King's College London

University Of London

PLACE this paper and any answer booklets **in the EXAM ENVELOPE provided**

Candidate No: .... **Desk No:** ....

MSc and MSci Examination

7CCMMS16T Algebraic Curves (MSc)

Summer 2019

Time Allowed: Two Hours

All questions carry equal marks.
Full marks will be awarded for complete answers to FOUR questions.
If more than four questions are attempted, then only the best FOUR will count.

NO calculators are permitted.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

1. (i) (10/25) Prove directly that the projective curve defined by $F(X, Y, Z) = XZ - Y^2$ is isomorphic to $\mathbb{P}^1$.
(ii) (10/25) Consider the family of elliptic curves in $\mathbb{P}^2$ given by

$$C_a = \{([X : Y : Z] \in \mathbb{P}^2 : X^3 + Y^3 + Z^3 = 3aXYZ\}, \quad a \in \mathbb{C}$$

Determine the values of a for which C_a is singular, and for those values of a for which C_a is singular, determine all the singular points of C_a.
(iii) (5/25) For singular C_a show that each irreducible component of C_a is isomorphic to $\mathbb{P}^1$. (HINT: Show that the lines in $\mathbb{C}P^2$ joining the singular points of C_a are components of C_a. It suffices to do this only for one singular curve C_a as the other cases are similar).

2. (i) (5/25) Let f be a holomorphic map $S_h \to S_g$ between compact Riemann surfaces of genus h and g respectively. State the definition of the branching index b_f. State the Riemann-Hurwitz formula.
(ii) (10/25) Show that if $h < g$ then the only holomorphic maps between S_h and S_g are the constant maps.
(iii) (10/25) Let X_F be the projective curve of degree d defined by the homogeneous polynomial $F(X, Y, Z) = X^d + Y^d + Z^d$, called the *Fermat curve*. Let $\pi : X_F \to \mathbb{P}^1$ be given by $\pi([X : Y : Z]) = [X : Y]$.

Check that the Fermat curve is smooth. Show that π is a well-defined map of degree d. Use the Riemann-Hurwitz formula to compute the genus of X to be

$$g(X_F) = \frac{(d-1)(d-2)}{2}.$$

3. (i) (10/25) Let $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. For fixed such τ, consider the following function

$$\vartheta(z;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi\, in^2\tau + 2\pi inz}$$

Show that the series defining $\vartheta(z;\tau)$ converges for all $z \in \mathbb{C}$.

(ii) Fix τ with $\mathrm{Im}(\tau) > 0$ and write $\vartheta(z) = \vartheta(z;\tau)$. Verify the following identities.

a) (5/25) $\vartheta(z+1) = \vartheta(z)$.

b) (5/25) $\vartheta(z+\tau) = e^{-\pi\, i\tau - 2\pi\, iz}\vartheta(z)$.

c) (5/25) For $a, b \in \mathbb{Z}$, $\vartheta(z+a+b\tau) = e^{-\pi\, ib^2\tau - 2\pi\, ibz}\vartheta(z)$. (HINT: Use induction. Note that b can be negative.)

4. (i) (5/25) State the weak form of Bezout's theorem and weak form of Hilbert's Nullstellensatz concerning two curves in $\mathbb{C}^2$ defined by polynomials $f(z,w)$, $g(z,w) \in \mathbb{C}[z,w]$. Pay close attention to the hypotheses.

(ii) (10/25) Prove that an irreducible affine curve has only finitely many singular points.

(iii) (10/25) Find the singular points of the quartic curve in $\mathbb{C}P^2$ given by $F = (X^2 - Z^2)^2 - Y^2Z(2Y + 3Z)$ and determine their multiplicities.

5. (i) (5/25) State Belyi's theorem.

(ii) (10/25) Suppose that f is a Belyi function on a Riemann surface S. Consider the degree d map $g(z) = z^d$ viewed as a holomorphic map $\mathbb{P}^1 \to \mathbb{P}^1$. Show that $g \circ f : S \to \mathbb{P}^1$ is also a Belyi function.

(iii) (10/25) Let $f(z) = -\frac{4z^2(z-1)^2}{(2z-1)^2}$ define a holomorphic map from $\mathbb{P}^1$ to itself. Show that f is a degree 4 Belyi function. Sketch the dessin d'enfant associated to f.