Homework 4:

1) Recall that $p : E \to B$ is a normal (or regular or Galois) covering if $\Pi(p)(\pi_1(E,e)) \subset \pi_1(B,b)$ is a normal subgroup. Show that $p : E \to B$ is normal if and only if $Aut(E/B)$ acts transitively on F_b, i.e. for any $e, e' \in F_b$, there exists an $f \in Aut(E/B)$ such that $f(e) = e'$.

2) Construct a path-connected but not normal (or regular) covering of $S^1 \vee S^1$. Identify the group of deck transformations.

3) Hatcher, page 79, problem 9.

4) Hatcher, page 80, problem 18.

5) Hatcher, page 81, problem 23.

6) Let C be the space of $(p,q) \in \mathbb{C}^2$ such that $4p^3 + 27q^2 \neq 0$. Note that for such (p,q) the equation $X^3 + pX + q = 0$ has 3 distinct complex roots. Let us also define the spaces :

$$B = \{(p,q,x) \in C \times \mathbb{C} : x^3 + px + q = 0\}$$

and

$$A = \{(x,y,z) \in \mathbb{C}^3 : x + y + z = 0, x \neq y, y \neq z, z \neq x\}$$

We define $\pi : A \to B$ to be the map $(x,y,z) \to (p,q,x)$ where p,q are defined by

$$X^3 + pX + q = (X - x)(X - y)(X - z)$$

We also let $p : B \to C$ be the projection $(p,q,x) \to (p,q)$.

i) Show that π, p and $p \circ \pi$ are finite sheeted covering maps.
ii) Show that A, B, C are path-connected.
iii) Let $a = (-1,0) \in C$. Show that the action of $\pi_1(C,a)$ on the fibre $p^{-1}(a)$ defines a surjective homomorphism from $\pi_1(C,a)$ to the symmetric group $\mathfrak{S}_3$.
iv) Among the coverings π, p, and $p \circ \pi$, which ones are normal? What are the automorphism groups of these coverings?
v) Let K be the subspace of the sphere $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 2\}$ defined by

$$K = \{(z,w) \in \mathbb{C}^2 : z^3 = w^2\}$$

and $b = (1,-1) \in S^3$. Show that C and $S^3 \backslash K$ have the same homotopy type. Show that K is homeomorphic to the circle $S^1 = \{(z,w) \in S^3 : w = 0\}$ but $S^3 \backslash K$ and $S^3 \backslash S^1$ are not homeomorphic.

7) (Optional) Read Chapter 4 of May's book.