Homework III

Due for 13 November 2023

This homework will constitute %2 of your final grade. I will choose 2 random ones and grade those ones. Thus, to guarantee that you get full marks you should submit correct solutions to all problems. No partial credit is given but minor mistakes will be tolerated. So, you get either %1 or %0 from each problem. Some problems may be harder and you may need to do some reading before being able to solve them. You are encouraged to work on the problem sets in groups but the final write-up should be yours.

1) Let k be an algebrically closed field. I an ideal in $k[X_1, \ldots, X_n]$ and G a Gröbner basis for I (for some monomial ordering). Let $X = V(I)$ be the affine variety defined by I. Show that the following are equivalent:

i) X is a finite set.

ii) For all $j \in \{1, \ldots, n\}$, there exists $g_j \in G$ such that $LM(g_j) = X_j^{m_j}$ with $m_j \geq 0$.

Under these conditions, show that the size of X satisfies $|X| \leq \prod_{j=1}^{n} m_j$.

2) Let X be an affine variety such that X is a finite set. Let $I = \mathcal{I}(X)$. Show that the Hilbert polynomial $HP_I(t)$ is constant and equal to $|X|$.

3) Compute the Hilbert polynomials for the following ideals

i. $(X^3Y, XY^2) \subset k[X, Y]$,

ii. $(X^3YZ^5, XY^3Z^2) \subset k[X, Y, Z]$

4) Let $A \subset B$ be commutative rings. Show that B is finitely generated as an A-module if and only if B is finitely generated as an A-algebra and integral over A.

5) Find the normalisation of $\mathbb{Z}[\sqrt{d}]$ for d a square-free integer.

6) Show that 1 is in the ideal $(X^2 + Y^2 - Z, (X^2 + Y^2 - Y)X, Y^5 - Y^4Z + Z^2Y^3, XYZ - 1)$ in $\mathbb{C}[X, Y, Z]$.

7) Consider a 2-by-2 matrix M with entries X, Y, Z, W. If we then want to solve $M^2 = 0$, we get four equations and let's make that into an ideal $I = (X^2+YZ, XY+YW, XZ+WZ, W^2+YZ) \in k[X, Y, Z, W]$. Is I a radical ideal? Show that $\sqrt{I} = (X + W, XW - YZ)$.

8) Show that if $f : R \to S$ is a ring homomorphism between finitely generated k-algebras R and S, then $f^{-1}(\mathfrak{m})$ is a maximal ideal of R for all maximal ideals $\mathfrak{m}$ of S.