

Homework I

Due for 16 October 2023

This homework will constitute 20% of your final grade. I will choose 2 random ones and grade those ones. Thus, to guarantee that you get full marks you should submit correct solutions to all 8. No partial credit is given but minor mistakes will be tolerated. So, you get either 10% or 0% from each problem. Some problems may be harder and you may need to do some reading before being able to solve them. You are encouraged to work on the problem sets in groups but the final write-up should be yours.

- 1) Show that $x \in \mathcal{J}(R)$ if and only if $1 - xy$ is a unit of R for all $y \in R$ (uses Zorn's lemma).
- 2) Show that if a prime ideal $\mathfrak{p} = I \cap J$ for ideals I and J , then $\mathfrak{p} = I$ or $\mathfrak{p} = J$.
- 3) Show that S is a saturated multiplicative set if and only if the complement of S is union of prime ideals (uses Zorn's lemma).
- 4) Show that $\mathbb{Z}[i]$ is an Euclidean domain using the norm $|a + bi| = a^2 + b^2$. (Use the fact that disks of unit norm centred at Gaussian integers cover all of the complex plane.)
- 5) Show that (a) and (ai) for $a \in \mathbb{Z}$ are prime ideals in $\mathbb{Z}[i]$ if and only if a is prime number of the form $4n + 3$. $(a + bi)$ with $a, b \neq 0$ is a prime ideal if and only if $a^2 + b^2$ is a prime number.
- 6) Show that the ideal $(2, 1 + \sqrt{-3})$ is not principal in the ring $\mathbb{Z}[\sqrt{-3}]$.
- 7) Let k be an infinite field, and $f \in k[X_1, \dots, X_n]$. Show that $f = 0$ if and only if $f(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in k^n$. Give a counter-example to the statement when $k = \mathbb{F}_2$ (finite field with 2 elements).
- 8) Let $\mathfrak{p} \subset \mathbb{Z}[X]$ be a prime ideal. Suppose $\mathfrak{p} \neq \mathbb{Z}[X]$ or (0) . Show that $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal in \mathbb{Z} . Thus, $\mathfrak{p} \cap \mathbb{Z} = (p)$ where $p = 0$ or p is a prime number.
 - (i) Suppose $\mathfrak{p} \cap \mathbb{Z} = \{0\}$, then $\mathfrak{p} = (f)$ where $f \in \mathbb{Z}[X]$ is an irreducible polynomial.
 - (ii) Suppose $\mathfrak{p} \cap \mathbb{Z} = (p)$ with p a prime number. Then, $\mathfrak{p} = (p, f)$ where $f = 0$ or $f \in \mathbb{Z}[X]$ is a monic polynomial (leading coefficient is one) and its mod p reduction $\bar{f} \in \mathbb{F}_p[X]$ is irreducible.