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BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May – June 2023

MATH70061 Commutative Algebra

The following information must be completed:

Is the paper suitable for resitting students from previous years: Yes (though some new material was covered in this year's lectures)

Category A marks: available for basic, routine material (excluding any mastery question) (35):
1(a) 6 marks; 3(a) 3 marks; 3(b) 5 marks; 4(a) 2 marks; 4(b), (ii) 6 marks; 5(a), (i),(ii),(iii) 2,2,3 marks; 5(b), (i), (ii) 3,3 marks;

Category B marks: Further 31 for demonstration of a sound knowledge of a good part of the material and the solution of straightforward problems and examples with reasonable accuracy (excluding mastery question):
1(b) 7 marks; 1(c) 7 marks; 3(c) 5 marks; 4(b) (i) 6 marks; 4(c) 6 marks.

Category C marks: the next 20 percent of the marks for parts of questions at the high 2:1 or 1st class level (excluding mastery question):
2(a) 10 marks; 2(b) 10 marks.

Category D marks: Most challenging 14 percent of the paper (excluding mastery question):
3(d) 7 marks; 5(c) 7 marks.

Signatures are required for the final version:

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BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2023

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

Commutative Algebra

Date: ??

Time: ??

Time Allowed: 2.5 Hours

This paper has *5 Questions*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. Let k be a field. Suppose that we have n points $V = \{(a_1, b_1), \dots, (a_n, b_n)\} \subset k^2$ where $a_1, \dots, a_n$ are distinct. Let

$$f(X) = \sum_{i=1}^{n} b_i \prod_{j \neq i} \frac{X - a_j}{a_i - a_j} \in k[X]$$

be the Lagrange interpolation polynomial.

(a) Show that $f(X)$ is the unique polynomial of degree $\leq n-1$ satisfying $f(a_i) = b_i$ for $i = 1, \dots, n$. (6 marks)

(b) Prove that $\mathcal{I}(V) = (Y - f(X), g(X)) \subset k[X, Y]$ where $g(X) = \prod_{i=1}^{n}(X - a_i)$. (7 marks)

(c) Prove that $\{Y - f(X), g(X)\}$ is the reduced Gröbner basis for $\mathcal{I}(V)$ for the lex order with $Y > X$. (7 marks)

(Total: 20 marks)

2. Let A be a commutative ring with unit.

(a) Suppose that all the prime ideals of A are finitely generated. Show that A is Noetherian.

[*Hint: Let* Γ *be the set of all ideals of* A *which are not finitely generated. If* $\Gamma \neq \emptyset$*, use Zorn's lemma to show that* Γ *contains a maximal element* I*. As* I *is not a prime ideal, there exist* $x, y \in A$ *such that* $x \notin I$*,* $y \notin I$ *but* $xy \in I$*. Consider the ideals* $I + (y)$ *and* $I : y$*. Show that these two ideals are finitely generated and deduce that* I *is finitely generated to arrive at a contradiction.*] (10 marks)

(b) Suppose that A is an integral domain and all the prime ideals of A are principal. Show that A is a principal ideal domain (PID).

[*Hint: Follow a similar approach to the one in Part (a).*]

(10 marks)

(Total: 20 marks)

3. (a) Define the **Krull dimension** of a ring. (3 marks)

(b) What is the Krull dimension of $\mathbb{Z}[i]$? Justify your answer. (5 marks)

(c) Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$. Show that if there exist elements $x_1, \dots, x_n \in \mathfrak{m}$ such that $R/(x_1, \dots, x_n)$ is Artinian, then $\dim R \leq n$. (5 marks)

(d) Conversely, let R be a local Noetherian ring and $\mathfrak{m}$ be the maximal ideal of height $n = \dim R$. Show that there exist elements $x_1, \dots, x_n \in \mathfrak{m}$ such that $\mathfrak{m}$ is minimal among prime ideals containing $(x_1, \dots, x_n)$.

[*Hint: To argue by induction, suppose that you have constructed elements $x_1, \dots, x_k$ such that any prime minimal among prime ideals containing $(x_1, \dots, x_k)$ has height k.*]

(7 marks)

(Total: 20 marks)

4. (a) Describe the set of all polynomials in $\mathbb{C}[X, Y]$ that vanish on $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 - 1 = 0\}$. Justify your answer. (2 marks)

(b) (i) Let k be a field that is not algebraically closed and $I \subset k[X_1, \dots, X_n]$ an arbitrary ideal. Show that the variety $\mathcal{V}(I) \subset k^n$ can be written as the zero set of a single polynomial in $k[X_1, \dots, X_n]$.

[*Hint: If $I = (f_1, \dots, f_m)$, consider $\phi_m(f_1, \dots, f_m)$ where $\phi_m \in k[X_1, \dots, X_m]$ whose only zero is $(0, \dots, 0) \in k^m$. To find such ϕ_m, first construct $\phi_2(X, Y) \in k[X, Y]$ which only vanishes at $(0, 0)$ and then apply induction by defining $\phi_m(X_1, \dots, X_m) = \phi_2(\phi_{m-1}(X_1, \dots, X_{m-1}), X_m)$.*] (6 marks)

(ii) Give an example of an affine variety in $\mathbb{C}^2$ that cannot be written as the zero set of a single polynomial. Justify your answer. (6 marks)

(c) Let $\mathfrak{m} \subset \mathbb{R}[X, Y]$ be a maximal ideal, what are the possible fields that one can get as $\mathbb{R}[X, Y]/\mathfrak{m}$? Justify your answer. For each such field, give a specific example of a maximal ideal. (6 marks)

(Total: 20 marks)

5. (a) (i) Define what it means for a ring to be a **valuation ring**. (2 marks)

(ii) Let K be a field, define the notion of a **valuation** of K. What is the valuation ring R associated to such a valuation? In which case is R called a **discrete valuation ring (DVR)**? (2 marks)

(iii) Show that a DVR is normal. (3 marks)

(b) Which of the following sets is a valuation ring of the field $\mathbb{C}(X, Y)$? Justify your answer.

(i) The subring $R = \mathbb{C}[X, Y]_{(Y)}$. (3 marks)

(ii) The subring $\mathbb{C}[X, Y]_{(X,Y)}$. (3 marks)

(c) If R is a valuation ring with field of fractions K and $\dim R = 1$ then show that R is maximal as a subring of K. Conversely, show that a maximal proper subring of a field is a valuation ring of dimension 1.

[*Hint: To show the converse, first prove that a maximal proper subring R of a field K is integrally closed. Now take $x \in K \setminus R$, and show that $x^{-1} \in R$.*]

(7 marks)

(Total: 20 marks)