Homework II Solutions
Problems 1 and 6

1) Show that

$$R = \{f(X,Y) = \sum a_{ij} X^i Y^j | i,j \geq 0, \text{ and } i > 0 \text{ if } j \neq 0\}$$

is a subring of the Noetherian ring $k[X,Y]$ but R is not Noetherian.

$\triangleright$ By definition the ring R is given by polynomials $f(X,Y) \in k[X,Y]$ which do not have a term of the form aY^i for $a \in k, i > 0$. Adding, subtracting, or multiplying such polynomials together cannot introduce such terms (and $0, 1 \in R$), hence R is indeed a subring. We can also describe this subring as $R = k \cdot 1 + Xk[X,Y]$ or $R = k[X] + XYk[X,Y]$ or $k[X, XY, XY^2, XY^3, \ldots]$. The latter description requires an argument (as given below).

To see that R is not Noetherian, suppose that the infinitely generated ideal

$$\mathfrak{m} = (X, XY, XY^2, \ldots)$$

of R could be finitely generated. Thus, we can write $\mathfrak{m} = (f_1, f_2, \ldots, f_s)$ for some $f_i \in R$. Since each f_i has to be in the original ideal, $f_1, f_2, \ldots, f_s$ are generated by at most finitely many of the original generators, so we deduce that there exist some i such that

$$\mathfrak{m} = (X, XY, XY^2, \ldots, XY^i)$$

This means, in particular, that there are polynomials $g_0, g_1, \ldots, g_i \in R$ such that

$$XY^{i+1} = g_0 X + g_1 XY + g_2 XY^2 + \ldots + g_i XY^i$$

Now, the only way the monomial XY^{i+1} can appear on the right hand side is if $g_j XY^j$ has XY^{i+1} as a term for some j, but that can only happen if g_j has a term Y^{i+1-j}. However, $g_j \in R$ so it has no terms of that form. We arrive at a contradiction.

6) Suppose that V is a linear space, that is

$$V = \mathcal{V}(\{f_j = \sum_{i=1}^{n} a_{ij} X_i : 1 \leq j \leq m\})$$

Show that $\dim_k V = \deg HP_I(t)$, where $I = (f_1, f_2, \ldots, f_m)$.

$\triangleright$ Solution 1 Let $A = (a_{ij})$ be the $m \times n$ matrix with entries a_{ij} and let $B = (b_{ij})$ be its reduced row echelon form (obtained after the process of Gauss elimination). Define linear forms

$g_j = \sum_{i=1}^n b_{ij} X_i$ for $j = 1, \ldots, r$ corresponding to non-zero rows of B, where we note that r is equal to rank$A$. It is clear that $I = (g_1, \ldots, g_r)$ since the linear forms $\{g_i\}_{i=1}^r$ are k-linear combinations of the linear forms $\{f_i\}_{i=1}^m$ and vice versa.

Note that $LT(g_i) = X_{\sigma(i)}$ for where $\sigma : \{1, \ldots, r\} \to \{1, \ldots n\}$ is defined such that the first non-zero entry of the i^{th}-row in the reduced row echelon form appears in column $\sigma(i)$.

Claim: $g_1, \ldots, g_r$ is a Gröbner basis for I with respect to the lexicographic monomial order with $X_1 > X_2 > \ldots > X_n$.

Computing S-polynomials, we get

$$S(g_i, g_j) = X_j g_i - X_i g_j = X_j(g_i - LT(g_i)) - X_i(g_j - LT(g_j))$$

Now all the terms that appear in $g_i - LT(g_i)$ and $g_j - LT(g_j)$ are not divisible by any of $LT(g_k)$ for $k = 1, \ldots, r$. Hence, performing division with $\{g_1, \ldots, g_r\}$ gives

$$S(g_i, g_j) = (g_i - LT(g_i))g_j - (g_j - LT(g_j))g_i$$

which shows that the remainder is zero for all $1 \leq i, j \leq r$. Hence, we proved our claim.

Now, we can use the Gröbner basis to deduce that the initial ideal is given by

$$\mathfrak{in}(I) = (X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(r)})$$

Thus, the complement $C(\mathfrak{in}(I))$ is generated as a k-vector space by monomials X^α for $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = 0$ if $i \in \mathrm{Im}(\sigma)$.

Using the graded lexicographic monomial order (with the above ordering of variables), and appealing to Macaulay's lemma, we deduce that

$$HF_I(s) = |C(\mathfrak{in}(I))_{\leq s}| = \binom{n-r+s}{s} = \frac{1}{(n-r)!} s^{n-r} + \ldots$$

from which we find that

$$\deg HP_I = n - r.$$

On the other hand, as a linear space V is given by the solutions to the linear equation

$$A \cdot X = 0$$

which by the rank-nullity theorem has dimension given by

$$\dim_k V = \dim_k \ker A = n - \mathrm{rank} A = n - r$$

Solution 2 This solution instead uses the column reduction. However, we have to be careful and remember that column operations do not preserve the null space, or the ideal I. Indeed, column reduction modifies a matrix A by multiplying from the right by a sequence of elementary matrices. Thus, the matrix A is related to its reduced column echelon form C via

$$C = A \cdot E$$

where E is some invertible matrix given as a product of matrices that perform the column operations. On the other hand, we can re-write the equation $A \cdot X = 0$ as follows:

$$A \cdot X = (A \cdot E) \cdot (E^{-1} X) = 0$$

Therefore, let us define new variables Y by

$$E^{-1} \cdot \begin{pmatrix} X_1 \\ \cdot \\ \cdot \\ X_n \end{pmatrix} = \begin{pmatrix} Y_1 \\ \cdot \\ \cdot \\ Y_n \end{pmatrix}$$

Since this is a linear change of variables (using an invertible matrix), it induces a ring isomorphism

$$k[X_1, \ldots, X_n] \to k[Y_1, \ldots, Y_n]$$

sending the ideal $I = (f_1, \ldots, f_m)$ in $k[X_1, \ldots, X_n]$ to the ideal $J = (h_1, h_2, \ldots, h_m)$ in $k[Y_1, \ldots, Y_n]$ given by the rows of C.

Now, we can observe that from the shape of the reduced column echelon form, it is easy to deduce that $J = (Y_1, \ldots, Y_r)$ where r is the number of non-zero columns in C which is again equal to the rank of A since column rank and row rank of a matrix are equal. Hence, we conclude that

$$HP_J(s) = \frac{1}{(n-r)!} s^{n-r} + \ldots$$

Alternatively, we can observe that since none of the h_i involve the variables $Y_{r+1}, \ldots, Y_n$, we conclude that $J \cap k[Y_{r+1}, \ldots, Y_n] = \{0\}$, and we also have to show that this is the maximal number of variables that J avoids but this again follows from inspecting the reduced column echelon matrix. In either way, we deduce that

$$\deg HP_J = n - r$$

It looks as if we are done, but now we have to pay back our debt since column operations changed the ideal I (as well as the null space of A). However, since we made a linear change of variables there is a k-linear bijection $C(I)_{\leq s}$ and $C(J)_{\leq s}$. Therefore, it follows that

$$HF_I(s) = HF_J(s)$$

Hence, we finally conclude that

$$\deg HP_I = n - r$$

Bonus question: Let us consider a non-linear isomorphism between polynomial rings. For example, consider the ring isomorphism $k[X_1, X_2] \to k[Y_1, Y_2]$ given by $X_1 \to Y_1 + Y_2^r$ and

$X_2 \to Y_2$ for some $r \in \mathbb{N}$. Since this is clearly a ring isomorphism, it sends any ideal I of $k[X_1, X_2]$ to an ideal J of $k[Y_1, Y_2]$ and induces an isomorphism of rings

$$k[X_1, X_2]/I \simeq k[Y_1, Y_2]/J$$

However, it is not a degree preserving isomorphism. Hence, $C(I)_{\leq s}$ may very well be not isomorphic to $C(J)_{\leq s}$. Can you construct an example of an ideal such that under such an isomorphism

$$HF_I(s) \neq HF_J(s)$$

What about $\deg HP_I(s)$ and $\deg HP_J(s)$?

7) Let $I = (x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1)$ be the ideal in $\mathbb{C}[x, y, z]$. Find a Gröbner basis for I with respect to lexicographic order $x > y > z$, and determine $V(I)$ explicitly using the Gröbner basis.

▷ Discussion

We can use Lex order and compute the reduced Gröbner basis

$$I = (z^6 - 4z^4 + 4z^3 - z^2, 2yz^2 + z^4 - z^2, y^2 - y - z^2 + z, x + y + z^2 - 1)$$

Since the first polynomial only depends on z, we can solve that and then use the others to solve for x and y. (This is what the Lex order is good for.)

When you work it out, you should get the folllowing 5 solutions.

$$(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1 + \sqrt{2}, -1 + \sqrt{2}, -1 + \sqrt{2}), (-1 - \sqrt{2}, -1 - \sqrt{2}, -1 - \sqrt{2})$$

On the other hand, we have

$$\dim_k k[x, y, z]/I = 8$$

To see this, we use the GLex order and use Macaulay's lemma to say that $HF_I(s) = HF_{\text{in}(I)}(s) = |C(\text{in}(I))|_{<s}$

It turns out that with respect to GLex order, we have that the original description is already a Gröbner basis:

$$I = (x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1)$$

Thus, with respect to GLex, we have

$$\text{in}(I) = (x^2, y^2, z^2),$$

from which it is easy to see that $C(\text{in}(I))$ is generated as a k-vector space by $\{1, x, y, z, xy, xz, yz, xyz\}$. Hence, for $s >= 3$, $HF_I(s) = 8$. Note that we also have $HF_I(0) = 1, HF_I(1) = 4, HF_I(2) = 7$. So, the Hilbert polynomial $HP_I(s)$ is constant and equal to 8.

This is all fine and correct. On the other hand, we can also look at $\text{in}(I)$ with respect to the Lex order. Then we have

$$\text{in}(I) = (x, y^2, yz^2, z^6)$$

from which we see that $C(\text{in}(I))$ is generated as a k-vector space $\{1, y, z, yz, z^2, z^3, z^4, z^5\}$. This is also 8, as it should! Because this is what the second part of Theorem 4.8 in lecture notes gives us. (Note that Theorem 4.8 is valid for any monomial ordering.)

On the other hand, since Macaulay's lemma is not valid for the Lex order. If we use the Lex order, we would get the wrong Hilbert function. Indeed, this gives us $HF_I(0) = 1, HF_I(1) = 3, HF_I(2) = 5, HF_I(3) = 6, HF_I(4) = 7, HF_I(s) = 8$ for $s >= 5$. As you can observe this does not agree with the above (correct) computation of Hilbert function. (Recall that Hilbert function does not depend on the monomial ordering.)

So we have $5 = |V(I)| < 8 = \dim_k k[x, y, z]/I$. In general, when $V(I)$ is finite, one has that $|V(I)| = \dim_k k[X_1, .., X_s]/I$ if I is radical (a fact that you can prove using Nullstellensatz, assuming k is algebraically closed here). So, this means that $I = (x^2 + y + z - 1, x + y^2 + z - 1, x+y+z^2-1)$ is not a radical ideal. We can also confirm this with Macaulay 2 as follows:

```
R= QQ[x,y,z, w,MonomialOrder => GLex ]
I = ideal (x^2+y+z-1, x+y^2+z-1,x+y+z^2-1)
I == radical I
```

which outputs false! Fun!

So, what's an element in $\sqrt{I}$, that is not in I? This is where we can apply Nullstellensatz. $\sqrt{I}$ is the ideal of functions in $k[x, y, z]$ that vanish on our 5 points. It looks like it is easy to write down some functions that vanish on our 5 points. Here is one:

$$xy - yz$$

So, $xy - yz$ is in $\sqrt{I}$ but is it in I? We are in luck, we already have a Gröbner basis at hand for I. We need to divide $xy - yz$ by $\{x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1\}$ and see if we get a 0 remainder. But, we don't! Because, the $LT(xy - yz) = xy$ which is not divisble by x^2 or y^2 or z^2. Happy.

If you really wanted to finish this off, you can also find a Gröbner basis for $\sqrt{I}$. I can't resist the temptation to enter this to Macaulay 2.

```
R= QQ[x,y,z, MonomialOrder => Lex ]
I = ideal (x^2+y+z-1, x+y^2+z-1,x+y+z^2-1)
gens gb radical I
```

which gives me

$$\sqrt{I} = (z^4 + z^3 - 3z^2 + z, 2yz + z^3 - z, y^2 - y - z^2 + z, x + y + z^2 - 1)$$

and

$$\text{in}(\sqrt{I}) = (z^4, yz, y^2, x)$$

Hence, $C(\text{in}(\sqrt{I}))$ is generated as k-vector space $1, y, z, z^2, z^3$ which has 5 elements. Thus, as we claimed:

$$|V(I)| = \dim_k k[x, y, z]/\sqrt{I}.$$