Homework IV Solutions
Problems 4 and 6

4) Let A be a finitely generated k-algebra (k a field), then show that A is an Artinian ring if and only if A is finite-dimensional as a k-vector space.

▷ Suppose that A is finite-dimensional as a k-vector space, consider a descending chain of submodules $A = M \supset M_1 \supset M_2 \ldots$. Since $k \subset A$, each M_i is in particular a k-vector space. Since A is finite-dimenisonal as a k-vector space, M_i are finite-dimensional as k-vector spaces. Now, it must be that the chain stabilises, as otherwise, we would have an infinite descending chain of finite-dimensional k-vector spaces, which is a contradiction. Thus, A is Artinian.

Conversely, suppose that A is Artinian. We give two different arguments for showing that A is finite dimensional k-vector space. Note that as A is a finitely generated algebra, by Hilbert basis theorem, we know that it is Noetherian.

First argument: By Noether normalisation, we can find a polynomial ring $k[a_1, \ldots, a_d] \subset A$ such that A is integral over $k[a_1, \ldots, a_d]$. But $\dim A = 0$, so $d = 0$ by Cohen-Seidenberg, which means A is integral over k. Now, A is a finitely generated algebra over k and is integral over k, hence it has to be a finitely generated k-module, which means it is a finite dimensional k-vector space.

Second argument: Since A is Artinian, it has a composition series $A = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_n = \{0\}$ such that each module M_i/M_{i+1} is a simple A-module. Simple A-modules are isomorphic to some $A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$. But, $A/\mathfrak{m}$ is a field that is finitely generated over k, since A is finitely generated over k, hence by Zariski's lemma, $A/\mathfrak{m}$ is a finite field extension of k, hence, it a finite-dimensional vector space over k. But now, we see by induction that, as a k-vector space A is isomorphic to $M_0/M_1 \oplus M_1/M_2 \oplus \ldots \oplus M_{n-1}/M_n$ which is a finite-dimensional k-vector space since each M_i/M_{i+1} is so.

6) Suppose that A is a ring with the property that $A_\mathfrak{p}$ has no nilpotent elements for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Show that A has no nilpotent elements. If each $A_\mathfrak{p}$ is an integral domain, must A be an integral domain?

▷ Suppose that A has a nilpotent element, that is $x \in A$ such that $x^n = 0$ for som $n > 1$. Let $\mathrm{ann}(x) = \{a \in A : ax = 0\}$ be the annihilator ideal of x. In fact, it is a proper ideal since $1 \notin \mathrm{ann}(x)$. Let $\mathfrak{m} \supset \mathrm{ann}(x)$ be a maximal ideal containing $\mathrm{ann}(x)$. Consider the localisation $A_\mathfrak{m}$. We claim that $\frac{x}{1} \neq 0 \in A_\mathfrak{m}$. Indeed, if $\frac{x}{1} = 0 \in A_\mathfrak{m}$, there exists $s \in A \setminus \mathfrak{m}$ such that $sx = 0$, but such $s \in \mathrm{ann}(x) \subset \mathfrak{m}$. Therefore, it follows that $\frac{x}{1} \neq 0 \in A_\mathfrak{m}$ but $(\frac{x}{1}) = \frac{x^n}{1} = 0$, hence $\frac{x}{1}$ is a nilpotent element in $A_\mathfrak{m}$, which is a contradiction to the hypothesis of the problem. Hence, A has no nilpotent elements.

Take any field k (for example $k = \mathbb{F}_2$) and consider $A = k \times k$. Then, $(1, 0) \cdot (0, 1) = 0$ hence A is not an integral domain, but the only prime ideals in A are $k \times (0)$ and $(0) \times k$ (since for any ideal $I \subset k \times k$, $I \cap k \times \{0\}$ and $I \cap \{0\} \times k$ are ideals), and their localisations are easily seen to be isomorphic to k, hence are integral domains (in fact, they are isomorphic to fields).