Homework II

Due for 2 November 2022

This homework will constitute %2 of your final grade. I will choose 2 random ones and grade those ones. Thus, to guarantee that you get full marks you should submit correct solutions to all problems. No partial credit is given but minor mistakes will be tolerated. So, you get either %1 or %0 from each problem. Some problems may be harder and you may need to do some reading before being able to solve them. You are encouraged to work on the problem sets in groups but the final write-up should be yours.

1) Show that

$$R = \{f(X,Y) = \sum a_{ij}X^iY^j | i,j \geq 0, \text{ and } i > 0 \text{ if } j \neq 0\}$$

is a subring of the Noetherian ring $k[X,Y]$ but R is not Noetherian.

2) Prove that if R is Noetherian, $R[[T]]$ is also Noetherian. (Use a variation of the proof of Hilbert basis theorem.)

3) Prove that if U is an affine variety, then $U = \mathcal{V}(\mathcal{I}(U))$.

4) Take $I = (f_1, f_2)$ with $f_1 = X^3 - 2XY$ and $f_2 = X^2Y - 2Y^2 + X$ and use $\leq_{grlex}$. Show that $(LT(f_1), LT(f_2))$ is strictly contained in $\mathfrak{in}(I)$. In other words, show that $(LT(f_1), LT(f_2)) \subsetneq \mathfrak{in}(I)$.

5) Let $f = x^2y + xy^2 + y^2$, $f_1 = xy - 1$ and $f_2 = y^2 - 1$ in $k[x,y]$. For the lexicographic order $x > y$ divide f by f_1, f_2 and then by f_2, f_1.

6) Determine a Gröbner basis for the ideal $I = (x^3 - 2xy, x^2y - 2y^2 + x) \subset \mathbb{C}[x,y]$ for the lexicographic order $x > y$.

7) Let $I \subset k[x,y,z]$ be the ideal generated by $y - x^2$ and $z - x^3$.

a) Determine a Gröbner basis for the lexicographic order $x < y < z$.

b) Determine a Gröbner basis for the lexicographic order $x > y > z$.

8) For the lexicographic order $x > y > z > w$, determine a Gröbner basis for the ideal

$$(3x - 6y - 2z, 2x - 4y + 4w, x - 2y - z - w) \subset \mathbb{Q}[x,y,z,w]$$

Compare the result with the *reduced row echelon form* (Gaussian elimination) of the system of linear equations:

$$\begin{aligned} 3x - 6y - 2z &= 0 \\ 2x - 4y + 4w &= 0 \\ x - 2y - z - w &= 0 \end{aligned}$$

9) Suppose that V is a linear space, that is

$$V = \mathcal{V}(\{f_j = \sum_{i=1}^{n} a_{ij} X_i : 1 \leq j \leq m\})$$

Show that $\dim_k V = \deg HP_I(t)$, where $I = (f_1, f_2, \ldots, f_m)$.

10) Let $I = (x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1)$ be the ideal in $\mathbb{C}[x, y, z]$. Find a Gröbner basis for I with respect to lexicographic order $x > y > z$, and determine $V(I)$ explicitly using the Gröbner basis.

11) Let $I \subset k[X_1, \ldots, X_n]$ be a non-zero ideal, G a Gröbner basis for I with respect to the $\leq_{lex}$ order. Show that $G \cap k[X_{l+1}, \ldots, X_n]$ generates the ideal $I \cap k[X_{l+1}, \ldots, X_n]$ for all $0 \leq l \leq n$.