Lecture 14. Irreducible decomposition; the Zariski topology

In the last lecture we saw that, for k algebraically closed, the radical of a proper ideal $I \subset k[X_1, \ldots, X_n]$ is the intersection of the maximal ideals containing I (Corollary 4.36). On the other hand, we also have the following decomposition in terms of finitely many prime ideals containing I, which will turn out to be the minimal ones.

Corollary 4.37. *Let k be algebraically closed and I a proper ideal in $k[X_1, \ldots, X_n]$. Then there exist prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r \subset k[X_1, \ldots, X_n]$ such that*

$$\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_r.$$

Moreover,

$$\deg HP_I(t) = \deg HP_{\sqrt{I}}(t) = \max\{\deg HP_{\mathfrak{p}_i}(t) : 1 \leq i \leq r\}.$$

Definition 4.38. *A non-empty affine variety V is said to be* **irreducible** *if V cannot be written as a union $V = V_1 \cup V_2$ where $V_1, V_2 \subsetneq V$ are affine varieties.*

Lemma 4.39. *A non-empty affine variety V is irreducible if and only if $\mathcal{I}(V)$ is a prime ideal. In particular, if k is algebraically closed, then $\mathcal{V}(\mathfrak{p})$ is irreducible for every prime ideal $\mathfrak{p}$.*

Proof. Suppose $V = \mathcal{V}(I)$ is irreducible and let $f, g \in k[X_1, \ldots, X_n]$ with $fg \in \mathcal{I}(V)$. Then

$$V = (V \cap \mathcal{V}(f)) \cup (V \cap \mathcal{V}(g)) = \mathcal{V}(I + (f)) \cup \mathcal{V}(I + (g))$$

is a union of two affine varieties, so one of them is all of V, that is, $f \in \mathcal{I}(V)$ or $g \in \mathcal{I}(V)$. Since $V \neq \emptyset$, $\mathcal{I}(V)$ is a proper ideal, hence prime.

Conversely, suppose $\mathcal{I}(V)$ is prime and $V = V_1 \cup V_2$ with $V_1, V_2 \subsetneq V$ affine varieties. Since $\mathcal{V}(\mathcal{I}(V_i)) = V_i \neq V$, we have $\mathcal{I}(V_i) \supsetneq \mathcal{I}(V)$, so we can choose $f_i \in \mathcal{I}(V_i) \setminus \mathcal{I}(V)$. Then $f_1 f_2$ vanishes on $V_1 \cup V_2 = V$, so $f_1 f_2 \in \mathcal{I}(V)$, contradicting the primality of $\mathcal{I}(V)$.

For the last statement, $\mathcal{V}(\mathfrak{p})$ is non-empty by the weak Nullstellensatz, and $\mathcal{I}(\mathcal{V}(\mathfrak{p})) = \sqrt{\mathfrak{p}} = \mathfrak{p}$ by the strong Nullstellensatz. □

Proof of Corollary 4.37. We first show that every non-empty affine variety V can be written as a union $V = V_1 \cup V_2 \cup \ldots \cup V_r$ of finitely many irreducible affine varieties. Suppose, for contradiction, that some non-empty affine variety cannot be written in this way, and consider the ideals $\mathcal{I}(W)$ of all such varieties W. Since $k[X_1, \ldots, X_n]$ is Noetherian, this non-empty collection of ideals has a maximal element $\mathcal{I}(W_0)$ (see the remark after the proof). As $\mathcal{V}(\mathcal{I}(W)) = W$ for every affine variety W, this means that W_0 is minimal among the varieties that are not finite unions of irreducible varieties. Now W_0 is not irreducible, so $W_0 = W_1 \cup W_2$ with $W_1, W_2 \subsetneq W_0$ affine varieties, both non-empty since their union is W_0. By the minimality of W_0, both W_1 and W_2 are finite unions of irreducible varieties, hence so is W_0, a contradiction.

Now let $V = \mathcal{V}(I)$, which is non-empty by the weak Nullstellensatz, and write $V = V_1 \cup \ldots \cup V_r$ with V_i irreducible. By the strong Nullstellensatz,

$$\sqrt{I} = \mathcal{I}(V) = \bigcap_{i=1}^{r} \mathcal{I}(V_i),$$

and by Lemma 4.39 the ideals $\mathfrak{p}_i = \mathcal{I}(V_i)$ are prime.

Next, $\sqrt{I} \subset \mathfrak{p}_i$ implies $HF_{\mathfrak{p}_i}(s) \leq HF_{\sqrt{I}}(s)$ for all s, hence $\deg HP_{\mathfrak{p}_i}(t) \leq \deg HP_{\sqrt{I}}(t)$ for all i, and $\deg HP_{\sqrt{I}}(t) = \deg HP_I(t)$ by Corollary 4.35. Thus

$$\max\{\deg HP_{\mathfrak{p}_i}(t) : 1 \leq i \leq r\} \leq \deg HP_{\sqrt{I}}(t) = \deg HP_I(t).$$

For the reverse inequality, let $d \coloneqq \deg HP_{\sqrt{I}}(t)$ and suppose that $\deg HP_{\mathfrak{p}_i}(t) < d$ for all i. By Corollary 4.18, there exist $1 \leq i_1 < i_2 < \ldots < i_d \leq n$ such that $\sqrt{I} \cap k[X_{i_1}, \ldots, X_{i_d}] = \{0\}$, while $\mathfrak{p}_i \cap k[X_{i_1}, \ldots, X_{i_d}] \neq \{0\}$ for every i, since $\deg HP_{\mathfrak{p}_i}(t) < d$. Choose non-zero $f_i \in \mathfrak{p}_i \cap k[X_{i_1}, \ldots, X_{i_d}]$. Then $f = f_1 f_2 \cdots f_r$ is non-zero and

$$f \in \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_r \subset \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_r = \sqrt{I},$$

so f is a non-zero element of $\sqrt{I} \cap k[X_{i_1}, X_{i_2}, \ldots, X_{i_d}]$, a contradiction. □

► The proof used the Noetherian property of $k[X_1, \ldots, X_n]$ in the form that every non-empty collection of ideals has a maximal element. This is usually deduced from the ascending chain condition by building a chain, which requires infinitely many successive choices, that is, the Axiom of Dependent Choice. For polynomial rings, Gröbner bases avoid this. Monomial ideals are determined by their finitely many minimal generators, so they form a countable set which can be enumerated explicitly, and every ascending chain of them is finite by the Hilbert basis theorem. Hence a collection of monomial ideals without a maximal element would yield an infinite ascending chain by always passing to the first strictly larger member of the collection, a contradiction. Finally, if $I \subsetneq J$ then $\mathfrak{in}(I) \subsetneq \mathfrak{in}(J)$, since otherwise a Gröbner basis of I would be a Gröbner basis of J and would generate J by Theorem 4.8; so an ideal in a collection whose initial ideal is maximal among the initial ideals is maximal in the collection. Together with Theorem 4.31, this makes Corollary 4.37 independent of the Axiom of Choice.

The decomposition of $\sqrt{I}$ as an intersection of finitely many prime ideals is unique in the following sense. Call a decomposition $\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_r$ **minimal** if $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$; any decomposition becomes minimal after discarding those $\mathfrak{p}_i$ that contain another $\mathfrak{p}_j$. The key observation is that if $\mathfrak{q}$ is a prime ideal containing I, then $\mathfrak{q} \supset \sqrt{I} \supset \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_r$, and hence $\mathfrak{q} \supset \mathfrak{p}_i$ for some i: otherwise we could choose $a_i \in \mathfrak{p}_i \setminus \mathfrak{q}$ for each i, and the product $a_1 a_2 \cdots a_r$ would lie in $\mathfrak{q}$, contradicting the primality of $\mathfrak{q}$. Now suppose $\sqrt{I} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ are two minimal decompositions. By the observation, each $\mathfrak{q}_j$ contains some $\mathfrak{p}_i$, which in turn contains some $\mathfrak{q}_l$; then $\mathfrak{q}_l \subset \mathfrak{q}_j$ forces $l = j$ by minimality, so $\mathfrak{q}_j = \mathfrak{p}_i$. Thus every $\mathfrak{q}_j$ is one of the $\mathfrak{p}_i$ and, by symmetry, every $\mathfrak{p}_i$ is one of the $\mathfrak{q}_j$: $r = s$ and the two collections coincide up to reordering. The observation also shows that the $\mathfrak{p}_i$ in a minimal decomposition are exactly the minimal prime ideals containing I, that is, the prime ideals containing I which do not properly contain

another prime ideal containing I. Geometrically, the $V_i = \mathcal{V}(\mathfrak{p}_i)$ are the maximal irreducible subvarieties of $\mathcal{V}(I)$, called its **irreducible components**.

▶ Let $I = (XZ, YZ) \subset k[X, Y, Z]$. A point lies in $\mathcal{V}(I)$ if and only if $Z = 0$ or $X = Y = 0$, so $\mathcal{V}(I)$ is the union of the plane $\mathcal{V}(Z)$ and the line $\mathcal{V}(X, Y)$, see Figure 6. Correspondingly, $I = (Z) \cap (X, Y)$ is the minimal prime decomposition; in particular I is radical. The two components have dimensions 2 and 1, and indeed $\deg HP_I(t) = 2$: the standard monomials are $X^a Y^b$ and Z^c, so $HF_I(s) = \binom{s+2}{2} + s$.

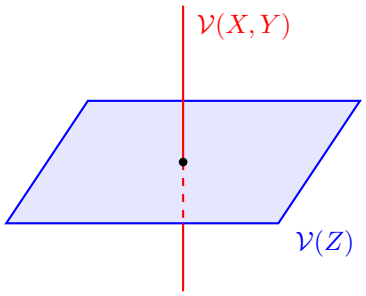


Figure 6: The variety $\mathcal{V}(XZ, YZ)$: its irreducible components are the plane $\mathcal{V}(Z)$ and the line $\mathcal{V}(X, Y)$.

Here is how you can obtain the minimal prime decomposition in Macaulay2:

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R = QQ[x,y,z,w,t]
I = ideal(x^2*y-z^3, x*y*w-z^2*t, x*w*t-z*t^2)
J = radical I
C = minprimes J
netList C
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Let us now summarize the correspondence between algebra and geometry with the following table (we consider $k[X_1, \ldots, X_n]$ with k algebraically closed):

Algebra	Geometry
radical ideal $\sqrt{I}$	affine variety $\mathcal{V}(\sqrt{I})$
$\sqrt{\sqrt{I} + \sqrt{J}} = \sqrt{I + J}$	$\mathcal{V}(\sqrt{I}) \cap \mathcal{V}(\sqrt{J})$
$\sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$	$\mathcal{V}(\sqrt{I}) \cup \mathcal{V}(\sqrt{J})$
$\sqrt{I} : \sqrt{J}$	$\overline{\mathcal{V}(\sqrt{I}) \setminus \mathcal{V}(\sqrt{J})}$
prime ideal $\mathfrak{p}$	irreducible affine variety $\mathcal{V}(\mathfrak{p})$
maximal ideal $\mathfrak{m} = (X_1 - a_1, \ldots, X_n - a_n)$	point $(a_1, \ldots, a_n) \in k^n$
prime decomposition $\sqrt{I} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$	irreducible components $\mathcal{V}(\mathfrak{p}_1), \ldots, \mathcal{V}(\mathfrak{p}_r)$

Note that $\sqrt{I} + \sqrt{J}$ need not be radical: for $I = (Y - X^2)$ and $J = (Y)$ in $k[X, Y]$ we have $I + J = (Y, X^2)$, whose variety is the single point $(0, 0)$ at which the parabola is tangent to the line. We covered all the items in the table except the fourth row, so let us explain that next.

◁ Show that any finite union and any intersection of affine varieties is an affine variety: $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(IJ)$ and $\bigcap_\alpha \mathcal{V}(I_\alpha) = \mathcal{V}(\sum_\alpha I_\alpha)$. (An arbitrary union need not be: $\mathbb{Z} \subset \mathbb{C}$ is not an affine variety.)

This shows that we can define a topology on k^n whose closed sets are the affine varieties. This is called the **Zariski topology**. If $V \subset k^n$ is an affine variety, it can be endowed with the subspace topology: an open set of V is the intersection with V of an open set of k^n, and the closed sets of V are the affine varieties contained in V. If $f \in k[X_1, X_2, \ldots, X_n]$ then $U_f = k^n \setminus \mathcal{V}(f)$ is called a **basic open set**.

◁ Show that the basic open sets U_f form a basis of the Zariski topology: any open set is a union of basic open sets.

◁ Show that the Zariski topology on k^n is quasi-compact: any open cover of k^n has a finite subcover.

Note that when $k = \mathbb{R}$ or $\mathbb{C}$, the Zariski topology is quite different from the usual (Euclidean) topology. It is much coarser: it has far fewer closed sets, since every affine variety is closed in the Euclidean topology, polynomials being continuous. It is not Hausdorff; in fact, any two non-empty open subsets of k^n intersect.

◁ Show that an affine variety V is irreducible if and only if any two non-empty open subsets of V intersect.

Proposition 4.40. *The closure in the Zariski topology of $\mathcal{V}(I) \setminus \mathcal{V}(J)$ is contained in $\mathcal{V}(I : J)$, that is,*

$$\overline{\mathcal{V}(I) \setminus \mathcal{V}(J)} \subset \mathcal{V}(I : J).$$

Moreover, equality holds if $I = \sqrt{I}$ is a radical ideal and k is algebraically closed.

◁ Prove Proposition 4.40. (Hint: for the equality, show that $I : J = \mathcal{I}(\mathcal{V}(I) \setminus \mathcal{V}(J))$ when I is radical, using the strong Nullstellensatz.)

▶ Let $I = (XY)$ and $J = (X)$ in $k[X, Y]$, so that $\mathcal{V}(I)$ is the union of the two coordinate axes and $\mathcal{V}(J)$ is the Y-axis. Then $\mathcal{V}(I) \setminus \mathcal{V}(J)$ is the X-axis with the origin removed, which is not an affine variety. Its Zariski closure is the whole X-axis, and indeed $I : J = (XY) : (X) = (Y)$, see Figure 7. So the colon ideal removes the component $\mathcal{V}(J)$, and the closure puts back the points of the other components lying on it.

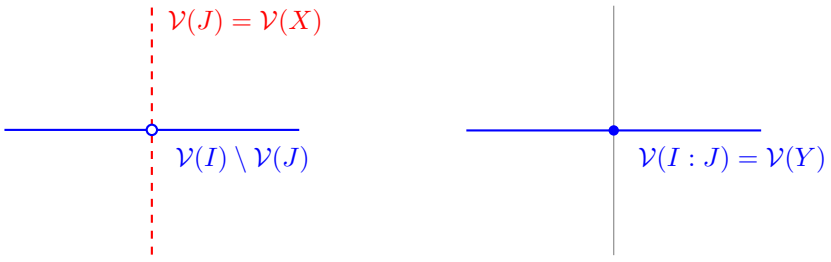


Figure 7: For $I = (XY)$ and $J = (X)$, the set $\mathcal{V}(I) \setminus \mathcal{V}(J)$ is the X-axis without the origin. Its Zariski closure is the whole X-axis, which is $\mathcal{V}(I : J)$ with $I : J = (Y)$.

▶ Consider the set of four points $S = \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{C}^2$. Its ideal is $\mathcal{I}(S) = (X^2 - X, Y^2 - Y)$. To remove the two points lying on the line $X = Y$, we compute $\mathcal{I}(S) : (X - Y) = (X + Y - 1, Y^2 - Y)$, whose variety is $\{(1,0),(0,1)\}$. Here is how you can do this in Macaulay2:

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R= QQ[x,y]
ideal(x^2-x,y^2-y) : ideal (x-y)
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