)\tau)+\tau) = e^{-\pi\, i\tau - 2\pi\, i(z+(b-1)\tau)}\vartheta(z+(b-1)\tau) \\ &= e^{-(2b-1)\pi\, i\tau - 2\pi\, iz}\vartheta(z+(b-1)\tau)\end{aligned}$$

By induction, this is equal to

$$= e^{-(2b-1)\pi\, i\tau - 2\pi\, iz - \pi(b-1)^2\tau - 2\pi\, i(b-1)z}\vartheta(z) = e^{-\pi\, ib^2\tau - 2\pi\, ibz}\vartheta(z)$$

as required.

If $b < 0$, by what we proved we have

$$\vartheta(z - b\tau) = e^{-\pi\, ib^2\tau + 2\pi\, ibz}\vartheta(z)$$

Now, let $z \to z + b\tau$, to conclude that

$$\vartheta(z+b\tau) = e^{\pi\, ib^2\tau - 2\pi\, ib(z+b\tau)}\vartheta(z) = e^{-\pi\, ib^2\tau - 2\pi\, ibz}\vartheta(z)$$

as required.

4) (i) Let $f(z,w), g(z,w) \in \mathbb{C}[z,w]$ be two polynomials.

(Weak form of Bezout's theorem) If f and g are relatively prime, then the curves $f(z,w) = 0$ and $g(z,w) = 0$ intersect only at finitely many points.

(Weak form of Hilbert's Nullstellensatz) If f is irreducible and g vanishes at all points of the curve $f(z,w) = 0$ then f divides g.

(ii) A singularity of the affine curve defined by a polynomial $f(z,w)$ is given by common zeros of $f(z,w), f_z(z,w), f_w(z,w)$. Suppose $f(z,w)$ and $f_z(z,w)$ have infinitely many common zeros, then by weak form of Bezout's theorem, $f(z,w)$ and $f_z(z,w)$ must have a common divisor but $f(z,w)$ is irreducible, so $f(z,w)$ has to divide $f_z(z,w)$ but this is impossible for degree reasons unless $f_z(z,w)$ is identically zero, in which case we can argue in a similar way using $f_w(z,w)$.

(iii) We compute the derivatives

$$\partial_X F = 4(X^2 - Z^2)X$$

$$\partial_Y F = -6YZ(Y+Z)$$

$$\partial_Z F = -4(X^2 - Z^2)Z - 6Y^2Z - 2Y^3$$

From which we conclude $\epsilon_1 = [-1;0;1]$, $\epsilon_2 = [1;0;1]$, $\epsilon_3 = [0;-1;1]$ are the singular points.

We compute the multiplicities in the affine chart $\{Z = 1\}$ which includes all the singular points. So, we let $f(x,y) = F(X,Y,1) = (x^2-1)^2 - y^2(2y+3)$.

$$\partial_{xx} f = 12x^2 - 4$$

$$\partial_{xy} f = 0$$

$$\partial_{yy} f = -12y$$

Thus we see that ∂_{xx} does not vanish for any of the ϵ_i. Hence, the multiplicity of each of the singular points is 2.

5)(i) Let S be a compact Riemann surface. The following statements are equivalent: 1) S can be defined over $\overline{\mathbb{Q}}$. 2) S admits a morphism $f : S \to \mathbb{P}^1$ with at most 3 branch points.

(ii) Recall that $Branch(g \circ f) = Branch(g) \cup g(Branch(f))$.

We have $g(z) = z^d$, so $Branch(g) = \{0, \infty\}$. We have $Branch(f) = \{0, 1, \infty\}$, thus $g(Branch(f)) = \{0, 1, \infty\}$. Therefore, we see that $Branch(g \circ f) = \{0, 1, \infty\}$, hence $g \circ f$ is a Belyi function.

(iii) The branch points of f are b such that

$$f^{-1}(b)$$

consists of less than 4 distinct solutions (including possibly $b = \infty$). We immediately note that $f^{-1}(\infty) = \{1/2, \infty\}$, which shows that $b = \infty$ is a branch point. For $z \neq \infty$, we compute

$$f'(z) = -4\frac{(z-1)z(4z^2 - 4z + 2)}{(2z-1)^3}$$

So, ramification points are $\{0, 1/2, 1, \frac{1 \pm i}{2}\}$. The branch points are obtained by calculating the value of f at these points, which gives $f(0) = f(1) = 0$, $f(1/2) = \infty$, $f(\frac{1 \pm i}{2}) = 1$.

Thus, the white vertices of the dessin are at 0 and 1, and the black vertices are at $\frac{1 \pm i}{2}$. Each vertex have degree 2, hence the corresponding dessin looks as follows: