Homework V

Due for 11 December 2023

This homework will constitute %2 of your final grade. I will choose 2 random ones and grade those ones. Thus, to guarantee that you get full marks you should submit correct solutions to all problems. No partial credit is given but minor mistakes will be tolerated. So, you get either %1 or %0 from each problem. Some problems may be harder and you may need to do some reading before being able to solve them. You are encouraged to work on the problem sets in groups but the final write-up should be yours.

1) Supppose A is Noetherian, and $\sqrt{(0)} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s$ is the minimal prime decomposition of the nilradical. Show that $\cup_{i=1}^{s} \mathfrak{p}_i$ consists of zero divisors alone. Use this to show that, $(a) \subset A$ a proper principal ideal with a not a zero-divisor, and $\mathfrak{p}$ a prime ideal minimal among prime ideals such that $a \in \mathfrak{p}$. Then $h(\mathfrak{p}) = 1$.

2) Let A be a Noetherian integral domain. Then, A is a UFD if and only if every prime ideal $\mathfrak{p} \subset A$ with $h(\mathfrak{p}) = 1$ is principal.

3) Suppose that A is a finitely generated k-algebra and an integral domain. Show that $h(\mathfrak{m}) = \dim A$ for all maximal ideals $\mathfrak{m} \subset A$. (Hint: Show that you can reduce to the special case $A = k[X_1, \ldots, X_n]$.) Deduce that for a prime ideal $\mathfrak{p} \subset A$, one has $\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} = \dim A$.

4) Let $I = (f_1, \ldots, f_r) \subset k[X_1, \ldots, X_n]$, then for any irreducible component W of $V = \mathscr{V}(I)$, one has $\dim(W) \geq n - r$.

5) We call a local Noetherian ring A with maximal ideal $\mathfrak{m}$ regular if $\dim A = \mu(A)$. Show that the ring $(k[x,y]/(y - x^2))_{(x,y)}$, the localisation of $k[x,y]/(y - x^2)$ at $\mathfrak{m} = (x, y)$ is regular, but $(k[x,y]/(x^2 - y^3))_{(x,y)}$ is not regular.

6) Let B be an integral extension of A. Let $\mathfrak{m} \subset B$ be a maximal ideal and $\mathfrak{n} = \mathfrak{m} \cap A$ be the corresponding maximal ideal in A. Is $B_{\mathfrak{m}}$ integral over $A_{\mathfrak{n}}$? (Hint: Consider the example $A = k[x^2 - 1]$ and $B = k[x]$, where k is a field with characteristic other than 2, and $\mathfrak{m} = (x - 1)$. Can the element $1/(x + 1)$ be integral?)