Lecture 22. The Cohen–Seidenberg theorems

Let $V \subset k^n$ be an irreducible affine variety, so that $\mathcal{I}(V) \subset k[X_1, \dots, X_n]$ is a prime ideal, and let $A = k[X_1, \dots, X_n]/\mathcal{I}(V)$ be its coordinate ring. We now have two notions of dimension: the dimension $\dim V = \deg HP_{\mathcal{I}(V)}(t)$ of the variety, defined by the Hilbert polynomial, and the Krull dimension $\dim A$ of its coordinate ring, defined by chains of prime ideals. How do they compare?

Theorem 6.26. $\dim V = \dim A$.

Let us see what is needed for the proof. By the Noether normalisation lemma (Theorem 4.25), there exist algebraically independent elements $a_1, \dots, a_d \in A$ such that A is integral over $k[a_1, \dots, a_d]$. Then every element of A, and hence of $\mathrm{Frac}(A)$, is algebraic over the field $k(a_1, \dots, a_d)$, so $\{a_1, \dots, a_d\}$ is a transcendence basis of $\mathrm{Frac}(A)$ over k, and $d = \dim V$ by Corollary 4.28. On the other hand, $k[a_1, \dots, a_d]$ is a polynomial ring in d variables, so $\dim k[a_1, \dots, a_d] = d$ by Corollary 6.25. Thus Theorem 6.26 follows once we know that $\dim A = \dim k[a_1, \dots, a_d]$, that is, that the Krull dimension does not change under integral extensions. This is part of the following theorem.

Theorem 6.27. *(Cohen–Seidenberg[13]) Let $A \subset B$ be rings such that B is integral over A.*

(1) (Lying over) For every prime ideal $\mathfrak{p} \subset A$, there exists a prime ideal $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. In other words, the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, $\mathfrak{q} \mapsto \mathfrak{q} \cap A$, is surjective.

(2) (Incomparability) If $\mathfrak{q}_1 \subset \mathfrak{q}_2$ are prime ideals of B such that $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A$, then $\mathfrak{q}_1 = \mathfrak{q}_2$.

(3) (Going up) Given prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1$ of A and a prime ideal $\mathfrak{q}_0 \subset B$ such that $\mathfrak{q}_0 \cap A = \mathfrak{p}_0$, there exists a prime ideal $\mathfrak{q}_1 \subset B$ such that $\mathfrak{q}_0 \subset \mathfrak{q}_1$ and $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$.

As a consequence of (1), (2) and (3), we have $\dim A = \dim B$.

If $\mathfrak{q} \cap A = \mathfrak{p}$, we say that $\mathfrak{q}$ **lies over** $\mathfrak{p}$.

▶ The ring of Gaussian integers $\mathbb{Z}[i]$ is integral over $\mathbb{Z}$. Since $5 = (2+i)(2-i)$, the prime ideals $(2+i)$ and $(2-i)$ of $\mathbb{Z}[i]$ both lie over (5), and neither contains the other. The only prime ideal lying over (3) is (3) itself, and the only one lying over (2) is $(1+i)$.

▶ Recall the two examples of Lecture 10. The ring $B = k[X,Y]/(X^2 - Y^2)$ is integral over $A = k[Y]$, and the prime ideals of B lying over $(Y - c)$ are $(X - c, Y - c)$ and $(X + c, Y - c)$: the points of the two lines $X = \pm Y$ above the point c of the Y-axis, see Figure 18. On the other hand, $C = k[X,Y]/(XY - 1)$ is not integral over $k[X]$, and no prime ideal of C lies over (X), since X is a unit in C: no point of the hyperbola lies above 0. Incomparability can also fail for extensions which are not integral: in $k \subset k[X]$, the prime ideals $(0) \subset (X)$ both lie over (0).

[13] I. S. Cohen and A. Seidenberg, Prime ideals and integral dependence, Bull. Amer. Math. Soc. 52 (1946), 252–261.

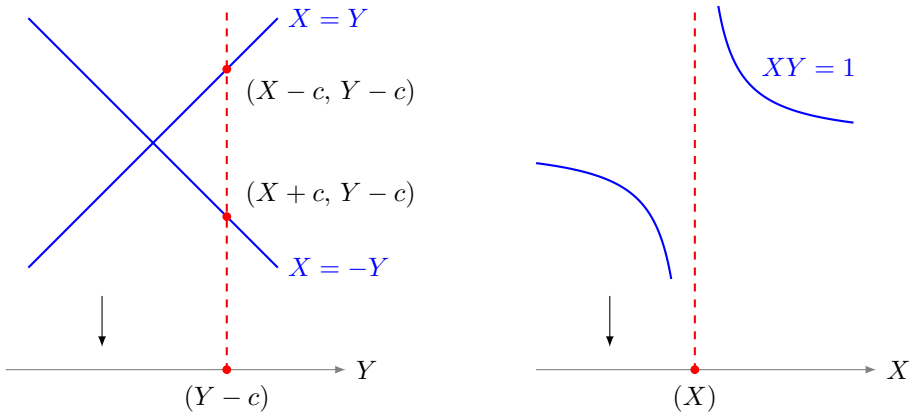

Figure 18: Left: the two prime ideals of $B = k[X,Y]/(X^2 - Y^2)$ lying over the prime ideal $(Y - c)$ of $A = k[Y]$ are the two points of $\mathcal{V}(X^2 - Y^2)$ which project to the point c of the Y-axis. Right: no prime ideal of $C = k[X,Y]/(XY - 1)$ lies over the prime ideal (X) of $k[X]$, since no point of the hyperbola projects to the point 0 of the X-axis.

We need two lemmas for the proof. Let $A \subset B$ be rings. For an ideal $I \subset B$ and a multiplicative set $S \subset A$, the homomorphisms $A/(I \cap A) \to B/I$ and $S^{-1}A \to S^{-1}B$ induced by the inclusion are injective; for the second one, note that $\frac{a}{s} = 0$ means in both rings that $ta = 0$ for some $t \in S$. So we may regard $A/(I \cap A)$ as a subring of B/I, and $S^{-1}A$ as a subring of $S^{-1}B$.

Lemma 6.28. *Let $A \subset B$ be rings such that B is integral over A.*

(i) Let I be an ideal of B. Then B/I is integral over $A/(I \cap A)$.

(ii) Let $S \subset A$ be a multiplicative set. Then $S^{-1}B$ is integral over $S^{-1}A$.

Proof. Let $b \in B$. There exist $n \geq 1$ and $a_0, a_1, \ldots, a_{n-1} \in A$ such that

$$b^n + a_{n-1}b^{n-1} + \ldots + a_1 b + a_0 = 0.$$

(i) Applying the quotient homomorphism $B \to B/I$, $b \mapsto \overline{b}$, to this equation, we see that $\overline{b}$ is integral over $A/(I \cap A)$.

(ii) Let $s \in S$. Dividing the equation by s^n, we get

$$\left(\frac{b}{s}\right)^n + \frac{a_{n-1}}{s}\left(\frac{b}{s}\right)^{n-1} + \ldots + \frac{a_1}{s^{n-1}} \cdot \frac{b}{s} + \frac{a_0}{s^n} = 0$$

in $S^{-1}B$, so $\frac{b}{s}$ is integral over $S^{-1}A$, and every element of $S^{-1}B$ is of this form. □

Lemma 6.29. *Let $A \subset B$ be rings such that B is integral over A, and let $\mathfrak{q} \subset B$ be a prime ideal. Then $\mathfrak{q}$ is a maximal ideal of B if and only if $\mathfrak{q} \cap A$ is a maximal ideal of A.*

Proof. By Lemma 6.28(i), the integral domain $B/\mathfrak{q}$ is integral over $A/(\mathfrak{q} \cap A)$. By Corollary 4.23(d), $B/\mathfrak{q}$ is a field if and only if $A/(\mathfrak{q} \cap A)$ is a field. □

Proof of Theorem 6.27. Proof of (1) and (2). Let $\mathfrak{p} \subset A$ be a prime ideal and $S = A \setminus \mathfrak{p}$. By Lemma 6.28(ii), the ring $S^{-1}B$ is integral over the local ring $S^{-1}A = A_\mathfrak{p}$, whose maximal ideal is $\mathfrak{p}A_\mathfrak{p}$. Consider the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A_\mathfrak{p} & \longrightarrow & S^{-1}B \end{array}$$

where α and β are the localisation maps and the horizontal maps are the inclusions. For a prime ideal $\mathfrak{n} \subset S^{-1}B$ it gives

$$\beta^{-1}(\mathfrak{n}) \cap A = \alpha^{-1}(\mathfrak{n} \cap A_\mathfrak{p}).$$

By Lemma 6.14(iii), $\mathfrak{p}A_\mathfrak{p}$ is the only prime ideal of $A_\mathfrak{p}$ whose preimage under α is $\mathfrak{p}$. Hence $\beta^{-1}(\mathfrak{n})$ lies over $\mathfrak{p}$ if and only if $\mathfrak{n}$ lies over $\mathfrak{p}A_\mathfrak{p}$.

For (1), note that $S^{-1}B \neq \{0\}$, since it contains $A_\mathfrak{p}$, so it has a maximal ideal $\mathfrak{n}$ by Proposition 2.4. By Lemma 6.29, $\mathfrak{n} \cap A_\mathfrak{p}$ is a maximal ideal of $A_\mathfrak{p}$, so it is equal to $\mathfrak{p}A_\mathfrak{p}$. Hence $\mathfrak{q} = \beta^{-1}(\mathfrak{n})$ is a prime ideal of B lying over $\mathfrak{p}$.

For (2), let $\mathfrak{q}_1 \subset \mathfrak{q}_2$ be prime ideals of B lying over $\mathfrak{p}$. They do not meet S, so by Lemma 6.14(iii), applied to B, we have $\mathfrak{q}_j = \beta^{-1}(\mathfrak{n}_j)$ for the prime ideals $\mathfrak{n}_j = S^{-1}\mathfrak{q}_j$ of $S^{-1}B$, and $\mathfrak{n}_1 \subset \mathfrak{n}_2$. Both $\mathfrak{n}_1$ and $\mathfrak{n}_2$ lie over the maximal ideal $\mathfrak{p}A_\mathfrak{p}$, so they are maximal ideals of $S^{-1}B$ by Lemma 6.29. Hence $\mathfrak{n}_1 = \mathfrak{n}_2$, and therefore $\mathfrak{q}_1 = \mathfrak{q}_2$.

Proof of (3). By Lemma 6.28(i), the ring $B/\mathfrak{q}_0$ is integral over $A/\mathfrak{p}_0$. By (1), applied to this extension, there is a prime ideal of $B/\mathfrak{q}_0$ lying over the prime ideal $\mathfrak{p}_1/\mathfrak{p}_0$ of $A/\mathfrak{p}_0$. It is of the form $\mathfrak{q}_1/\mathfrak{q}_0$ for a prime ideal $\mathfrak{q}_1 \supset \mathfrak{q}_0$ of B. Its intersection with $A/\mathfrak{p}_0$ is $(\mathfrak{q}_1 \cap A)/\mathfrak{p}_0$, so $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$.

Proof of $\dim A = \dim B$. Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n$ be a chain of prime ideals of A. By (1), there is a prime ideal $\mathfrak{q}_0$ of B lying over $\mathfrak{p}_0$, and applying (3) n times we obtain prime ideals $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_n$ of B with $\mathfrak{q}_i \cap A = \mathfrak{p}_i$. These inclusions are strict, since the $\mathfrak{p}_i$ are distinct. Hence $\dim B \geq \dim A$. Conversely, let $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \ldots \subsetneq \mathfrak{q}_n$ be a chain of prime ideals of B. Then the $\mathfrak{p}_i = \mathfrak{q}_i \cap A$ are prime ideals of A with $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$, and these inclusions are strict by (2). Hence $\dim A \geq \dim B$. □

This also completes the proof of Theorem 6.26.

◁ Let k be algebraically closed. Show that $\dim V = \dim k[V]$ for every non-empty affine variety $V \subset k^n$, irreducible or not.

▶ We can now justify the geometric picture of integral extensions from Lectures 10 and 11. Let k be algebraically closed, and let $\phi : V \to W$ be a regular mapping of affine varieties such that $\phi^* : k[W] \to k[V]$ is injective and $k[V]$ is integral over the subring $\phi^*(k[W])$. Then ϕ is

surjective. Indeed, let $\mathbf{w} \in W$, with maximal ideal $\mathfrak{n}_{\mathbf{w}} \subset k[W]$. By lying over, there is a prime ideal $\mathfrak{q} \subset k[V]$ with $(\phi^*)^{-1}(\mathfrak{q}) = \mathfrak{n}_{\mathbf{w}}$, and $\mathfrak{q}$ is a maximal ideal by Lemma 6.29. So $\mathfrak{q} = \mathfrak{m}_{\mathbf{v}}$ for some point $\mathbf{v} \in V$, and $\phi(\mathbf{v}) = \mathbf{w}$, because $(\phi^*)^{-1}(\mathfrak{m}_{\mathbf{v}}) = \mathfrak{n}_{\phi(\mathbf{v})}$, as we saw in Lecture 15. In particular, the map $\pi : V \to k^d$ given by a Noether normalisation of $k[V]$ is surjective.

◁ In the situation of the previous remark, show that the fibres $\phi^{-1}(\mathbf{w})$ are finite.

▶ There is also a **going-down** theorem. Let $A \subset B$ be integral domains such that B is integral over A and A is normal. Given prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1$ of A and a prime ideal $\mathfrak{q}_1 \subset B$ lying over $\mathfrak{p}_1$, there exists a prime ideal $\mathfrak{q}_0 \subset \mathfrak{q}_1$ of B lying over $\mathfrak{p}_0$. The proof is harder; see [3], Theorem 5.16.

◁ Let A be a subring of B (not necessarily an integral extension). Show that if the going-up property holds for the ring extension $A \subset B$, then the lying-over property also holds.