Homework 11:

1) Using the fact that $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^{n+1})$ with the grading given by $|\beta| = 2$, show that there is no retraction of $\mathbb{C}P^n$ onto $\mathbb{C}P^k$ for $0 < k < n$.

2) Use the result of the previous exercise to show that the Hopf map $f : S^{2n-1} \cong \{(z_0, z_1, \dots z_{n-1}) \in \mathbb{C}^n : \sum_i |z_i|^2 = 1\} \to \mathbb{C}P^{n-1}$ defined by

$$(z_0, z_1, \dots, z_{n-1}) \to [z_0 : z_1 : \dots : z_{n-1}]$$

is not null-homotopic. In particular, conclude that $\pi_3(S^2) \neq 0$.

3) Compute the cohomology ring of $S^2 \vee S^4$. Show that $S^2 \vee S^4$ is not homotopy equivalent to $\mathbb{C}P^2$.

4) Construct a diagonal approximation for S^1. Recall that this is a chain map

$$C_*(S^1) \to C_*(S^1) \otimes C_*(S^1)$$

with the property that a 0-chain σ is sent to $\sigma \otimes \sigma$. You might prefer to use the cellular chain complex as a replacement for $C_*(S^1)$. Use this to construct a diagonal approximation for $S^1 \times S^1$ (again via cellular chain complexes). Use your diagonal approximation on cellular chains to compute the cup-product structure on $H^*(S^1 \times S^1)$.

5) Show that a ring homomorphism $R \to S$, induces a ring homomorphism from $H^*(X, A; R) \to H^*(X, A; S)$ for any (X, A). Using this for $\mathbb{R}P^n$ deduce the ring structure of $H^*(\mathbb{R}P^n;\mathbb{Z})$ from the ring structure of $H^*(\mathbb{R}P^n;\mathbb{Z}_2)$, which we have computed in class.

6) Hatcher, page 229, problem 6.