Lecture 23. Valuation rings

6.5 Valuation rings

Throughout the course, we have developed the notion of dimension, first for affine varieties via the Hilbert polynomial and then for arbitrary rings via the Krull dimension. We saw that the 0-dimensional Noetherian rings are precisely the Artinian rings. We next want to explore the structure of 1-dimensional Noetherian rings. We begin with a class of local rings which, as we shall see in the next two lectures, contains the most important local rings of dimension one.

Definition 6.30. *Let R be an integral domain and K its field of fractions. We say that R is a* **valuation ring***, or a valuation ring of K, if for all $x \in K \setminus \{0\}$ we have $x \in R$ or $x^{-1} \in R$.*

Note that if a subring R of a field K has this property, then K is the field of fractions of R.

Obviously, K itself is a valuation ring of K. Here is a more interesting example. Let p be a prime number and consider the local ring

$$\mathbb{Z}_{(p)} = \{\tfrac{a}{b} \in \mathbb{Q} : p \nmid b\}$$

of Lecture 4, the localisation of $\mathbb{Z}$ at the prime ideal (p). It is a valuation ring of $\mathbb{Q}$: write $x \in \mathbb{Q}\setminus\{0\}$ as $\frac{a}{b}$ in lowest terms; if $p \nmid b$, then $x \in \mathbb{Z}_{(p)}$, and otherwise $p \nmid a$, so that $x^{-1} = \frac{b}{a} \in \mathbb{Z}_{(p)}$. The same argument shows that if A is a UFD with field of fractions K and $\pi \in A$ is a prime element, then

$$A_{(\pi)} = \{\tfrac{a}{b} : a, b \in A, \pi \nmid b\}$$

is a valuation ring of K. For example, $k[X]_{(X)}$ is a valuation ring of $k(X)$.

► $k[[X]]$ is a valuation ring. Its field of fractions is the field of formal Laurent series $k((X)) := k[[X]][X^{-1}]$. Indeed, every non-zero element of $k((X))$ is of the form $X^n u$ with $n \in \mathbb{Z}$ and u a power series with non-zero constant term, that is, a unit of $k[[X]]$; it lies in $k[[X]]$ if $n \geq 0$, and its inverse $X^{-n}u^{-1}$ lies in $k[[X]]$ if $n \leq 0$.

► The ring $\mathbb{Z}$ is not a valuation ring, since neither $\frac{2}{3}$ nor $\frac{3}{2}$ is an integer. A local integral domain need not be a valuation ring either: neither $\frac{X}{Y}$ nor $\frac{Y}{X}$ lies in $k[X, Y]_{(X,Y)}$.

Note that it is obvious from the definition that if R is a valuation ring of K and R' is a ring with $R \subset R' \subset K$, then R' is also a valuation ring of K.

Proposition 6.31. *Let R be a valuation ring with field of fractions K.*

(i) If I and J are ideals of R, then $I \subset J$ or $J \subset I$.

(ii) R is a local ring. Its maximal ideal, which we denote by $\mathfrak{m}_R$, is the set of non-units of R.

(iii) $K \setminus R = \{x \in K \setminus \{0\} : x^{-1} \in \mathfrak{m}_R\}$.

Proof. (i) Suppose that there exists $x \in I$ with $x \notin J$; we have to show that $J \subset I$. Let $0 \neq y \in J$. Then $xy^{-1} \notin R$, since otherwise $x = y(xy^{-1}) \in J$. Hence $x^{-1}y \in R$, and so $y = x(x^{-1}y) \in I$.

(ii) By (i), any two maximal ideals of R are comparable, hence equal, so R is a local ring. The complement of its maximal ideal $\mathfrak{m}_R$ is the set of units of R, by the exercise in Lecture 3.

(iii) Let $x \in K \setminus R$. Then $x \neq 0$ and $x^{-1} \in R$. If $x^{-1} \notin \mathfrak{m}_R$, then x^{-1} is a unit of R, hence $x = (x^{-1})^{-1} \in R$, a contradiction. Thus $x^{-1} \in \mathfrak{m}_R$. Conversely, let $x \in K \setminus \{0\}$ with $x^{-1} \in \mathfrak{m}_R$. If $x \in R$, then $1 = xx^{-1} \in \mathfrak{m}_R$, a contradiction. So $x \notin R$. □

◁ Prove that in a valuation ring, any proper radical ideal is prime.

◁ Suppose R is a valuation ring with field of fractions K. Show that any subring R' of K with $R \subset R' \subset K$ is equal to $R_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset R$. (Hint: Consider $\mathfrak{p} = \mathfrak{m}' \cap R$, where $\mathfrak{m}' \subset R'$ is the maximal ideal of R'.)

Our next aim is to show that valuation rings exist in abundance. Let $A \subset B$ be rings. We say that an ideal $I \subset A$ **survives in** B if $IB \neq B$, or equivalently $1 \notin IB$.

Lemma 6.32. *Let R be a subring of a field K, let I be a proper ideal of R, and let $x \in K \setminus \{0\}$. Then I survives in $R[x]$ or in $R[x^{-1}]$.*

Proof. Suppose that the conclusion fails. Then we have equations

$$1 = a_0 + a_1x + \ldots + a_nx^n \text{ with } \ a_i \in I,$$

$$1 = b_0 + b_1x^{-1} + \ldots + b_mx^{-m} \text{ with } \ b_i \in I,$$

and we choose them with n and m as small as possible. Since $1 \notin I$, we have $n, m \geq 1$, and interchanging x and x^{-1} if necessary, we may assume that $n \geq m$. Multiplying the first equation by $1 - b_0$, we get

$$1 = b_0 + (1 - b_0)a_0 + (1 - b_0)a_1x + \ldots + (1 - b_0)a_nx^n.$$

Multiplying the second equation by x^n, we get

$$(1 - b_0)x^n = b_1x^{n-1} + \ldots + b_mx^{n-m},$$

where no negative powers of x occur because $n \geq m$. Using this to substitute for $(1 - b_0)x^n$ in the equation before, we arrive at

$$1 = b_0 + (1 - b_0)a_0 + (1 - b_0)a_1x + \ldots + (1 - b_0)a_{n-1}x^{n-1} + a_n(b_1x^{n-1} + \ldots + b_mx^{n-m}),$$

which is an equation of the first kind of degree less than n with all coefficients in I, a contradiction. □

Theorem 6.33. *Let K be a field, $A \subset K$ a subring, and $\mathfrak{p} \subset A$ a prime ideal. Then there exists a valuation ring R of K such that $A \subset R$ and $\mathfrak{m}_R \cap A = \mathfrak{p}$.*

Proof. We may replace A by its localisation $A_{\mathfrak{p}} \subset K$ and $\mathfrak{p}$ by $\mathfrak{p}A_{\mathfrak{p}}$: if R is a valuation ring of K containing $A_{\mathfrak{p}}$ with $\mathfrak{m}_R \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, then $\mathfrak{m}_R \cap A = \mathfrak{p}A_{\mathfrak{p}} \cap A = \mathfrak{p}$ by Lemma 6.14(iii). So we assume that A is a local ring with maximal ideal $\mathfrak{p}$.

Let

$$\mathcal{F} := \{B \subset K : B \text{ is a subring, } A \subset B \text{ and } \mathfrak{p} \text{ survives in } B\}.$$

We have $A \in \mathcal{F}$. If $\{B_\lambda\}_{\lambda \in \mathcal{T}}$ is a non-empty family of elements of $\mathcal{F}$ that is totally ordered by inclusion, then $B = \bigcup_{\lambda \in \mathcal{T}} B_\lambda$ is a subring of K, and $\mathfrak{p}$ survives in B: an equation $1 = \sum_i p_i b_i$ with $p_i \in \mathfrak{p}$ and $b_i \in B$ involves only finitely many elements b_i, and these all lie in one of the B_λ. So $B \in \mathcal{F}$ is an upper bound for the family. Hence we can apply Zorn's lemma and find a maximal element $R \in \mathcal{F}$ with respect to inclusion.

We claim that R is a valuation ring of K. Let $x \in K \setminus \{0\}$ with $x \notin R$; we want to see that $x^{-1} \in R$. Since $R \subsetneq R[x]$, we have $R[x] \notin \mathcal{F}$ by the maximality of R, that is, $\mathfrak{p}R$ does not survive in $R[x]$. By Lemma 6.32, applied to the proper ideal $\mathfrak{p}R$ of R, it survives in $R[x^{-1}]$. So $R[x^{-1}] \in \mathcal{F}$, and by the maximality of R we conclude that $R = R[x^{-1}]$, hence $x^{-1} \in R$.

Finally, $\mathfrak{p}R$ is a proper ideal of the local ring R, so $\mathfrak{p} \subset \mathfrak{m}_R \cap A$. This is a proper ideal of A, and $\mathfrak{p}$ is a maximal ideal of A, hence $\mathfrak{m}_R \cap A = \mathfrak{p}$. □

Theorem 6.34. *Let K be a field.*

(i) Every valuation ring R of K is integrally closed in K. In other words, valuation rings are normal.

(ii) Let $A \subset K$ be a subring. Then

$$\overline{A}_K = \{x \in K : x \text{ integral over } A\} = \bigcap_{A \subset R \subset K} R,$$

where the intersection is taken over all valuation rings R of K containing A.

Proof. (i) Let $x \in K$ be integral over R, and suppose that $x \notin R$. There exist $m \geq 1$ and $a_i \in R$ such that

$$x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 = 0.$$

By Proposition 6.31(iii), we have $x^{-1} \in \mathfrak{m}_R$. Dividing by x^m, we get

$$1 + a_{m-1}x^{-1} + \ldots + a_1 x^{1-m} + a_0 x^{-m} = 0.$$

But this implies $1 \in \mathfrak{m}_R$, which is a contradiction. Hence $x \in R$.

(ii) If $x \in K$ is integral over A, then it is integral over every valuation ring R of K containing A, so $x \in R$ by (i). This gives $\overline{A}_K \subset \bigcap_{A \subset R \subset K} R$.

Conversely, let $x \in K$ and suppose that x is not integral over A; in particular $x \neq 0$. We must find a valuation ring R of K containing A such that $x \notin R$. We claim that the ideal $x^{-1}A[x^{-1}] \subset A[x^{-1}]$ does not contain 1. Indeed, if $1 \in x^{-1}A[x^{-1}]$, then

$$1 = a_1 x^{-1} + \ldots + a_n x^{-n} \text{ with } a_i \in A.$$

Multiplying by x^n gives

$$x^n = a_1 x^{n-1} + \ldots + a_n,$$

which implies that x is integral over A, a contradiction. Thus there exists a maximal ideal $\mathfrak{m} \subset A[x^{-1}]$ with $x^{-1} \in \mathfrak{m}$.

By Theorem 6.33, there exists a valuation ring R of K with $A[x^{-1}] \subset R$ and $\mathfrak{m}_R \cap A[x^{-1}] = \mathfrak{m}$. Therefore $x^{-1} \in \mathfrak{m}_R$, and hence $x \notin R$ by Proposition 6.31(iii), as desired. □

◁ Show that the valuation rings of $\mathbb{Q}$ are $\mathbb{Q}$ itself and the rings $\mathbb{Z}_{(p)}$ for the prime numbers p.

▶ Here is Theorem 6.34(ii) for $A = \mathbb{Z}$ and $K = \mathbb{Q}$. By the exercise, the intersection of the valuation rings of $\mathbb{Q}$ is $\bigcap_p \mathbb{Z}_{(p)}$, the set of rational numbers whose denominator in lowest terms is not divisible by any prime number. This is $\mathbb{Z}$, which is indeed the integral closure of $\mathbb{Z}$ in $\mathbb{Q}$, as we saw in Lecture 10.

◁ Show that if A is normal, then $A[X]$ is also normal. (Hint: Use Theorem 6.34 to reduce to the case where A is a valuation ring.)