Homework I Solutions
Problems 8 and 13

8) Show that in a local ring R, the units in R are precisely the elements in $R - \mathfrak{m}$. (uses Zorn's lemma); conversely, a ring R whose non-units form an ideal is a local ring.

A unit cannot belong to a proper ideal, since any ideal containing a unit contains 1. Therefore, the units $R^\times \subset R - \mathfrak{m}$. On the other hand, if x is a non-unit, then the ideal (x) is proper, hence is contained in a maximal ideal (by Zorn's lemma). But $\mathfrak{m}$ is the unique maximal ideal, hence $R - \mathfrak{m} \subset R^\times$. Therefore, we conclude that $R^\times = R - \mathfrak{m}$.

Suppose $\mathfrak{m} = R - R^\times$ is an ideal. Since any proper ideal cannot contain a unit, any proper ideal is a subset of $\mathfrak{m}$. Hence, $\mathfrak{m}$ is the unique maximal ideal of R.

13) Let k be an infinite field, and $f \in k[X_1, \ldots, X_n]$. Show that $f = 0$ if and only if $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in k^n$. What about if k is a finite field? Consider $X^2 - X$ in $\mathbb{F}_2[X]$.

Clearly, $f(X) = X^2 - X$ vanishes for all $X \in \mathbb{F}_2$ but $f \neq 0$. So, let's assume k is an infinite field.

If $f = 0$, it follows trivially that $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in k^n$. Conversely, suppose $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in k^n$. We want to show that $f = 0$. We argue by induction. The case $n = 1$ is elementary: a degree d polynomial in $k[X]$ can have at most d roots, since for every root a, we must have that $(X - a) \mid f(X)$. Since the filed k is infinite, it follows that if $f(x) = 0$ for all $x \in k$, then $f = 0$. Suppose now that the statement is true for $n - 1$. Given a polynomial $f \in k[X_1, \ldots, X_n]$, write it as $f(X_1, \ldots, X_n) = \sum_{i=1}^{s} f_i(X_1, \ldots, X_{n-1}) X_n^i$ for polynomials $f_i(X_1, \ldots, X_{n-1}) \in k[X_1, \ldots, X_{n-1}]$. Suppose that $f \neq 0$, then there exists an i such that $f_i \neq 0$. By induction hypothesis, there exists $(a_1, \ldots, a_{n-1}) \in k^{n-1}$ such that $f_i(a_1, \ldots, a_{n-1}) \neq 0$. Then, the polynomial $g(X) \in k[X]$ defined by $g(X) = f(a_1, \ldots, a_{n-1}, X)$ is a nonzero polynomial. So, it can only have finitely many zeros (by the $n = 1$ case). This contradicts the assumption that $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n)$.