Lecture 19. Local rings: Nakayama's lemma

We begin with some basic properties of localisation. Throughout, $S \subset A$ is a multiplicative set and $\iota : A \to S^{-1}A$, $a \mapsto \frac{a}{1}$, is the canonical homomorphism.

Lemma 6.14. *(i) Let $J \subset S^{-1}A$ be an ideal and $I = \iota^{-1}(J) \subset A$. Then $J = S^{-1}I$. In particular, $S^{-1}A$ is Noetherian if A is Noetherian. On the other hand, for an ideal $I \subset A$ we have $\iota^{-1}(S^{-1}I) = \{a \in A : ta \in I \text{ for some } t \in S\}$.*

(ii) (Localisation commutes with quotients) Let I be an ideal in A, and write $\overline{S}$ for the image of S under the canonical ring homomorphism $A \to A/I$. Then $\overline{S}^{-1}(A/I) \simeq S^{-1}A/S^{-1}I$.

(iii) The map $\iota^{-1} : \mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is an inclusion-preserving bijection onto the subset $\{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$, with inverse given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.

Proof. (i) Since J is an ideal containing $\iota(I)$, we have $S^{-1}I \subset J$. Conversely, let $x = \frac{a}{s} \in J$. Then $\frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s} \in J$, hence $a \in \iota^{-1}(J) = I$ and $x \in S^{-1}I$. If A is Noetherian, then $I = (a_1, \dots, a_n)$ for some $a_i \in A$, and $J = S^{-1}I$ is generated by $\frac{a_1}{1}, \dots, \frac{a_n}{1}$; so every ideal of $S^{-1}A$ is finitely generated. For the last statement, let $a \in \iota^{-1}(S^{-1}I)$, say $\frac{a}{1} = \frac{b}{r}$ with $b \in I$ and $r \in S$. Then $u(ar - b) = 0$ for some $u \in S$, so $ta = ub \in I$ for $t = ur \in S$. Conversely, if $ta \in I$ with $t \in S$, then $\frac{a}{1} = \frac{ta}{t} \in S^{-1}I$.

(ii) Write $\overline{a}$ for the class of $a \in A$ in A/I. We define $h : S^{-1}A \to \overline{S}^{-1}(A/I)$ by $h(\frac{a}{s}) = \frac{\overline{a}}{\overline{s}}$. This is well defined: if $\frac{a}{s} = \frac{b}{r}$, then $u(ar - bs) = 0$ for some $u \in S$, hence $\overline{u}\,(\overline{a}\,\overline{r} - \overline{b}\,\overline{s}) = 0$ with $\overline{u} \in \overline{S}$, that is, $\frac{\overline{a}}{\overline{s}} = \frac{\overline{b}}{\overline{r}}$. It is a ring homomorphism, since sums and products of fractions are given by the same formulas on both sides, and it is clearly surjective. We claim that $\ker(h) = S^{-1}I$; the first isomorphism theorem then gives $S^{-1}A/S^{-1}I \simeq \overline{S}^{-1}(A/I)$. Since $\frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s}$ and $\frac{a}{s} = \frac{1}{s} \cdot \frac{a}{1}$, we have $\frac{a}{s} \in S^{-1}I$ if and only if $a \in \iota^{-1}(S^{-1}I)$, that is, by (i), if and only if $ta \in I$ for some $t \in S$. On the other hand, $h(\frac{a}{s}) = 0$ if and only if $\overline{t}\,\overline{a} = 0$ for some $t \in S$, that is, $ta \in I$ for some $t \in S$. This proves the claim.

(iii) Let $\mathfrak{p} \subset A$ be a prime ideal with $\mathfrak{p} \cap S = \emptyset$. Then $A/\mathfrak{p}$ is an integral domain and $0 \notin \overline{S}$, so $\overline{S}^{-1}(A/\mathfrak{p})$ is a subring of $\mathrm{Frac}(A/\mathfrak{p})$; in particular, it is a nonzero integral domain. By (ii), $S^{-1}\mathfrak{p}$ is the kernel of a surjective homomorphism from $S^{-1}A$ to a nonzero integral domain, hence a prime ideal of $S^{-1}A$. Moreover, $\iota^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ by (i), since $ta \in \mathfrak{p}$ with $t \in S$ forces $a \in \mathfrak{p}$.

Conversely, let $\mathfrak{q} \subset S^{-1}A$ be a prime ideal and $\mathfrak{p} = \iota^{-1}(\mathfrak{q})$, a prime ideal of A by Proposition 2.5. Then $\mathfrak{p} \cap S = \emptyset$, since otherwise $\mathfrak{q}$ would contain a unit of $S^{-1}A$, and $S^{-1}\mathfrak{p} = \mathfrak{q}$ by part (i). So the two maps are inverse to each other, and both clearly preserve inclusions. □

Part (iii) of Lemma 6.14 describes the spectra of the two localisations A_f and $A_{\mathfrak{p}}$ of Lecture 18.

Corollary 6.15. *(i) For $f \in A$, the map ι^{-1} identifies* $\mathrm{Spec}(A_f)$ *with the basic open set* $D(f) = \{\mathfrak{q} \in \mathrm{Spec}(A) : f \notin \mathfrak{q}\}$.

(ii) For a prime ideal $\mathfrak{p} \subset A$, the map ι^{-1} identifies $\mathrm{Spec}(A_\mathfrak{p})$ *with* $\{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{q} \subset \mathfrak{p}\}$, *preserving inclusions. In particular,* $\dim A_\mathfrak{p} = h(\mathfrak{p})$.

Proof. A prime ideal $\mathfrak{q}$ is disjoint from $\{1, f, f^2, \ldots\}$ if and only if $f \notin \mathfrak{q}$, and disjoint from $A \setminus \mathfrak{p}$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. For the last statement, every chain of prime ideals contained in $\mathfrak{p}$ can be extended to end at $\mathfrak{p}$, so the supremum of their lengths is $h(\mathfrak{p})$. □

Thus the localisation A_f plays the role of the ring of "regular functions" on the open set $D(f)$.

▶ The passage from A to $A_\mathfrak{p}$ cuts out all the prime ideals except those contained in $\mathfrak{p}$. This is in contrast to the passage from A to $A/\mathfrak{p}$, which cuts out all the prime ideals except those containing $\mathfrak{p}$. In Figure 10, for $\mathfrak{p} = (Y - X^2)$ the first keeps the generic points of the plane and of the parabola, the second keeps the parabola with all its points.

◁ Given prime ideals $\mathfrak{q} \subset \mathfrak{p}$, we can also localise with respect to $\mathfrak{p}$ and quotient with respect to $\mathfrak{q}$. Show that $A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p} \simeq (A/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}}$, and that the prime ideals of this ring are those lying between $\mathfrak{q}$ and $\mathfrak{p}$ (use Lemma 6.14(ii)).

The ring $A_\mathfrak{p}$ is the algebraic analogue of the germs of functions defined near the point $\mathfrak{p}$ of $\mathrm{Spec}(A)$. The dimension of A can be computed from these local rings: by definition $\dim A = \sup_\mathfrak{p} h(\mathfrak{p})$, and $h(\mathfrak{p}) \leq h(\mathfrak{m})$ when $\mathfrak{p} \subset \mathfrak{m}$, so

$$\dim A = \sup_{\mathfrak{p} \in \mathrm{Spec}(A)} \dim A_\mathfrak{p} = \sup_{\mathfrak{m} \in \mathrm{mSpec}(A)} \dim A_\mathfrak{m}.$$

6.4 Dimension for local rings

Next, we come to a very useful lemma in the study of local rings, which will later on help us understand how to compute $\dim A_\mathfrak{p}$.

Lemma 6.16. *(Nakayama's Lemma) Let A be a local ring with maximal ideal $\mathfrak{m}$. Let M be a finitely generated A-module and $N \subset M$ be a submodule with the property that $M = N + \mathfrak{m}M$. Then $M = N$.*

Proof. The A-module $\overline{M} = M/N$ is finitely generated and satisfies $\overline{M} = \mathfrak{m}\overline{M}$. We want to show that $\overline{M} = \{0\}$. If not, let $x_1, \ldots, x_n \in \overline{M}$ be generators, $\overline{M} = Ax_1 + Ax_2 + \ldots + Ax_n$, with $n \geq 1$ minimal. Since $\overline{M} = \mathfrak{m}\overline{M}$, we have $x_n = \sum_{i=1}^n a_i x_i$ with $a_i \in \mathfrak{m}$. Then $(1 - a_n)x_n = \sum_{i=1}^{n-1} a_i x_i$, and $1 - a_n \notin \mathfrak{m}$ is a unit, because A is local and every non-unit lies in a maximal ideal. It follows that $x_n \in Ax_1 + Ax_2 + \ldots + Ax_{n-1}$, contradicting the minimality of n. □

This is a simple but important result that allows us to make deductions about properties of M as an A-module from the properties of $M/\mathfrak{m}M$ as an $A/\mathfrak{m}$-vector space. Thus, many properties

of the finitely generated modules over a local ring A can be studied via linear algebra over the field $A/\mathfrak{m}$.

Here is a first application of Nakayama's lemma. For a finitely generated module M, write

$$\mu(M) \coloneqq \min\{n \geq 0 : M = Ax_1 + \ldots + Ax_n \text{ for some } x_1, \ldots, x_n \in M\}$$

for the minimal number of generators of M.

Proposition 6.17. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$, and let M be a finitely generated A-module. Elements $x_1, \ldots, x_n \in M$ generate M if and only if their images $\overline{x}_1, \ldots, \overline{x}_n$ span the k-vector space $M/\mathfrak{m}M$. In particular, $\mu(M) = \dim_k M/\mathfrak{m}M$.*

Proof. If $M = Ax_1 + \ldots + Ax_n$, then $M/\mathfrak{m}M$ is spanned by $\overline{x}_1, \ldots, \overline{x}_n$. Conversely, suppose that $\overline{x}_1, \ldots, \overline{x}_n$ span $M/\mathfrak{m}M$, and let $N = Ax_1 + \ldots + Ax_n \subset M$. By Nakayama's lemma, it suffices to show that $M = N + \mathfrak{m}M$. Let $x \in M$. Then $\overline{x} = \sum_{i=1}^n \overline{a}_i\overline{x}_i$ for some $a_i \in A$, so $x - \sum_{i=1}^n a_i x_i \in \mathfrak{m}M$, and $x \in N + \mathfrak{m}M$. Finally, a k-vector space spanned by n elements has dimension at most n, and a basis of $M/\mathfrak{m}M$ lifts to a generating set of M; hence $\mu(M) = \dim_k M/\mathfrak{m}M$. □

Now let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Then $\mathfrak{m}$ is a finitely generated A-module, and we write

$$\mu(A) \coloneqq \mu(\mathfrak{m}) = \dim_k \mathfrak{m}/\mathfrak{m}^2$$

for the minimal number of generators of $\mathfrak{m}$.

◁ Let $\mathfrak{m}$ be a maximal ideal of a ring A. Show that $A_\mathfrak{m}/\mathfrak{m}^2A_\mathfrak{m} \simeq A/\mathfrak{m}^2$, and deduce that $\mathfrak{m}A_\mathfrak{m}/(\mathfrak{m}A_\mathfrak{m})^2 \simeq \mathfrak{m}/\mathfrak{m}^2$ (use Lemma 6.14(ii)).

▶ Here is the geometric meaning of $\mu(A)$. Let $V \subset k^n$ be an affine variety with $\mathcal{I}(V) = (f_1, \ldots, f_m)$, let $\mathbf{v} \in V$, and let $A = k[V]_{\mathfrak{m}_\mathbf{v}}$ be the local ring of V at $\mathbf{v}$, so that $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m}_\mathbf{v}/\mathfrak{m}_\mathbf{v}^2$ by the exercise. Every $g \in k[X_1, \ldots, X_n]$ can be written as $g = g(\mathbf{v}) + d_\mathbf{v}g + r$, where $d_\mathbf{v}g = \sum_i \frac{\partial g}{\partial X_i}(\mathbf{v})(X_i - v_i)$ is the linear part of g at $\mathbf{v}$, which we view as a linear form on k^n, and $r \in (X_1 - v_1, \ldots, X_n - v_n)^2$. Then $g \mapsto d_\mathbf{v}g$ induces an isomorphism $\mathfrak{m}_\mathbf{v}/\mathfrak{m}_\mathbf{v}^2 \simeq (k^n)^*/\langle d_\mathbf{v}f_1, \ldots, d_\mathbf{v}f_m\rangle$, and the right hand side is the dual of the **tangent space**

$$T_\mathbf{v}V = \{\mathbf{w} \in k^n : d_\mathbf{v}f_j(\mathbf{w}) = 0 \text{ for } j = 1, \ldots, m\}.$$

So $\mu(A) = \dim_k T_\mathbf{v}V$. For the parabola $V = \mathcal{V}(Y - X^2)$, the tangent space at the origin is the line $Y = 0$, so $\mu(A) = 1 = \dim A$. For the cusp $V = \mathcal{V}(Y^2 - X^3)$ of Figure 8, we have $d_\mathbf{0}(Y^2 - X^3) = 0$, so the tangent space at the origin is all of k^2 and $\mu(A) = 2$, while $\dim A = 1$. At every other point of the cusp the tangent space is a line, see Figure 14.

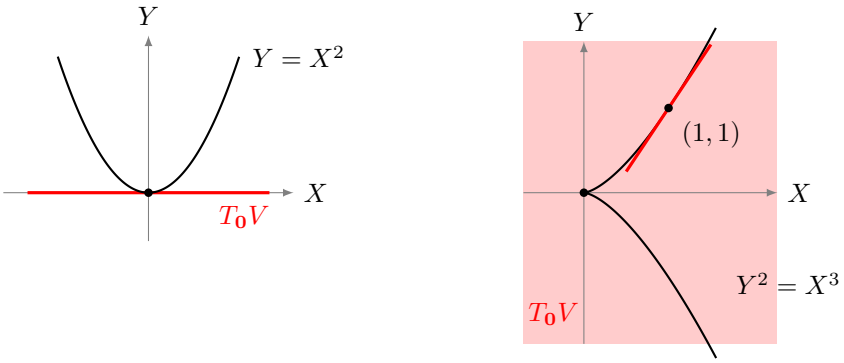

Figure 14: Tangent spaces (red). At the origin of the parabola, $T_{\mathbf{0}}V$ is the line $Y = 0$ and $\mu = 1 = \dim$. At the origin of the cusp, $T_{\mathbf{0}}V$ is the whole plane and $\mu = 2 > 1 = \dim$; at the point $(1, 1)$ the tangent space is again a line.