Lecture 16. The spectrum of a ring

6 Abstract commutative algebra

We now begin studying commutative rings via more abstract methods. In particular, we leave the constructive mindset behind for now and admit the consequences of Zorn's lemma. We generally have Noetherian rings in mind, but we will not assume this at the outset. The first question we address is the following: given a general commutative ring A, can we define a space on which A is the ring of algebraic functions?

6.1 $\mathrm{Spec}(A)$

Let A be a commutative ring with unity. The **maximal spectrum** of A is

$$\mathrm{mSpec}(A) \coloneqq \{\mathfrak{m} \subset A : \mathfrak{m} \text{ is a maximal ideal}\}.$$

The constructions $\mathcal{V}$ and $\mathcal{I}$ have the following analogues. For an ideal $I \subset A$, define

$$\mathcal{V}(I) \coloneqq \{\mathfrak{m} \in \mathrm{mSpec}(A) : I \subset \mathfrak{m}\}$$

and for $S \subset \mathrm{mSpec}(A)$

$$\mathcal{I}(S) \coloneqq \bigcap_{\mathfrak{m} \in S} \mathfrak{m}.$$

If $A = k[X_1, X_2, \ldots, X_n]$ and k is algebraically closed, then the weak Nullstellensatz (Theorem 4.29) identifies $\mathrm{mSpec}(A)$ with k^n, and these constructions coincide with the geometric ones defined before.

For a general commutative ring, it turns out that using prime ideals is more fruitful. The main problem with the maximal spectrum is that for a ring homomorphism $f : A \to B$ and a maximal ideal $\mathfrak{m} \subset B$, the ideal $f^{-1}(\mathfrak{m})$ need not be maximal: for the inclusion $\mathbb{Z} \to \mathbb{Q}$ and $\mathfrak{m} = (0)$ we get $f^{-1}(\mathfrak{m}) = (0)$. By Proposition 2.5, on the other hand, $f^{-1}(\mathfrak{p})$ is prime whenever $\mathfrak{p}$ is prime.

The **spectrum** of the ring A is

$$\mathrm{Spec}(A) \coloneqq \{\mathfrak{p} \subset A : \mathfrak{p} \text{ is a prime ideal}\}.$$

The analogues of the constructions $\mathcal{V}$ and $\mathcal{I}$ are as follows. For an ideal $I \subset A$, define

$$\mathscr{V}(I) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}(A) : I \subset \mathfrak{p}\}$$

and for $S \subset \mathrm{Spec}(A)$

$$\mathscr{I}(S) \coloneqq \bigcap_{\mathfrak{p} \in S} \mathfrak{p}.$$

We changed the notation from $\mathcal{V}, \mathcal{I}$ to $\mathscr{V}, \mathscr{I}$ since the new constructions do not reproduce the previous ones when $A = k[X_1, X_2, \ldots, X_n]$, even when k is algebraically closed: by Lemma 4.39, $\mathscr{V}(I)$ is in bijection with the irreducible affine varieties contained in $\mathcal{V}(I)$, not with its points.

- If $I_1, \ldots, I_r \subset A$ are ideals, then $\mathscr{V}(I_1 \cap \ldots \cap I_r) = \mathscr{V}(I_1) \cup \ldots \cup \mathscr{V}(I_r)$.
- $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals in A, then $\mathscr{V}(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} \mathscr{V}(I_\lambda)$.
- $\mathscr{V}(\{0\}) = \mathrm{Spec}(A)$ and $\mathscr{V}(A) = \emptyset$.

These properties imply that $\mathrm{Spec}(A)$ is a topological space with closed sets given by $\mathscr{V}(I)$ for I an ideal in A. This is called the (abstract) Zariski topology.

$\lhd$ Suppose M is a matrix with entries in $\mathbb{C}$. Consider the $\mathbb{C}$-algebra $\mathbb{C}[M] \coloneqq \{f(M) : f \in \mathbb{C}[X]\}$ where multiplication is given by matrix multiplication. Show that $\mathrm{Spec}(\mathbb{C}[M])$ coincides with the eigenvalues of the matrix M. This is where the name "spectrum" comes from. (Eigenvalues of a Hermitian matrix is known as the spectrum of that Hermitian operator; terminology used in quantum mechanics which can then be traced back to wave spectrum and light.)

$\lhd$ From Proposition 2.5, we see that if $f : A \to B$ is a ring homomorphism we get a map $f^* : \mathrm{Spec}B \to \mathrm{Spec}A$. It is easy to check that this map is continuous in the Zariski topology (check it on basic open sets!).

$\lhd$ Show that the natural map $A \to A/\sqrt{(0)}$ induces a homeomorphism between the prime spectra.

Lemma 6.1. *For $S \subset \mathrm{Spec}(A)$ we have $\mathscr{V}(\mathscr{I}(S)) = \overline{S}$.*

Proof. We have $S \subset \mathscr{V}(\mathscr{I}(S))$. Conversely, let $C = \mathscr{V}(I) \subset \mathrm{Spec}(A)$ be a closed set such that $S \subset C$, we need to show that $\mathscr{V}(\mathscr{I}(S)) \subset C$. Since $S \subset C$, for all $\mathfrak{p} \in S$ we have $I \subset \mathfrak{p}$. Then $I \subset \cap_{\mathfrak{p} \in S} \mathfrak{p} = \mathscr{I}(S)$. Hence $C = \mathscr{V}(I) \supset \mathscr{V}(\mathscr{I}(S))$, as required. □

Lemma 6.2. *Let $I \subset A$ be an ideal. Then, $\mathscr{I}(\mathscr{V}(I)) = \sqrt{I}$.*

Proof. This is just the same statement as the one given in Proposition 2.7. □

Note that this is only a formal analogue of the Nullstellensatz that we proved in the case $A = k[X_1, \ldots, X_n]$. It is a much easier result not as deep as the Nullstellensatz. The main difference is that

$$\mathscr{I}(\mathscr{V}(I)) = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p} \text{ and } \mathcal{I}(\mathcal{V}(I)) = \bigcap_{I \subset \mathfrak{m}} \mathfrak{m}.$$

These two expressions coincide for Jacobson rings (by definition) but not in general.

Question: How can A be viewed as functions on $\mathrm{Spec}A$?

Let $f \in A$ and $\mathfrak{p} \in \mathrm{Spec}A$, we define the "value" of f at $\mathfrak{p}$ as follows. Since $\mathfrak{p}$ is a prime ideal, $A/\mathfrak{p}$ is an integral domain, so we let $k(\mathfrak{p}) = Frac(A/\mathfrak{p})$ be the fraction field. This field is called

the **residue field** at $\mathfrak{p}$. Then, we define $f(\mathfrak{p})$ as the image of f under the composition of the ring homomorphisms:

$$A \to A/\mathfrak{p} \to k(\mathfrak{p})$$

Thus, an element $f \in A$ can be "evaluated" at a point $\mathfrak{p}$ as $f(\mathfrak{p}) \in k(\mathfrak{p})$. Moreover, $f(\mathfrak{p}) = 0$ if and only $(f) \subset \mathfrak{p}$, or equivalently $\mathfrak{p} \in \mathscr{V}((f))$.

Conversely, one can characterize prime ideals in A as the kernel of homomorphisms $A \to k$ where k can be arbitrary field. Thus, we add a point $\mathfrak{p}$ for every possible evaluation homomorphism from A to some field $k(\mathfrak{p})$.

In the geometric case, when $A = k[X_1, \ldots, X_n]/I$ is a finitely generated k-algebra and k is algebraically closed, a maximal ideal is given by $\mathfrak{m} = (X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n)$ and one can canonically identify $k(\mathfrak{m}) = k[X_1, \ldots, X_n]/\mathfrak{m}$ with k via the homomorphism $A \to k$ that corresponds to the evaluation homomorphism

$$A \to k \quad , \quad f \to f(a_1, a_2, \ldots, a_n)$$

So the above assignment gives back the usual notion of functions on $\mathcal{V}(I)$ under the bijection between maximal ideals in A and the points of $\mathcal{V}(I)$.

In general, the construction views A as a generalized function on $\mathrm{Spec}(A)$ since it possibly takes values in different fields $k(\mathfrak{p})$ depending on the point $\mathfrak{p}$. This is the starting point of Grothendieck's scheme theory. You should take a course on Algebraic Geometry to see how this idea develops further.

▶ Let's consider the case $A = \mathbb{Z}$. Then, $\mathrm{Spec}(\mathbb{Z})$ consist of maximal ideal (p) for prime $p \in \mathbb{Z}$ and the non-maximal, prime ideal (0). An element $n \in \mathbb{Z}$ is viewed as a function which sends

$$(0) \to n \in \mathbb{Q} \quad , \quad (p) \to n \pmod{p} \in \mathbb{F}_p$$

◁ Let A be an integral domain. Prove that $\{(0)\}$ is a dense point in $\mathrm{Spec}(A)$ in the Zariski topology.

◁ Prove that $\mathrm{Spec}\mathbb{Z}$ and $\mathrm{Spec}\overline{\mathbb{Q}}[X]$ are homeomorphic.

Definition 6.3. *Suppose X is a topological space.*

(a) We say that X is Noetherian if all the chains $Y_1 \supset Y_2 \supset \ldots$ with $Y_i \subset X$ closed subspaces stabilise: There exists N such that $Y_N = Y_{N+i}$ for all $i > 0$.

(b) We say that X is reducible if $X = X_1 \cup X_2$ with X_1, X_2 proper closed subspaces of X. Otherwise, we say X is irreducible.

The following statements are proven exactly the same way as in Corollary 4.37 and Lemma 4.39.

▶ Let A be a Noetherian ring, then $X = \mathrm{Spec}A$ is a Noetherian topological space.

▶ Let $Y \subset \mathrm{Spec}(A)$ for a ring A be a closed subset. Then, Y is irreducible if and only if $\mathscr{I}(Y)$ is prime.

▶ If X is a Noetherian topological space, then $X = Y_1 \cup Y_2 \cup \ldots \cup Y_r$ with Y_i closed and irreducible subspaces.

▶ If A is a Noetherian ring and I is a proper ideal. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ with $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$ such that $\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_r$. Moreover, if $\mathfrak{q} \in \mathrm{Spec}(A)$ with $I \subset \mathfrak{q}$, then there exists i such that $\mathfrak{p}_i \subset \mathfrak{q}$. In particular, A contains finitely many minimal primes.