

Homework IV

Due for 27 November 2023

This homework will constitute 20% of your final grade. I will choose 2 random ones and grade those ones. Thus, to guarantee that you get full marks you should submit correct solutions to all problems. No partial credit is given but minor mistakes will be tolerated. So, you get either 100% or 0% from each problem. Some problems may be harder and you may need to do some reading before being able to solve them. You are encouraged to work on the problem sets in groups but the final write-up should be yours.

1) A group G is called residually finite if for every non-identity element $g \in G$, there exists a group homomorphism $\pi : G \rightarrow F$ to a finite group F such that $\pi(g) \neq 1$. Let $GL(n, \mathbb{C})$ be the general linear group with \mathbb{C} coefficients. Any group homomorphism from $GL(n, \mathbb{C})$ to a finite group is constant¹, thus $GL(n, \mathbb{C})$ is not residually finite. Show that every finitely generated subgroup of $GL(n, \mathbb{C})$ is residually finite (Hint: use the “arithmetic” version of Nullstellensatz).

2) Show that for any ring A , $\text{Spec}(A)$ and $\text{mSpec}(A)$ are topological spaces. Precisely, prove the following for $V = \mathcal{V}$ or \mathcal{V} :

(i) If $I_1, \dots, I_r \subset A$ are ideals, then $V(I_1 \cap \dots \cap I_r) = V(I_1) \cup \dots \cup V(I_r)$.

(ii) $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals in A , then $V(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

(iii) $V(\{0\}) = \text{Spec}(A)$ or $\text{mSpec}(A)$, and $V(A) = \emptyset$.

3) i) For any ring A , show that $\text{Spec}(A)$ is compact, that is each of its open covers has a finite subcover. (Sometimes, this is referred to as quasi-compact as $\text{Spec}(A)$ is almost never Hausdorff.)

ii) If $f : A \rightarrow B$ is a ring homomorphism we get a map $f^* : \text{Spec} B \rightarrow \text{Spec} A$. Show that this map is continuous in the Zariski topology. Show that $A \rightarrow A/\sqrt{(0)}$ gives a homeomorphism between the spectrum of these rings.

iii) Suppose A is an integral domain. Prove that $\{(0)\}$ is a dense point in $\text{Spec}(A)$ in the Zariski topology.

4) Let A be a finitely generated k -algebra (k a field), then show that A is an Artinian ring if and only if A is finite-dimensional as a k -vector space.

¹You can assume this fact and here is a proof. Let m be the order of the group F . Given any non-identity element a in $GL(n, \mathbb{C})$ we can find b such that $b^m = a$, but then the image of the homomorphism should send a to 1. To extract an m -th root of an element of $GL(n, \mathbb{C})$ we consider its Jordan normal form. Diagonalisable elements are obviously m^{th} powers. Up to a non-zero multiple, a Jordan block is $1 + N$, where N is nilpotent. Write $(1 + a_1 N + a_2 N^2 + \dots)^m = 1 + N$ and equate the coefficients at the powers of N . We determine $a_1 = 1/m$, then a_2, a_3 , and so on.

5) Let $f \in A$ and $S := \{f^n : n = 1, 2, \dots\} \cup \{1\}$ multiplicatively closed set. Then we write $A_f := S^{-1}A$. Prove that A_f is isomorphic to the ring $A[X]/(Xf - 1)$.

6) Suppose that A is a ring with the property that $A_{\mathfrak{p}}$ has no nilpotent elements for all $\mathfrak{p} \in \text{Spec}(A)$. Show that A has no nilpotent elements. If each $A_{\mathfrak{p}}$ is an integral domain, must A be an integral domain?