Associative Yang-Baxter equation and Fukaya categories of square-tiled surfaces

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We show that all strongly non-degenerate trigonometric solutions of the associative Yang-Baxter equation (AYBE) can be obtained from triple Massey products in the Fukaya categories of square-tiled surfaces. Along the way, we give a classification result for cyclic A_∞-algebra structures on a certain Frobenius algebra associated with a pair of 1-spherical objects in terms of the equivalence classes of the corresponding solutions of the AYBE. As an application, combining our results with homological mirror symmetry for punctured tori (cf. [17]), we prove that any two simple vector bundles on a cycle of projective lines are related by a sequence of 1-spherical twists and their inverses.

Introduction

The associative Yang-Baxter equation (AYBE) is the equation

$$r^{12}(-u', v)r^{13}(u+u', v+v') - r^{23}(u+u', v')r^{12}(u, v) + r^{13}(u, v+v')r^{23}(u', v') = 0, \tag{0–1}$$

where $r : \mathbb{C}^2 \to \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{Mat}_n(\mathbb{C})$ is a meromorphic function of two complex variables (u, v) in a neighborhood of $(0, 0)$ taking values in $\mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{Mat}_n(\mathbb{C})$, and $\mathrm{Mat}_n(\mathbb{C})$ is the matrix algebra. Here we use the notation $r^{12} = r \otimes 1 \in \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{Mat}_n(\mathbb{C})$, etc. The equation (0–1) is usually coupled with the *skew-symmetry* (also called *unitarity*) condition

$$r^{21}(-u, -v) = -r(u, v) \tag{0–2}$$

where r^{21} is obtained from r by the transposition of tensor factors $a_2 \otimes a_1 \to a_1 \otimes a_2$. Note that the constant version of the AYBE was studied by Aguiar [2] in connection with infinitesimal Hopf algebras.

The AYBE is an analog in the world of associative algebras of the well-known classical Yang-Baxter equation (CYBE),

$$[r^{12}(v), r^{13}(v+v')] + [r^{12}(v), r^{23}(v')] + [r^{13}(v+v'), r^{23}(v')] = 0,$$

where $r(v)$ is a meromorphic function in a neighborhood of 0 taking values in a Lie algebra $\mathfrak{g} \otimes \mathfrak{g}$. Solutions of the CYBE lead to Poisson-Lie groups and classical integrable systems (see for ex. [9], [7]). There is a direct relation between the two equations in the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$: if $r(u, v)$ is a skew-symmetric solution of the AYBE such that the limit $\overline{r}(v) = (pr \otimes pr)r(u, v)|_{u=0}$ exists (where pr is the projection away from the identity to traceless matrices), then $\overline{r}(v)$ is a solution of the CYBE for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

It was discovered in [19] that solutions of the AYBE often arise from 1-Calabi-Yau A_∞-categories. More precisely, assume we have such a minimal A_∞-category $\mathscr{C}$ and two sets of isomorphism classes of objects in $\mathscr{C}$, $\mathcal{X}$ and $\mathcal{Y}$, such that for every pair of distinct objects $x_1, x_2 \in \mathcal{X}$ (resp. $y_1, y_2 \in \mathcal{Y}$), $\mathrm{Hom}^*(x_1, x_2) = 0$ (resp. $\mathrm{Hom}^*(y_1, y_2) = 0$). Furthermore, assume $\mathrm{Hom}^{\neq 0}(x, y) = 0$ (and so $\mathrm{Hom}^{\neq 1}(y, x) = 0$) for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then dualising the triple product [1]

$$\mathfrak{m}_3 : \mathrm{Hom}^0(x_2, y_2) \otimes \mathrm{Hom}^1(y_1, x_2) \otimes \mathrm{Hom}^0(x_1, y_1) \to \mathrm{Hom}^0(x_1, y_2),$$

where $x_1, x_2 \in \mathcal{X}$, $y_1, y_2 \in \mathcal{Y}$, using the Calabi-Yau pairing, we get a tensor

$$r^{x_1,x_2}_{y_1,y_2} : \mathrm{Hom}^0(x_2, y_2) \otimes \mathrm{Hom}^0(x_1, y_1) \to \mathrm{Hom}^0(x_1, y_2) \otimes \mathrm{Hom}^0(x_2, y_1).$$

defined by

$$\langle r^{x_1,x_2}_{y_1,y_2}(f_{22} \otimes f_{11}), g_{21} \otimes g_{12} \rangle = \langle \mathfrak{m}_3(f_{22}, g_{12}, f_{11}), g_{21} \rangle. \tag{0–3}$$

where $f_{ii} \in \mathrm{Hom}^0(x_i, y_i)$, $g_{ij} \in \mathrm{Hom}^1(y_i, x_j)$. Note that by the cyclicity of the A_∞-structure, this tensor satisfies the following *skew-symmetry condition*:

$$(r^{x_1x_2}_{y_1y_2})^{21} = -r^{x_2x_1}_{y_2y_1}. \tag{0–4}$$

Now, let x_1, x_2, x_3 (resp., y_1, y_2, y_3) be distinct elements of $\mathcal{X}$ (resp., $\mathcal{Y}$). Then taking into account the assumptions on $\mathcal{X}, \mathcal{Y}$ the relevant A_∞-relation involving $\mathfrak{m}_3$ takes the form

$$\mathfrak{m}_3(\mathfrak{m}_3(f_{33}, g_{23}, f_{22}), g_{12}, f_{11}) + \mathfrak{m}_3(f_{33}, \mathfrak{m}_3(g_{23}, f_{22}, g_{12}), f_{11}) - \mathfrak{m}_3(f_{33}, g_{23}, \mathfrak{m}_3(f_{22}, g_{12}, f_{11})) = 0, \tag{0–5}$$

where $f_{ii} \in \mathrm{Hom}^0(x_i, y_i)$, $g_{ij} \in \mathrm{Hom}^1(y_i, x_j)$. Note that here the first and the the third terms can be immediately expressed in terms of the tensor $r^{x_1x_2}_{y_1y_2}$. To do this for the middle term, one has to use the cyclic symmetry of the A_∞-structure, which gives

$$\langle f_{31}, \mathfrak{m}_3(g_{23}, f_{22}, g_{12}) \rangle = \langle g_{12}, \mathfrak{m}_3(f_{31}, g_{23}, f_{22}) \rangle.$$

Taking into account the cyclic symmetry, we can rewrite the above A_∞-relation as follows (see [19, Thm. 1]):

$$(r^{x_1x_2}_{y_1y_3})^{13}(r^{x_2x_3}_{y_2y_3})^{12} + (r^{x_3x_2}_{y_1y_2})^{23}(r^{x_1x_3}_{y_1y_3})^{13} - (r^{x_1x_3}_{y_2y_3})^{12}(r^{x_1x_2}_{y_1y_2})^{23} = 0. \tag{0–6}$$

This is viewed as an equation on

$$\mathrm{Hom}^0(x_3, y_3) \otimes \mathrm{Hom}^0(x_2, y_2) \otimes \mathrm{Hom}^0(x_1, y_1) \to \mathrm{Hom}^0(x_2, y_3) \otimes \mathrm{Hom}^0(x_1, y_2) \otimes \mathrm{Hom}^0(x_3, y_1)$$

Permuting the second and third factors in the tensor product, and swapping x_1 with x_2 and y_1 and y_2 (and also taking into account the skew-symmetry (0–4)), the equation (0–6) is equivalent to the following equation

$$(r^{x_2x_1}_{y_2y_3})^{12}(r^{x_1x_3}_{y_1y_3})^{13} - (r^{x_1x_3}_{y_1y_2})^{23}(r^{x_2x_3}_{y_2y_3})^{12} + (r^{x_2x_3}_{y_1y_3})^{13}(r^{x_1x_2}_{y_1y_2})^{23} = 0. \tag{0–7}$$

[1] Our convention is that in an A_∞-category, we read the compositions from right-to-left (as in [24]). This affects certain signs in computations. In particular, the A_∞-relations are given by:

$$\sum_{m,n} (-1)^{|a_1|+\ldots+|a_n|-n} \mathfrak{m}_{d-m+1}(a_d, \ldots, a_{n+m+1}, \mathfrak{m}_m(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots a_1) = 0$$

We will call the equation (0–7) the *general AYBE* (or simply *AYBE* when no confusion can arise).[2]

It was further shown in [19] that in the case when $\mathscr{C}$ is the derived category of coherent sheaves on an elliptic curve (or some of its degenerations) then there exist natural choices of $\mathcal{X}$ and $\mathcal{Y}$ as above, so that all the spaces $\mathrm{Hom}^0(x,y)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, can be identified with the fixed finite-dimensional vector space V. Furthermore, in this case $\mathcal{X}$ and $\mathcal{Y}$ have abelian group structures, and the obtained tensors $r^{x_1,x_2}_{y_1,y_2} : V^{\otimes 2} \to V^{\otimes 2}$ depend only on the differences $u = x_2 - x_1$, $v = y_2 - y_1$, which leads to the equation (0–1).

Note that different choices of trivialization of the Hom-spaces in the above construction correspond to the natural equivalence relation on solutions of the AYBE introduced in [19]. Namely, given a function φ^x_y with values in $\mathrm{GL}_n(\mathbb{C})$, we can transform a solution $r^{x_1x_2}_{y_1y_2}$ of (0–7) to the new solution

$$\widetilde{r}^{x_1x_2}_{y_1y_2} = (\varphi^{x_2}_{y_1} \otimes \varphi^{x_1}_{y_2}) r^{x_1x_2}_{y_1y_2} (\varphi^{x_1}_{y_1} \otimes \varphi^{x_2}_{y_2})^{-1}. \tag{0–8}$$

Our first result is that an analog of the above construction gives a bijection between formal solutions of the general AYBE and a class of A_∞-structures. Namely, we will consider deformations of the formal A_∞-category $\mathcal{A} = \mathcal{A}_n$ defined below. Note that we use the sign conventions of [24], so that the double compositions in the associated cohomology category differ from those induced by $\mathfrak{m}_2$ by a sign.

Definition 0.0.1 The A_∞-category $\mathcal{A} = \mathcal{A}_n$ has two objects X and Y, and the Hom-spaces

$$\mathrm{Hom}(X,Y) = \mathrm{Hom}^0(X,Y) = \mathbb{Z}\theta_1 \oplus \ldots \oplus \mathbb{Z}\theta_n, \;\; \mathrm{Hom}(Y,X) = \mathrm{Hom}^1(Y,X) = \mathbb{Z}\eta_1 \oplus \ldots \oplus \mathbb{Z}\eta_n,$$
$$\mathrm{Hom}^0(X,X) = \mathbb{Z}\,\mathrm{id}_X, \;\; \mathrm{Hom}^0(Y,Y) = \mathbb{Z}\,\mathrm{id}_Y, \;\; \mathrm{Hom}^1(X,X) = \mathbb{Z}\xi_X, \;\; \mathrm{Hom}^1(Y,Y) = \mathbb{Z}\xi_Y.$$

The elements id_X and id_Y act as strict units in the sense that

$$\mathfrak{m}_2(a, \mathrm{id}_X) = a \;\;,\;\; \mathfrak{m}_2(\mathrm{id}_X, a) = (-1)^{|a|} a \;\;,\;\; \mathfrak{m}_2(a, \mathrm{id}_Y) = a \;\;,\;\; \mathfrak{m}_2(\mathrm{id}_Y, a) = (-1)^{|a|} a,$$

whenever composition with $a \in \mathcal{A}$ is non-zero, where $|a|$ is the degree of a, and the other compositions are given by

$$\mathfrak{m}_2(\eta_\alpha, \theta_\beta) = \delta_{\alpha\beta}\xi_X, \;\; \mathfrak{m}_2(\theta_\alpha, \eta_\beta) = -\delta_{\alpha\beta}\xi_Y.$$

Note that one can view $\mathcal{A}$ as a graded category by defining the composition as:

$$a_2 \cdot a_1 = (-1)^{|a_1|}\mathfrak{m}_2(a_2, a_1)$$

We also define the symmetric perfect pairing on the Hom-spaces of $\mathcal{A}$ by

$$\langle \eta_\alpha, \theta_\beta \rangle = -\langle \theta_b, \eta_a \rangle = \delta_{\alpha\beta}, \;\; \langle \xi_X, \mathrm{id}_X \rangle = -\langle \mathrm{id}_X, \xi_X \rangle = \langle \xi_Y, \mathrm{id}_Y \rangle = -\langle \mathrm{id}_Y, \xi_Y \rangle = 1.$$

Let $\mathbf{k}$ be a field. We are going to consider A_∞-structures on $\mathcal{A} \otimes \mathbf{k}$, with given $\mathfrak{m}_2$, which are cyclic with respect to this pairing. Recall that a *strictly cyclic A_∞-category of dimension 1* is a strictly unital, proper A_∞-category together with nondegenerate pairings

$$\langle \; , \; \rangle : \mathit{hom}^*(X,Y) \otimes \mathit{hom}^{1-*}(Y,X) \to \mathbf{k}$$

[2] Our equation differs from [19, Eq. (1.2)] due to different conventions. The two equations become equivalent if we replace $r^{xx'}_{yy'}$ by $r^{x'x}_{y'y}$.

satisfying

$$\langle a_1, a_2\rangle = (-1)^{(|a_1|-1)(|a_2|-1)+1}\langle a_2, a_1\rangle$$

and the cyclic symmetry condition:

$$\langle a_{k+1}, \mathfrak{m}_k(a_k, a_{k-1}, \dots, a_1)\rangle = (-1)^{(|a_{k+1}|-1|)(|a_1|+|a_2|+\dots+|a_k|-k)}\langle a_k, \mathfrak{m}_k(a_{k-1}, a_{k-2}\dots, a_1, a_{k+1})\rangle$$

An A_∞-functor $\mathfrak{f} = (\mathfrak{f}^n)_{n\geq 1} : \mathscr{A} \to \mathscr{B}$ between cyclic A_∞ categories is said to be *cyclic* if the following hold:

$$\langle \mathfrak{f}^1(a_2), \mathfrak{f}^1(a_1)\rangle = \langle a_2, a_1\rangle$$

for any a_2, a_1 and

$$\sum_{k+l=n} \langle \mathfrak{f}^l(a_n, \dots, a_{k+1}), \mathfrak{f}^k(a_k, \dots a_1)\rangle = 0$$

for any sequence of composable morphisms $a_n, \dots, a_1$ with $n \geq 3$.

For a commutative $\mathbf{k}$-algebra R we denote by $\mathcal{M}_\infty(\mathcal{A}\otimes R)$ the set of cyclic, strictly unital, minimal A_∞-structures on $\mathcal{A}\otimes R$, up to a strict cyclic A_∞-equivalence (i.e., the one with $\mathfrak{f}^1 = \mathrm{id}$). Let us set

(0–9) $$\mathrm{P} := \sum_{i,j} e_{ij}\otimes e_{ji} \in \mathrm{Mat}_n(\mathbf{k})\otimes \mathrm{Mat}_n(\mathbf{k}),$$

where (e_{ij}) is the standard basis of $\mathrm{Mat}_n(\mathbf{k})$ defined by $e_{ij}(\mathbf{e}_k) = \delta_{jk}\mathbf{e}_i$ if $(\mathbf{e}_i)_{i=1}^n$ is a basis of $\mathbf{k}^n$. In other words, P is the transposition operator given by:

$$\mathrm{P}(x\otimes y) = y\otimes x.$$

Theorem A. *There is a natural explicit bijection between $\mathcal{M}_\infty(\mathcal{A}\otimes R)$ and the equivalence classes of formal skew-symmetric solutions $r^{x_1x_2}_{y_1y_2}$ of the general AYBE of the following type. We let x_1, x_2, y_1, y_2 to be formal variables and consider*

$$r^{x_1x_2}_{y_1y_2} \in \mathrm{Mat}_n(\mathbf{k})\otimes \mathrm{Mat}_n(\mathbf{k})\otimes R[[x_1,x_2,y_1,y_2]][(x_2-x_1)^{-1}(y_2-y_1)^{-1}]$$

of the form

(0–10) $$r^{x_1x_2}_{y_1y_2} \equiv \frac{\mathrm{id}\otimes\mathrm{id}}{x_2-x_1} + \frac{\mathrm{P}}{y_1-y_2} \quad \{\mathrm{mod}\ \ \mathrm{Mat}_n(\mathbf{k})\otimes \mathrm{Mat}_n(\mathbf{k})\otimes R[[x_1,x_2,y_1,y_2]]\},$$

such that (0–7) is satisfied in $\mathrm{Mat}_n(\mathbf{k})\otimes \mathrm{Mat}_n(\mathbf{k})\otimes R[[x_1,x_2,x_3,y_1,y_2,y_3]][\Delta^{-1}]$, where $\Delta = \prod_{i<j}(x_j - x_i)(y_j-y_i)$. The skew-symmetry is the equation (0–4). The equivalence between such solutions is given by (0–8), with

$$\varphi^x_y \in \mathrm{Id} + (x,y) \subset \mathrm{Mat}_n(R)[[x,y]].$$

Considering the more general equivalences, where the constant term of φ^x_y is only required to be an invertible matrix, corresponds to general cyclic A_∞-equivalences of the cyclic A_∞-structures.

The key idea in this theorem is to apply a version of the above construction of solutions of the AYBE to a pair $\mathcal{X}$, $\mathcal{Y}$ of formal deformations of objects X and Y in $\mathcal{A}$. A technical point is that these formal deformations are defined in the category of twisted objects over $\mathcal{A}$, which is non-minimal. Because of this one has to use certain triple Massey products instead of just $\mathfrak{m}_3$ (see Sec. 1.2). In particular, the singular terms in the expansion of $r^{x_1x_2}_{y_1y_2}$ are obtained naturally in this approach due to the definition of the Massey products.

Recall that Belavin and Drinfeld in the seminal paper [4] classified nondegenerate[3] solutions of the classical Yang-Baxter equation for simple complex Lie algebras, up to some natural equivalence. They showed that they can be either elliptic or trigonometric or rational, and further classified trigonometric solutions in terms of some combinatorial data, involving so called Belavin-Drinfeld triples.

Similarly, one can pose the problem of classifying nondegenerate solutions $r(u,v)$ of the AYBE (and of its formal general version). Partial results in this direction we obtained in [19] and [20]. If r is strongly nondegenerate (see Def. 1.4.3 and Prop. 1.4.4), the Laurent expansion of the solution at $u=0$ has the form

$$r(u,v) = \frac{1\otimes 1}{u} + r_0(v) + \dots \tag{0–11}$$

Under this assumption, it was shown that the projection $\bar{r}_0(v)$ of $r_0(v)$ to $\mathfrak{sl}_n(\mathbb{C})\otimes\mathfrak{sl}_n(\mathbb{C})$ is a nondegenerate solution of the CYBE, and that if $r_0(v)$ is either elliptic or trigonometric then $r(u,v)$ is determined by $\bar{r}_0(v)$, up to some natural transformations. Note that the Laurent expansion (0–11) appears naturally in the construction of Theorem A. It was shown in [19] that all elliptic solutions of the CYBE extend to those of the AYBE. In [22] Schedler observed that this is not the case for all the trigonometric solutions. Extending this work, it was proved in [20] that nondegenerate solutions of the AYBE, with the Laurent expansion at $u=0$ of the form (0–11) and such that $\bar{r}_0(v)$ is a trigonometric solution of the CYBE, admit a classification in terms of the following combinatorial data (see also Sec. 2.3 below).

Definition 0.0.2 An *associative Belavin-Drinfeld structure* (S,C_1,C_2,A) consists of a finite set S, a pair of transitive permutations $C_1,C_2 : S\to S$ and a proper subset $A\subset S$ such that for all $a\in A$, one has :

$$C_1(C_2(a)) = C_2(C_1(a)).$$

The reader familiar with the original Belavin-Drinfeld triples (defined in terms of Dynkin diagrams) may notice that the above associative analog of this notion is more elementary (the original definition in [20] is slightly different but is equivalent to the one above, see Sec. 2.3).

One can also ask which solutions of the AYBE can be realized by families $(\mathcal{X},\mathcal{Y})$ of objects in some geometric 1-Calabi-Yau-categories. A natural source is provided by the derived categories of coherent sheaves on elliptic curves and their degenerations. Then we can take as $\mathcal{X}$ a universal deformation of a simple vector bundle, and as $\mathcal{Y}$ the family of the structure sheaves of points.

It turns out that all the solutions of the AYBE for which $\bar{r}_0(v)$ is elliptic arise in this way from simple vector bundles on elliptic curve, and can be explicitly computed in terms of elliptic functions (see [19]). In [20], all the solutions coming from the nodal degenerations of elliptic curves, i.e., cycles of projective lines (aka standard m-gons), were computed and were shown to be trigonometric. However, it turned out, somewhat unexpectedly, that not all trigonometric solutions of the AYBE appear in this way. Namely, it was also shown in [20] that the trigonometric solution of the AYBE, corresponding to

[3] this means that the tensor $r(v)$ is nondegenerate for generic v

the data (S, C_1, C_2, A), arises from a simple vector bundle on a cycle of projective lines if and only if the corresponding transitive permutations C_1 and C_2 commute (equivalently, $C_2 = C_1^k$ for some k).

This raised a natural problem of finding other 1-Calabi-Yau A_∞-categories and objects in them, which would account for missing solutions. This problem is solved in the present paper by looking at appropriate Fukaya categories. Namely, starting from the data of an associative Belavin-Drinfeld structure (S, C_1, C_2, A), we construct a square-tiled surface Σ with a local symplectomorphism

$$\pi : \Sigma \to \mathbb{T}$$

to the square torus $\mathbb{T}$. In the case $A = \emptyset$, Σ is just the $n-$fold covering space of the punctured torus $\mathbb{T}_0$ associated to the permutations C_1, C_2 (see Section 2 for the general case). Lifts of standard Lagrangian curves in $\mathbb{T}$ to Σ give a pair of exact Lagrangians L_1 and L_2 in Σ such that

$$\bigoplus_{i,j=1}^{2} \mathrm{HF}^*(L_i, L_j) \simeq \mathcal{A} \otimes \mathbb{C}.$$

Now, we have complex push-offs of the Lagrangians L_1 and L_2 forming 1-parameter families L_1^x and L_2^y (see Definition 2.3.2). Taking these two families as families $\mathcal{X}$ and $\mathcal{Y}$ in our general construction of solutions of the AYBE, we get such a solution that records triple products between $(L_1^{x_1}, L_2^{y_1}, L_1^{x_2}, L_2^{y_2})$. We show that this gives exactly the trigonometric solution of the AYBE associated with (S, C_1, C_2, A). More precisely, we have:

Theorem B. *Let (Σ, L_1, L_2) be the square-tiled surface Σ and Lagrangians L_1 and L_2 associated with an associative Belavin-Drinfeld structure (S, C_1, C_2, A). Then the tensor $r_{y_1,y_2}^{x_1,x_2}$ obtained from the triple products in the Fukaya category $\mathcal{F}(\Sigma)$ only depends on $u = x_2 - x_1$, $v = y_2 - y_1$ and is a solution of the AYBE over $\mathbb{C}$ given explicitly by the following formula (for an appropriate choice of basis):*

(0–12)
$$\begin{aligned} r(u,v) = {} & \frac{1}{\exp(u)-1}\sum_i e_{ii}\otimes e_{ii} + \frac{1}{1-\exp(-v)}\sum_i e_{ii}\otimes e_{ii} \\ & + \frac{1}{\exp(u)-1}\sum_{0<k<n,i}\exp(\frac{ku}{n})e_{C_1^k(i),C_1^k(i)}\otimes e_{ii} + \frac{1}{\exp(v)-1}\sum_{0<m<n,i}\exp(\frac{mv}{n})e_{i,C_2^m(i)}\otimes e_{C_2^m(i),i} \\ & + \sum_{0<k,0<m;a\in A(k,m)}\Big\{\exp(-\frac{ku+mv}{n})e_{C_2^m(a),a}\otimes e_{C_1^k(a),C_1^kC_2^m(a)} - \exp(\frac{ku+mv}{n})e_{C_1^k(a),C_1^kC_2^m(a)}\otimes e_{C_2^m(a),a}\Big\}, \end{aligned}$$

where we denote by $A(k,m) \subset A$ the set of all $a \in A$ such that $C_1^iC_2^j(a) \in A$ for all $0 \le i < k, 0 \le j < m$.

We note that the surface Σ has genus 1 (i.e., is a punctured torus) if and only if C_1 and C_2 commute. This explains why only these solutions appeared from simple vector bundles on nodal degenerations of elliptic curves, which are mirror dual to punctured tori (see e.g., [17]).

As an application of the viewpoint developed in this paper (combined with the results of [20]) we derive the following result about simple vector bundles on cycles of projective lines.

Theorem C. *Let C be a cycle of projective lines (the standard n-gon) over $\mathbb{C}$. For any simple vector bundle V on C there exists a composition Φ of 1-spherical twists and their inverses such that $\Phi(\mathcal{O}_C) \simeq V$.*

Here we use the notion of the twist autoequivalence associated with an n-spherical object introduced in [27]. Recall that an n-spherical object E should satisfy $\mathrm{Hom}^*(E,E) = k \oplus k[-n]$ (together with an additional nondegeneracy condition). The corresponding twist autoequivalence T_E fits into an exact triangle

$$\mathrm{Hom}^*(E,X) \otimes E \to X \to T_E(X) \to \dots$$

In this paper we consider only 1-spherical objects and the corresponding twists.

Note that Theorem C is known in the case $n = 1$ by the work [6]. In this case the situation is very similar to the case of elliptic curves. The case $n > 1$ is much more complicated: in this case one can still classify all simple vector bundles on C (see [5]) but the relevant combinatorics is quite involved.

The idea of the proof of Theorem C is to consider the solution of the formal general AYBE associated with the pair $(\mathcal{O}, V)$ (where V is sufficiently positive), and to use Theorem A which states that the subcategory generated by $\mathcal{O}$ and V is determined by this solution. The point is that we know this solution of the AYBE to be the same as for the Lagrangians L_1, L_2 in the symplectic surface Σ of genus 1 associated with some associative Belavin-Drinfeld structure. We prove that in this situation the pair $(\mathcal{O}, V)$ (resp. (L_1, L_2)) split generates the perfect derived category of C (resp., the Fukaya category of Σ). Thus, we reduce the problem to a similar question about Lagrangians in the Fukaya category, where we can use the action of the mapping class group.

The paper is organized as follows. In Section 1 we study the relation between formal solutions of the general AYBE and A_∞-structures, in particular, proving Theorem A in 1.3. In addition, in Sec. 1.4 we discuss the natural involution on solutions of the AYBE, which allows us to deduce the pole conditions imposed in [20] for strongly nondegenerate solutions of the AYBE (see Prop. 1.4.4). Also, in Sec. 1.5 we explain, basing on ideas of [25], the connection between solutions of the AYBE coming from algebraic (or analytic) families of objects (see (0–3)) and the corresponding formal solutions from Theorem A. Section 2 is devoted to the construction of trigonometric solutions of the AYBE from Fukaya products on the square-tiled surfaces associated with Belavin-Drinfeld structures. In Section 3 we consider two applications of Theorems A and B to vector bundles over a (nodal) cycle C of projective lines. One is a criterion, in terms of some combinatorial data, for a pair of simple vector bundles on C to be related by a Fourier-Mukai autoequivalence (see Theorem 3.1.4). Another is Theorem C, proved in Sec. 3.2.

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1 A class of cyclic A_∞-structures and formal solutions of the general AYBE

1.1 Twisted objects over complete rings

Let us quickly review the definition of the A_∞-category of twisted objects with coefficients in a complete ring (see [11], [10], [16]), mostly following [24, Ch. 1] and [14, Sec. 7.6].

Let $\mathscr{C}$ be a topological A_∞-category over a complete ring R. We assume that R is topologized by a decreasing filtration (R_n), such that $R_m R_n \subset R_{m+n}$, and that the Hom-spaces in $\mathscr{C}$ are complete. We will only consider twisted objects of the following kind: (X, δ_X), where X is an object of $\mathscr{C}$ and $\delta_X \in R_1 \operatorname{Hom}^1(X, X)$ is an element satisfying the Maurer-Cartan equation

$$\sum_{n \geq 1} \mathfrak{m}_n(\delta_X^n) = 0.$$

Note that here the left-hand side converges in $\operatorname{Hom}^2(X, X)$. The Hom-space between two such objects (X, δ_X) and (Y, δ_Y) is simply $\operatorname{Hom}(X, Y)$. There are natural A_∞-products $(\mathfrak{m}_n^t)$ for the twisted objects, which are obtained by inserting the twisting elements δ any number of times wherever possible. More precisely, $\mathfrak{m}_d^t$ is given by

$$\mathfrak{m}_d^t(a_d, a_{d-1}, \dots, a_1) = \sum_{i_0, \dots, i_d \geq 0} \mathfrak{m}_{d+i_0+\dots+i_d}(\delta_{X_d}^{i_d}, a_d, \delta_{X_{d-1}}^{i_{d-1}}, a_{d-1}, \dots, a_1, \delta_{X_0}^{i_0})$$

(We follow the sign conventions of [24, Ch.1]).

Let us point out one additional feature of the A_∞-category of twisted objects: it easy to check that if we start with a cyclic A_∞-category then the corresponding A_∞-category inherits the cyclic structure.

An A_∞-functor $\mathfrak{f} = (\mathfrak{f}^n)_{n \geq 1} : \mathscr{C} \to \mathscr{C}'$ between A_∞-categories over R, as above, leads to an A_∞-functor $\mathcal{F} = (\mathcal{F}^n)_{n \geq 1}$ between their categories of twisted objects. Namely, (X, δ_X) maps to $(\mathcal{F}(X), \mathcal{F}(\delta_X))$, where

$$\mathcal{F}(X) = \mathfrak{f}(X) \ , \ \mathcal{F}(\delta_X) = \sum \mathfrak{f}^n(\delta_X^n).$$

The maps $\mathcal{F}^d$ on Hom-spaces between twisted objects is given by

$$\mathcal{F}^d(a_d, a_{d-1}, \dots, a_1) = \sum_{i_0, \dots, i_d \geq 0} \mathfrak{f}^{d+i_0+\dots+i_d}(\delta_{X_d}^{i_d}, a_d, \delta_{X_{d-1}}^{i_{d-1}}, a_{d-1}, \dots, a_1, \delta_{X_0}^{i_0})$$

1.2 Triple Massey products and a construction of formal solutions of the general AYBE

First, let us recall the general definition of the triple Massey product in an A_∞-category $\mathscr{C}$ following [19], but with different sign conventions. For a triple of $\mathfrak{m}_1$-closed composable morphisms

$$X_0 \xrightarrow{a_1} X_1 \xrightarrow{a_2} X_2 \xrightarrow{a_3} X_3$$

such that $\mathfrak{m}_2(a_2, a_1) = \mathfrak{m}_1(h_1)$, $\mathfrak{m}_2(a_3, a_2) = \mathfrak{m}_1(h_2)$, one sets

(1–1) $$\operatorname{MP}(a_3, a_2, a_1) = \mathfrak{m}_3(a_3, a_2, a_1) - \mathfrak{m}_2(h_2, a_1) - \mathfrak{m}_2(a_3, h_1) \quad \{\operatorname{mod} \operatorname{Im}(\mathfrak{m}_1)\},$$

which is well-defined as a coset for the image of

$$H^{|a_1|+|a_2|-1} \operatorname{Hom}(X_0, X_2) \oplus H^{|a_2|+|a_3|-1} \operatorname{Hom}(X_1, X_3) \xrightarrow{(\mathfrak{m}_2(a_3, ?), \mathfrak{m}_2(?, a_1))} H^{|a_1|+|a_2|+|a_3|-1} \operatorname{Hom}(X_0, X_3).$$

The main result about such triple Massey products is that they are preserved under A_∞-functors: if $\mathfrak{f} : \mathscr{C} \to \mathscr{C}'$ is an A_∞-functor then

$$\mathfrak{f}^1(\operatorname{MP}(a_3, a_2, a_1)) = \operatorname{MP}(\mathfrak{f}^1(a_3), \mathfrak{f}^1(a_2), \mathfrak{f}^1(a_1))$$

in the appropriate quotient-group (see [19, Prop. 1.1]). In particular, if $\mathscr{C}'$ is a minimal model of $\mathscr{C}$ obtained as a result of the homological perturbation procedure, then the Massey product $MP(a_3, a_2, a_1)$, computed in $\mathscr{C}$, agrees with $\mathfrak{m}_3^{\mathscr{C}'}(a_3, a_2, a_1)$ (since for a minimal A_∞-category our Massey product reduces to $\mathfrak{m}_3$).

Thus, the construction of solutions of the AYBE from two families of objects, presented in the introduction, has a version for non-minimal cyclic A_∞-categories, linear over some commutative ring R. More precisely, we have to replace in this construction $\mathfrak{m}_3$ by the triple Massey product MP and assume that cohomology of all the morphism spaces are projective R-modules (so that the homological perturbation can be applied). One technical problem is that the cyclic property of the A_∞-structure is not necessarily inherited by the minimal model. However, we have the following compatibility of the Massey products with the cyclic structure.

Lemma 1.2.1 *Suppose we are given a cycle of $\mathfrak{m}_1$-closed morphisms in a cyclic A_∞-category,*

$$X_0 \xrightarrow{a_1} X_1 \xrightarrow{a_2} X_2 \xrightarrow{a_3} X_3 \xrightarrow{a_4} X_0$$

such that

$$\mathfrak{m}_2(a_2, a_1) = \mathfrak{m}_1(h_1), \;\; \mathfrak{m}_2(a_3, a_2) = \mathfrak{m}_1(h_2), \;\; \mathfrak{m}_2(a_4, a_3) = \mathfrak{m}_1(h_3).$$

Assume also that the corresponding Massey products $\mathrm{MP}(a_3, a_2, a_1)$ *and* $\mathrm{MP}(a_4, a_3, a_2)$ *are univalued. Then*

$$\langle a_1, \mathrm{MP}(a_4, a_3, a_2)\rangle = (-1)^{(|a_1|-1)(|a_2|+|a_3|+|a_4|-1)}\langle a_4, \mathrm{MP}(a_3, a_2, a_1)\rangle.$$

Proof. Using (1–1), we see that it is enough to establish the following identities

$$\langle a_1, \mathfrak{m}_3(a_4, a_3, a_2)\rangle = (-1)^{(|a_1|-1)(|a_2|+|a_3|+|a_4|-1)}\langle a_4, \mathfrak{m}_3(a_3, a_2, a_1)\rangle,$$
$$\langle a_1, \mathfrak{m}_2(a_4, h_2)\rangle = (-1)^{(|a_1|-1)(|a_2|+|a_3|+|a_4|-1)}\langle a_4, \mathfrak{m}_2(h_2, a_1)\rangle,$$
$$\langle a_1, \mathfrak{m}_2(h_3, a_2)\rangle = (-1)^{(|a_1|-1)(|a_2|+|a_3|+|a_4|-1)}\langle a_4, \mathfrak{m}_2(a_3, h_1)\rangle.$$

The first two follow directly from the cyclic property of $\mathfrak{m}_3$ and $\mathfrak{m}_2$ (noting that $|h_2| = |a_2| + |a_3| - 1$). For the last one, we first rewrite the left-hand side using the cyclic property of $\mathfrak{m}_2$:

$$\langle a_1, \mathfrak{m}_2(h_3, a_2)\rangle = (-1)^{(|a_1|-1)(|a_2|+|a_3|+|a_4|-1)}\langle h_3, \mathfrak{m}_2(a_2, a_1)\rangle = (-1)^{(|a_1|-1)(|a_2|+|a_3|+|a_4|-1)}\langle h_3, \mathfrak{m}_1(h_1)\rangle.$$

Next, we use the cyclic property of $\mathfrak{m}_1$:

$$\langle h_3, \mathfrak{m}_1(h_1)\rangle = (-1)^{(|h_1|-1)(|h_3|-1)}\langle h_1, \mathfrak{m}_1(h_3)\rangle = (-1)^{(|h_1|-1)(|h_3|-1)}\langle h_1, \mathfrak{m}_2(a_4, a_3)\rangle.$$

It remains to apply the cyclic of $\mathfrak{m}_2$ to the last expression to get the required identity. □

Using Lemma 1.2.1, one can generalize the construction of the solutions of the AYBE to non-minimal cyclic A_∞-categories provided the appropriate vanishing assumptions hold on the cohomology level.

We want to apply this construction to certain Massey products involving twisted objects over an A_∞-structure on the category $\mathcal{A}$ (see Def. 0.0.1) More precisely, given a minimal cyclic A_∞-structure with the given m_2 on the category $\mathcal{A} \otimes R$, where R is a commutative ring, we extend the coefficients to

$$\widetilde{R} := R[[x_1, x_2, y_1, y_2]][(x_2 - x_1)^{-1}(y_2 - y_1)^{-1}]$$

and consider the following twisted objects of $\mathcal{A}\otimes\widetilde{R}$:

$$X_i=(X,x_i\xi_X),\quad Y_i=(Y,y_i\xi_Y),\quad \text{for } i=1,2. \tag{1–2}$$

Note that the Hom-spaces $\mathrm{Hom}(X_i,Y_j)$ and $\mathrm{Hom}(Y_j,X_i)$ are still concentrated in one degree and so have trivial $\mathfrak{m}_1^t$. We denote by θ_α^{ij} (resp., η_α^{ji}) the basis elements in $\mathrm{Hom}(X,Y)$ (resp, $\mathrm{Hom}(Y,X)$) viewed as elements of $\mathrm{Hom}(X_i,Y_j)$ (resp., $\mathrm{Hom}(Y_j,X_i)$). On the other hand, $\mathrm{Hom}(X_1,X_2)$ and $\mathrm{Hom}(Y_1,Y_2)$ now have a nontrivial differential:

$$\mathfrak{m}_1^t(\mathrm{id}_X)=(x_2-x_1)\xi_X,\quad \mathfrak{m}_1^t(\mathrm{id}_Y)=(y_2-y_1)\xi_Y,$$

so the corresponding cohomology vanishes (due to the localization in the definition of $\widetilde{R}$).

We consider the triple Massey product corresponding to the composable morphisms

$$X_1\xrightarrow{\theta_\alpha^{11}}Y_1\xrightarrow{\eta_\beta^{12}}X_2\xrightarrow{\theta_{\alpha'}^{22}}Y_2.$$

We claim that this Massey product is well-defined and univalued. Indeed, we have

$$\mathfrak{m}_2^t(\eta_\beta^{12},\theta_\alpha^{11})=\delta_{\alpha\beta}\xi_X=\frac{\delta_{\alpha\beta}}{x_2-x_1}\cdot\mathfrak{m}_1^t(\mathrm{id}_X),$$

$$\mathfrak{m}_2^t(\theta_{\alpha'}^{22},\eta_\beta^{12})=-\delta_{\alpha'\beta}\xi_Y=\frac{\delta_{\alpha'\beta}}{y_1-y_2}\cdot\mathfrak{m}_1^t(\mathrm{id}_Y),$$

hence it is well-defined. The fact that it is univalued follows immediately from the vanishing of $H^0\,\mathrm{Hom}(X_1,X_2)$ and $H^0\,\mathrm{Hom}(Y_1,Y_2)$. According to the formula (1–1) we have

$$\begin{aligned}\mathrm{MP}(\theta_{\alpha'}^{22},\eta_\beta^{12},\theta_\alpha^{11})&=\mathfrak{m}_3^t(\theta_{\alpha'}^{22},\eta_\beta^{12},\theta_\alpha^{11})-\frac{\delta_{\alpha\beta}}{x_2-x_1}\cdot\mathfrak{m}_2^t(\theta_{\alpha'}^{22},\mathrm{id}_X)-\frac{\delta_{\alpha'\beta}}{y_1-y_2}\cdot\mathfrak{m}_2^t(\mathrm{id}_Y,\theta_\alpha^{11})\\&=\mathfrak{m}_3^t(\theta_{\alpha'}^{22},\eta_\beta^{12},\theta_\alpha^{11})-\frac{\delta_{\alpha\beta}}{x_2-x_1}\cdot\theta_{\alpha'}^{22}-\frac{\delta_{\alpha'\beta}}{y_1-y_2}\cdot\theta_\alpha^{11},\end{aligned}$$

where the last equality follows since our A_∞-structure on $\mathcal{A}\otimes R$ is strictly unital, so the products $\mathfrak{m}_2^t$ involving the identity remain the same as $\mathfrak{m}_2$. Therefore, we have:

$$\langle\mathrm{MP}(\theta_{\alpha'}^{22},\eta_\beta^{12},\theta_\alpha^{11}),\eta_{\beta'}^{21}\rangle=\langle\mathfrak{m}_3^t(\theta_{\alpha'}^{22},\eta_\beta^{12},\theta_\alpha^{11}),\eta_{\beta'}^{21}\rangle+\frac{\delta_{\alpha\beta}\delta_{\alpha'\beta'}}{x_2-x_1}+\frac{\delta_{\alpha'\beta}\delta_{\alpha\beta'}}{y_1-y_2}. \tag{1–3}$$

Since the A_∞-category of twisted objects is still cyclic, one can show that the above triple Massey product gives rise to a solution of the general AYBE over $R[[x_1,x_2,x_3,y_1,y_2,y_3]][\Delta^{-1}]$, which would prove one part of Theorem A. Namely, one first shows that an analog of the A_∞-identity (0–5) holds for the Massey products by passing to an equivalent minimal A_∞-structure, and then uses Lemma 1.2.1 to rewrite the middle term. Even though this construction was what led us to Theorem A, we will use a different argument in its proof (since we need to show both directions).

1.3 Proof of Theorem A

To a minimal cyclic A_∞-structure on $\mathcal{A}\otimes R$ we associate the element $r^{x_1x_2}_{y_1y_2}\in \mathrm{Mat}_n(\mathbf{k})\otimes \mathrm{Mat}_n(\mathbf{k})\otimes \widetilde{R}$ obtained from $\mathrm{MP}(\theta^{22}_{\alpha'},\eta^{12}_{\beta},\theta^{11}_{\alpha})$ by dualization. In other words,

$$r^{x_1x_2}_{y_1y_2}=\sum_{\alpha,\alpha',\beta,\beta'}\langle \mathrm{MP}(\theta^{22}_{\alpha'},\eta^{12}_{\beta},\theta^{11}_{\alpha}),\eta^{21}_{\beta'}\rangle\cdot e_{\beta'\alpha'}\otimes e_{\beta\alpha}. \tag{1–4}$$

Let us set

$$f^{x_1x_2}_{y_1y_2}:=\sum_{\alpha,\alpha',\beta,\beta'}\langle \mathfrak{m}^t_3(\theta^{22}_{\alpha'},\eta^{12}_{\beta},\theta^{11}_{\alpha}),\eta^{21}_{\beta'}\rangle\cdot e_{\beta'\alpha'}\otimes e_{\beta\alpha},$$

which is an element of $\mathrm{Mat}_n(\mathbf{k})\otimes \mathrm{Mat}_n(\mathbf{k})\otimes R[[x_1,x_2,y_1,y_2]]$. We can rewrite (1–3) as

$$r^{x_1x_2}_{y_1y_2}=f^{x_1x_2}_{y_1y_2}+\frac{1}{x_2-x_1}\cdot\sum_{\alpha,\alpha'}e_{\alpha\alpha}\otimes e_{\alpha'\alpha'}+\frac{1}{y_1-y_2}\cdot\sum_{\alpha,\alpha'}e_{\alpha'\alpha}\otimes e_{\alpha\alpha'}=$$
$$f^{x_1x_2}_{y_1y_2}+\frac{\mathrm{id}\otimes\mathrm{id}}{x_2-x_1}+\frac{P}{y_1-y_2},$$

In particular, the singular part of $r^{x_1x_2}_{y_1y_2}$ has the required form. Also, the skew-symmetry equation (0–4) is equivalent to

$$(f^{x_1x_2}_{y_1y_2})^{21}=-f^{x_2x_1}_{y_2y_1},$$

which can be deduced from the cyclic symmetry equation as follows:

$$\begin{aligned}(f^{x_1x_2}_{y_1y_2})^{21}&=\sum_{\alpha,\alpha',\beta,\beta'}\langle \mathfrak{m}^t_3(\theta^{22}_{\alpha'},\eta^{12}_{\beta},\theta^{11}_{\alpha}),\eta^{21}_{\beta'}\rangle\cdot e_{\beta\alpha}\otimes e_{\beta'\alpha'}\\
&=-\sum_{\alpha,\alpha',\beta,\beta'}\langle \mathfrak{m}^t_3(\theta^{22}_{\alpha},\eta^{12}_{\beta'},\theta^{11}_{\alpha'}),\eta^{21}_{\beta}\rangle\cdot e_{\beta\alpha}\otimes e_{\beta'\alpha'}\\
&=-\sum_{\alpha,\alpha',\beta,\beta'}\langle \mathfrak{m}^t_3(\theta^{22}_{\alpha'},\eta^{12}_{\beta},\theta^{11}_{\alpha}),\eta^{21}_{\beta'}\rangle\cdot e_{\beta'\alpha'}\otimes e_{\beta\alpha}=-f^{x_2x_1}_{y_2y_1}.\end{aligned}$$

We claim that the general AYBE for $r^{x_1x_2}_{y_1y_2}$ is equivalent to the A_∞-constraints in $\mathcal{A}\otimes R$ applied to all possible strings of composable elements

$$\xi^f_{Y_1},\eta^{31}_{\alpha_3},\xi^e_{X_3},\theta^{33}_{\beta_2},\xi^d_{Y_3},\eta^{23}_{\alpha_2},\xi^c_{X_2},\theta^{22}_{\beta_1},\xi^b_{Y_2},\eta^{12}_{\alpha_1},\xi^a_{X_1}. \tag{1–5}$$

Note that because of the cyclic symmetry these constraints are equivalent to the full A_∞-constraints.

Recall that our general AYBE takes place over the ring $R[[x_1,x_2,x_3,y_1,y_2,y_3]][\Delta^{-1}]$. Over this ring we can define twisted objects (X_i,Y_i) for $i=1,2,3$ as in (1–2). We extend the notation θ^{ij}_α and η^{ji}_β for basis elements in the Hom-spaces to this case. Let us set for brevity

$$\lambda_{ij}=\frac{\mathrm{id}\otimes\mathrm{id}}{x_j-x_i},\quad \mu_{ij}=\frac{P}{y_i-y_j}.$$

Thus, the AYBE takes the following form:

$$\begin{aligned}&(f^{x_2x_1}_{y_2y_3}+\lambda_{21}+\mu_{23})^{12}(f^{x_1x_3}_{y_1y_3}+\lambda_{13}+\mu_{13})^{13}-(f^{x_1x_3}_{y_1y_2}+\lambda_{13}+\mu_{12})^{23}(f^{x_2x_3}_{y_2y_3}+\lambda_{23}+\mu_{23})^{12}\\
&+(f^{x_2x_3}_{y_1y_3}+\lambda_{23}+\mu_{13})^{13}(f^{x_1x_2}_{y_1y_2}+\lambda_{12}+\mu_{12})^{23}=0.\end{aligned}$$

A straightforward calculation shows that the terms depending quadratically on (λ_{ij},μ_{ij}) cancel out, so this is equivalent to an equation of the form

$$\mathrm{AYBE}[f]+\left[(f^{x_2x_1}_{y_2y_3})^{12}(\lambda_{13})^{13}-(\lambda_{13})^{23}(f^{x_2x_3}_{y_2y_3})^{12}\right]+\ldots=0, \tag{1–6}$$

where $\mathrm{AYBE}[f]$ is the left-hand side of (0–7) with f instead of r, and the remaining terms similarly combine the terms linear in λ_{ij} and μ_{ij}.

Now we claim that looking at the coefficients of the expansion of (1–6) in x_1,x_2,x_3,y_1,y_2,y_3 we get precisely the A_∞-constraints for the strings (1–5). These constraints have the form

$$\begin{aligned}
&-\sum_{f=f_2+f_1;c=c_2+c_1}\mathfrak{m}_*(\xi^{f_2}_{Y_1},\mathfrak{m}_*(\xi^{f_1}_{Y_1},\eta_{\alpha_3},\xi^{e}_{X_3},\theta_{\beta_2},\xi^{d}_{Y_3},\eta_{\alpha_2},\xi^{c_2}_{X_2}),\xi^{c_1}_{X_2},\theta_{\beta_1},\xi^{b}_{Y_2},\eta_{\alpha_1},\xi^{a}_{X_1})\\
&-\sum_{e=e_2+e_1;b=b_2+b_1}\mathfrak{m}_*(\xi^{f}_{Y_1},\eta_{\alpha_3},\xi^{e_2}_{X_3},\mathfrak{m}_*(\xi^{e_1}_{X_3},\theta_{\beta_2},\xi^{d}_{Y_3},\eta_{\alpha_2},\xi^{c}_{X_2},\theta_{\beta_1},\xi^{b_2}_{Y_2}),\xi^{b_1}_{Y_2},\eta_{\alpha_1},\xi^{a}_{X_1})\\
&+\sum_{d=d_2+d_1;a=a_2+a_1}\mathfrak{m}_*(\xi^{f}_{Y_1},\eta_{\alpha_3},\xi^{e}_{X_3},\theta_{\beta_2},\xi^{d_2}_{Y_3},\mathfrak{m}_*(\xi^{d_1}_{Y_3},\eta_{\alpha_2},\xi^{c}_{X_2},\theta_{\beta_1},\xi^{b}_{Y_2},\eta_{\alpha_1},\xi^{a_2}_{X_1}),\xi^{a_1}_{X_1})\\
&+\ldots=0,
\end{aligned}$$

where the additional terms appear when one of a,b,c,d,e,f is zero, and have the form either $\mathfrak{m}_*(\ldots,\mathfrak{m}_2,\ldots)$ or $\mathfrak{m}_2(\ldots,\mathfrak{m}_*,\ldots)$. Using cyclic symmetry, one can immediately check that the coefficients of $x_1^a y_2^b x_2^c y_3^d x_3^e y_1^f$ in the three terms in $-\mathrm{AYBE}[f]$ match the first three terms in the A_∞-constraint above.

Now let us show how the second term in (1–6) matches the terms with $f=0$ in the A_∞-constraint. The matching of the other terms is done similarly. First, we observe that

$$(f^{x_2x_1}_{y_2y_3})^{12}(\lambda_{13})^{13}-(\lambda_{13})^{23}(f^{x_2x_3}_{y_2y_3})^{12}=\frac{1}{x_3-x_1}(f^{x_2x_1}_{y_2y_3}-f^{x_2x_3}_{y_2y_3})^{12}.$$

If we expand $f^{x_2x_1}_{y_2y_3}$ in powers of x_1,

$$f^{x_2x_1}_{y_2y_3}=\sum_{n\geq 0}f^{x_2,n}_{y_2y_3}\cdot x_1^n,$$

then the above expression becomes

$$\sum_{n\geq 0}\frac{x_1^n-x_3^n}{x_3-x_1}\cdot(f^{x_2,n}_{y_2y_3})^{12}=-\sum_{a,e\geq 0}x_1^a x_3^e\cdot(f^{x_2,a+e+1}_{y_2y_3})^{12}.$$

Now one can easily check that this matches the contribution of the terms of the form

$$\mathfrak{m}_2(\eta_{\alpha_3},\mathfrak{m}_*(\xi^{e}_{X_3},\theta_{\beta_2},\xi^{d}_{Y_3},\eta_{\alpha_2},\xi^{c}_{X_2},\theta_{\beta_1},\xi^{b}_{Y_2},\eta_{\alpha_1},\xi^{a}_{X_1}))$$

in the A_∞-constraint.

Next, we claim that changing an A_∞-structure by a cyclic homotopy transforms the corresponding solution of general AYBE to an equivalent one, as in (0–8), and that all equivalences appear in this way. Namely, a cyclic homotopy $(\mathfrak{f}^n)$ gives rise to the formal series

$$\varphi^x_y:=\sum_{a,b\geq 0}\langle\mathfrak{f}^{a+b+1}(\xi^b_Y,\eta_\alpha,\xi^a_X),\theta_\beta\rangle\cdot x^a y^b\cdot e_{\beta\alpha}$$

in $\mathrm{Mat}_n(R)[[x,y]]$. Note that since $\mathfrak{f}^1$ is the identity, the constant term of φ_y^x is equal to $\mathrm{id} \in \mathrm{Mat}_n(R)$. One can easily check that if $(\mathfrak{m}'_n)$ is obtained from $(\mathfrak{m}_n)$ by a cyclic homotopy $(\mathfrak{f}_n)$ then the corresponding solutions of the general AYBE are related by (0–8).

Finally, we claim that a cyclic A_∞-equivalence $(\mathfrak{f}^n)$ is uniquely determined by the corresponding formal series φ_y^x, which could be an arbitrary series with the constant term equal to the identity. Indeed, recall that the condition for a strict A_∞-functor to be cyclic is that

$$\sum_{k+l=n} \langle \mathfrak{f}^l(a_n,\dots,a_{k+1}), \mathfrak{f}^k(a_k,\dots,a_1)\rangle = 0 \tag{1–7}$$

for any sequence of composable morphisms $a_1,\dots,a_n$, where $n \geq 3$. For our A_∞-category, the only potentially non-trivial values of $\mathfrak{f}^*$ are

$$\mathfrak{f}^*(\xi_Y^n,\eta_\alpha,\xi_X^m),\quad \mathfrak{f}^*(\xi_X^n,\theta_\beta,\xi_Y^m),\quad \mathfrak{f}^*(\xi_Y^p,\eta_\alpha,\xi_X^n,\theta_\beta,\xi_Y^m),\quad \mathfrak{f}^*(\xi_X^p,\theta_\beta,\xi_Y^n,\eta_\alpha,\xi_X^m).$$

The constraints between them are given by (1–7) applied to the following two kinds of composable strings

$$(\xi_X^c,\theta_\beta,\xi_Y^b,\eta_\alpha,\xi_X^a) \quad \text{and} \quad (\xi_Y^c,\eta_\alpha,\xi_X^b,\theta_\beta,\xi_Y^a),$$

where $a+b+c>0$. The first of these strings gives the identity

$$\begin{aligned}&\langle \xi, \mathfrak{f}^*(\xi_X^{c-1},\theta_\beta,\xi_Y^b,\eta_\alpha,\xi_X^a)\rangle + \langle \mathfrak{f}^*(\xi_X^c,\theta_\beta,\xi_Y^b,\eta_\alpha,\xi_X^{a-1}),\xi\rangle \\ &- \sum_{b_1+b_2=b,\gamma} \langle \mathfrak{f}^*(\xi_X^c,\theta_\beta,\xi_Y^{b_2}),\eta_\gamma\rangle \cdot \langle f^*(\xi_Y^{b_1},\eta_\alpha,\xi_X^a),\theta_\gamma\rangle \ = 0,\end{aligned}$$

where the first (resp., second) term appears only for $a \geq 1$ (resp., $c \geq 1$). We can rewrite these identities in terms of the generating series

$$\widetilde{\varphi}_y^x := \sum_{b,c\geq 0} \langle \mathfrak{f}^{b+c+1}(\xi_X^c,\theta_\beta,\xi_Y^b),\eta_\alpha\rangle \cdot x^c y^b \cdot e_{\beta\alpha},$$

$$\psi_y^{x_1,x_2} := \sum_{a,b,c\geq 0} \langle \mathfrak{f}^{a+b+c+2}(\xi_X^c,\theta_\beta,\xi_Y^b,\eta_\alpha,\xi_X^a),\xi_X\rangle \cdot x_1^a y^b x_2^c \cdot e_{\beta\alpha},$$

as follows:

$$(x_1-x_2)\psi_y^{x_1,x_2} - \widetilde{\varphi}_y^{x_2}\cdot\varphi_y^{x_1} - \mathrm{Id} = 0.$$

Note that subtracting Id here corresponds to avoiding the case $a=b=c=0$ in cyclic homotopy equation. Setting $x_2 = x_1$ we deduce from this that

$$\widetilde{\varphi}_y^x = -(\varphi_y^x)^{-1}, \tag{1–8}$$

and so the above identity can be solved for $\psi_y^{x_1,x_2}$:

$$\psi_y^{x_1,x_2} = \frac{\mathrm{Id} - \varphi_y^{x_1}(\varphi_y^{x_2})^{-1}}{x_1-x_2}.$$

Similarly, the constraints associated with the strings $(\xi_Y^c,\eta_\alpha,\xi_X^b,\theta_\beta,\xi_Y^a)$ boil down to (1–8) and to an equation expressing all $\mathfrak{f}^*(\xi_Y^c,\eta_\alpha,\xi_X^b,\theta_\beta,\xi_Y^a)$ in terms of the coefficients of φ_y^x. □

1.4 Involution on skew-symmetric solutions of the AYBE

In the following Proposition we define a natural involution on the skew-symmetric solutions of the general AYBE (0–7), where the variables x_i and y_i could be either distinct elements of some sets $\mathcal{X}$ and $\mathcal{Y}$, or formal variables, as in Theorem A.

Proposition 1.4.1 *Let $r^{x_1x_2}_{y_1y_2}$ be a solution of the general AYBE with values in $\mathrm{Mat}_n(\mathbf{k})\otimes\mathrm{Mat}_n(\mathbf{k})\otimes R$, where R is a commutative $\mathbf{k}$-algebra, satisfying the skew-symmetry condition (0–4). Set*

$$\hat{r}^{\hat{x}_1\hat{x}_2}_{\hat{y}_1\hat{y}_2} := (r^{\hat{y}_1\hat{y}_2}_{\hat{x}_2\hat{x}_1})^t\cdot \mathrm{P},$$

where $(a_1\otimes a_2)^t = a_1^t\otimes a_2^t$ (here a^t is the transpose of a matrix a) and P is given by (0–9), and in the non-formal case the arguments $\hat{x}_i$ (resp., $\hat{y}_i$) take values in $\mathcal{Y}$ (resp., $\mathcal{X}$). Then

(i) $\hat{r}^{\hat{x}_1\hat{x}_2}_{\hat{y}_1\hat{y}_2}$ is again a solution of the AYBE satisfying the skew-symmetry condition. Furthermore, one has $\hat{\hat{r}} = r$. If $\hat{x}_i$ are $\hat{y}_i$ are formal variables, and r has an expansion of the form (0–10) then so does $-\hat{r}^{\hat{x}_1\hat{x}_2}_{-\hat{y}_1,-\hat{y}_2}$.

(ii) In the context of Theorem A, assume that $r^{x_1x_2}_{y_1y_2}$ corresponds to an A_∞-structure on $\mathcal{A}\otimes R$. Formally setting $\hat{X} = Y$ and $\hat{Y} = \hat{X}[1]$, $\hat{\theta}_\alpha = \eta_\alpha$, $\hat{\eta}_\beta = \theta_\beta$, etc., we get a new A_∞-structure on $\mathcal{A}\otimes R$. Then the corresponding formal solution of the AYBE is precisely $\hat{r}$.

Proof. (i) Note that the map

$$\mathrm{Mat}_n(\mathbf{k})\otimes\mathrm{Mat}_n(\mathbf{k})\to\mathrm{Mat}_n(\mathbf{k})\otimes\mathrm{Mat}_n(\mathbf{k}) : x\mapsto \mathrm{P}\cdot x\cdot\mathrm{P}$$

is just the permutation of factors. Thus, the skew-symmetry condition can be rewritten as

$$\mathrm{P}r^{x_1x_2}_{y_1y_2}\mathrm{P} = -r^{x_2x_1}_{y_2y_1}.$$

Passing to the transposed matrices and making the substitution $x_i = \hat{y}_i$, $y_2 = \hat{x}_1$, $y_1 = \hat{x}_2$, we derive that $\hat{r}$ also satisfies (0–4). The fact that $\hat{\hat{r}} = r$ follows from the identity $\mathrm{P}\cdot\mathrm{P} = 1\otimes 1$.

The AYBE equation for $\hat{r}$ is the following equation for r:

$$(r^{x_2x_3}_{y_1y_2})^{12,t}\mathrm{P}^{12}(r^{x_1x_3}_{y_3y_1})^{13,t}\mathrm{P}^{13} - (r^{x_1x_2}_{y_3y_1})^{23,t}\mathrm{P}^{23}(r^{x_2x_3}_{y_3y_2})^{12,t}\mathrm{P}^{12} + (r^{x_1x_3}_{y_3y_2})^{13,t}\mathrm{P}^{13}(r^{x_1x_2}_{y_2y_1})^{23,t}\mathrm{P}^{23} = 0,$$

where we set $x_i = \hat{y}_i$, $y_i = \hat{x}_i$. Using the skew-symmetry of r (and the fact that $\mathrm{P}^t = \mathrm{P}$) we can rewrite this as

$$-\mathrm{P}^{12}(r^{x_3x_2}_{y_2y_1})^{12,t}(r^{x_1x_3}_{y_3y_1})^{13,t}\mathrm{P}^{13} + \mathrm{P}^{23}(r^{x_2x_1}_{y_1y_3})^{23,t}(r^{x_2x_3}_{y_3y_2})^{12,t}\mathrm{P}^{12} - \mathrm{P}^{13}(r^{x_3x_1}_{y_2y_3})^{13,t}(r^{x_1x_2}_{y_2y_1})^{23,t}\mathrm{P}^{23} = 0.$$

or passing to the transpose,

$$-\mathrm{P}^{13}(r^{x_1x_3}_{y_3y_1})^{13}(r^{x_3x_2}_{y_2y_1})^{12}\mathrm{P}^{12} + \mathrm{P}^{12}(r^{x_2x_3}_{y_3y_2})^{12}(r^{x_2x_1}_{y_1y_3})^{23}\mathrm{P}^{23} - \mathrm{P}^{23}(r^{x_1x_2}_{y_2y_1})^{23}(r^{x_3x_1}_{y_2y_3})^{13}\mathrm{P}^{13} = 0.$$

Using the fact that $\mathrm{P}\cdot\mathrm{P} = 1$, we can rewrite this as

$$-(r^{x_1x_3}_{y_3y_1})^{13}(r^{x_3x_2}_{y_2y_1})^{12} + \mathrm{P}^{13}\mathrm{P}^{12}(r^{x_2x_3}_{y_3y_2})^{12}(r^{x_2x_1}_{y_1y_3})^{23}\mathrm{P}^{23}\mathrm{P}^{12} - \mathrm{P}^{13}\mathrm{P}^{23}(r^{x_1x_2}_{y_2y_1})^{23}(r^{x_3x_1}_{y_2y_3})^{13}\mathrm{P}^{13}\mathrm{P}^{12} = 0.$$

Now we use the identities $\mathrm{P}^{13}\mathrm{P}^{12} = \mathrm{P}^{12}\mathrm{P}^{23}$, $\mathrm{P}^{13}\mathrm{P}^{23} = \mathrm{P}^{12}\mathrm{P}^{13}$ and the fact that $x\mapsto \mathrm{P}^{ij}x\mathrm{P}^{ij}$ acts as a transposition (ij), to rewrite this as

$$-(r^{x_1x_3}_{y_3y_1})^{13}(r^{x_3x_2}_{y_2y_1})^{12} + (r^{x_2x_3}_{y_3y_2})^{23}(r^{x_2x_1}_{y_1y_3})^{31} - (r^{x_1x_2}_{y_2y_1})^{12}(r^{x_3x_1}_{y_2y_3})^{32} = 0.$$

Swapping 2 and 3 we get the equation

$$-(r^{x_1x_3}_{y_3y_1})^{12}(r^{x_3x_2}_{y_2y_1})^{13}+(r^{x_2x_3}_{y_3y_2})^{32}(r^{x_2x_1}_{y_1y_3})^{21}-(r^{x_1x_2}_{y_2y_1})^{13}(r^{x_3x_1}_{y_2y_3})^{23}=0.$$

Using the skew-symmetry to rewrite the middle summand we get the equation obtained from the AYBE by the change of variables

$$(x_1,x_2,x_3;y_1,y_2,y_3)\mapsto(x_3,x_1,x_2;y_2,y_3,y_1).$$

(ii) This follows from the cyclic symmetry. Namely, if we set $y_1=\hat{x}_2$, $y_2=\hat{x}_1$, $x_i=\hat{y}_i$, for $i=1,2$, and use the corresponding identification between the twisted objects, we see that the new solution comes from the Massey products for the composable arrows $Y_2\to X_1\to Y_1\to X_2$. Furthermore, it is given by

$$\sum_{\alpha,\alpha',\beta,\beta'}\langle\mathfrak{m}_3^t(\eta^{12}_\beta,\theta^{11}_\alpha,\eta^{21}_{\beta'}),\theta^{22}_{\alpha'}\rangle\cdot e_{\alpha'\beta}\otimes e_{\alpha\beta'}+\ldots,$$

where the other terms are standard singular parts. Using the cyclic symmetry we can rewrite this as

$$\sum_{\alpha,\alpha',\beta,\beta'}\langle\mathfrak{m}_3^t(\theta^{22}_{\alpha'},\eta^{12}_\beta,\theta^{11}_\alpha),\eta^{21}_{\beta'}\rangle\cdot e_{\alpha'\beta}\otimes e_{\alpha\beta'}+\ldots.$$

This matches the formula for $(r^{x_1x_2}_{y_1y_2})^t\cdot\mathrm{P}$ due to the identity

$$(e_{\beta'\alpha'}\otimes e_{\beta\alpha})^t\cdot\mathrm{P}=e_{\alpha'\beta}\otimes e_{\alpha\beta'}.$$

□

Corollary 1.4.2 *Let $r(u,v)$ be a solution of the AYBE with values in $\mathrm{Mat}_n(\mathbf{k})\otimes\mathrm{Mat}_n(\mathbf{k})$, satisfying the skew-symmetry condition (0–2). Then $r(v,u)^t\cdot\mathrm{P}$ is again a solution of the AYBE satisfying the skew-symmetry condition.*

Proof. Apply Proposition 1.4.1(i) to $r^{x_1x_2}_{y_1y_2}=r(x_1-x_2,y_1-y_2)$. □

Recall that the nondegeneracy condition on solutions of the AYBE imposed in [20] is that the tensor $r(u,v)\in\mathrm{Mat}_n(\mathbf{k})\otimes\mathrm{Mat}_n(\mathbf{k})$ is nondegenerate for generic (u,v). Now we are going to use the above involution to show that the pole conditions for $r(u,v)$, imposed in the classification result of [20], are implied by the following stronger nondegeneracy condition, involving $r(u,v)$ and $r(u,v)^t\cdot\mathrm{P}$.

Definition 1.4.3 Let us say that an $\mathrm{Mat}_n(\mathbf{k})\otimes\mathrm{Mat}_n(\mathbf{k})$-valued function $r(u,v)$, meromorphic in a neighborhood of $(0,0)$, is *strongly nondegenerate* if the tensors $r(u,v)$ and $r(u,v)^t\cdot\mathrm{P}$ are nondegenerate for generic (u,v).

Proposition 1.4.4 *Assume $N>1$. Let $r(u,v)$ be a strongly nondegenerate skew-symmetric solution of the AYBE (meromorphic in a neighborhood of $(0,0)$). Then $r(u,v)$ has a simple pole at $u=0$ (resp., $v=0$) with the polar term $c\cdot\frac{1\otimes1}{u}$ (resp., $c'\cdot\frac{\mathrm{P}}{v}$), where c and c' are nonzero constants.*

Proof. First, we claim that the involution $r(u,v) \mapsto \hat{r}(u,v) := r(v,u)^t \cdot \mathrm{P}$ on skew-symmetric solutions of the AYBE preserves the notion of strong nondegeneracy. Indeed, this immediately follows from the observation that $\mathrm{P} \cdot r \cdot \mathrm{P} = r^{21}$, so it is nondegenerate if and only if r is nondegenerate.

Next, given $r(u,v)$, a strongly nondegenerate skew-symmetric solution of the AYBE, we claim that $r(u,v)$ necessarily has a pole at $u = 0$. Indeed, assume $r(u,v)$ is regular at $u = 0$. Then by [20, Lem. 1.2], we have an expansion $r(u,v) = r_0(v) + ur_1(v) + \dots$, where $r_0(v) = r(u,0)$ is a nondegenerate skew-symmetric solution of the AYBE. Then by [20, Thm. 0.2], $r_0(v)$ has a pole at $v = 0$, hence, $r(u,v)$ has a pole at $v = 0$. By [20, Lem. 1.3], this implies that $r(u,v)$ has a simple pole at $v = 0$ with the polar part $c \cdot \frac{\mathrm{P}}{v}$. Hence, $\hat{r}(u,v)$ has a simple pole at $u = 0$ with the polar part $c \cdot \frac{1 \otimes 1}{u}$. Since $\hat{r}(u,v)$ is still a nondegenerate skew-symmetric solution of the AYBE, by [20, Lem. 1.5], $\hat{r}(u,v)$ has a simple pole at $v = 0$. Equivalently, $r(u,v)$ has a simple pole at $u = 0$, which is a contradiction.

Thus, we know that $r(u,v)$ has a pole at $u = 0$, or equivalently, $\hat{r}(u,v)$ has a pole at $v = 0$. By [20, Lem. 1.3], this implies that $\hat{r}(u,v)$ has a simple pole at $v = 0$ with the polar part $c \cdot \frac{\mathrm{P}}{v}$. Hence, $r(u,v)$ has a simple at $u = 0$ with the polar part $c \cdot \frac{1 \otimes 1}{u}$. Now the assertion follows from [20, Lem. 1.5]. □

1.5 From algebraic/analytic to formal solutions of the general AYBE

In this section we want to consider the solutions of the general AYBE arising, as described in Introduction, from two algebraic families of objects $\mathcal{X}$ and $\mathcal{Y}$. We want to show how to pass from these solutions to the corresponding formal solutions associated to picking one object in each family.

We will use the formalism from [25, Sec. 1] concerning families of objects in A_∞-categories.

Let $\mathscr{A}$ be an A_∞-category over $\mathbf{k}$, and let $\mathcal{X}$ and $\mathcal{Y}$ be smooth affine curves over $\mathbf{k}$, such that we have perfect families of $\mathscr{A}$-modules $\mathcal{M}$ and $\mathcal{N}$ parametrized by $\mathcal{X}$ and $\mathcal{Y}$. We assume that for $x \neq x'$ (resp., $y \neq y'$) one has $\mathrm{Hom}^*(M_x, M_{x'}) = 0$ (resp., $\mathrm{Hom}^*(N_y, N_{y'}) = 0$), that each M_x (resp., N_y) is 1-spherical, and that $\mathrm{Hom}(M_x, N_y)$ are concentrated in degree 0. Furthermore, we assume that $\mathcal{H}om^*(p_\mathcal{X}^*\mathcal{M}, p_\mathcal{Y}^*\mathcal{N})$ is a vector bundle over $\mathcal{X} \times \mathcal{Y}$.

Recall (see [25, (1h)]) that one can associate with the families $\mathcal{M}$ and $\mathcal{N}$ the deformation classes

$$Def(\mathcal{M}) \in \Omega^1_\mathcal{X} \otimes \mathrm{Hom}^1(\mathcal{M}, \mathcal{M}), \ \ Def(\mathcal{N}) \in \Omega^1_\mathcal{Y} \otimes \mathrm{Hom}^1(\mathcal{N}, \mathcal{N}).$$

Let $\mathcal{U} \subset \mathcal{X}^2 \times \mathcal{Y}^2$ be the complement to the diagonals $\Delta_\mathcal{X} \times \mathcal{Y}^2 \cup \mathcal{X}^2 \times \Delta_\mathcal{Y}$. Then over $\mathcal{U}$ we have the induced families $\mathcal{M}(x_1)$, $\mathcal{M}(x_2)$, $\mathcal{N}(y_1)$ and $\mathcal{N}(y_2)$ (pull-backs from the families over $\mathcal{X}$ and $\mathcal{Y}$), such that

$$\mathcal{H}om^*(\mathcal{M}(x_1), \mathcal{M}(x_2)) = \mathcal{H}om^*(\mathcal{N}(y_1), \mathcal{N}(y_2)) = 0.$$

Thus, we have a well defined triple Massey product

$$\mathrm{MP}(x_1, x_2; y_1, y_2) : \mathcal{H}om^0(\mathcal{M}(x_1), \mathcal{N}(y_1)) \otimes \mathcal{H}om^1(\mathcal{N}(y_1), \mathcal{M}(x_2)) \otimes \mathcal{H}om^0(\mathcal{M}(x_2), \mathcal{N}(y_1)) \to \mathcal{H}om^0(\mathcal{M}(x_1), \mathcal{N}(y_2)),$$

which is a morphism of vector bundles over $\mathcal{U}$.

On the other hand, let us fix points $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$, and let us fix generators $\xi_{x_0} \in \mathrm{Hom}^1(M_{x_0}, M_{x_0})$ and $\xi_{y_0} \in \mathrm{Hom}^1(N_{y_0}, N_{y_0})$. Assume that

$$Def(\mathcal{M})|_{x_0} \in \Omega^1_{\mathcal{X}}|_{x_0} \otimes \mathrm{Hom}^1(M_{x_0}, M_{x_0}) \simeq \Omega^1_{\mathcal{X}}|_{x_0}$$

is nonzero, and similarly $Def(\mathcal{N})|_{y_0} \in \Omega^1_{\mathcal{Y}}|_{y_0}$ is nonzero. Then we can choose a formal parameter t near x_0 on $\mathcal{X}$ (resp., parameter s near y_0 on $\mathcal{Y}$) such that $Def(\mathcal{M}) = dt$ in a formal neighborhood of x_0 (resp., $Def(\mathcal{N}) = ds$ near y_0).

Let $\hat{\mathcal{M}}_{x_0}$ be the family over $k[[t]]$ obtained from $\mathcal{M}$ by restriction to a formal disk around x_0, and let $\hat{\mathcal{N}}_{y_0}$ be the similar family over $k[[s]]$. On the other hand, we have twisted objects $(M_{x_0}, t\xi_{x_0})$ and $(N_{y_0}, s\xi_{y_0})$, over $k[[t]]$ and $k[[s]]$, respectively. These twisted objects produce the same deformation classes dt and ds, so by the proof of [25, Prop. 1.21], we derive the existence of quasi-isomorphisms of families

$$\hat{\mathcal{M}}_{x_0} \simeq (M_{x_0}, t\xi_{x_0}), \ \ \hat{\mathcal{N}}_{y_0} \simeq (N_{y_0}, s\xi_{y_0}).$$

By the functoriality of Massey products, this implies that the formal expansion of the Massey product $\mathrm{MP}(x_1, x_2, y_1, y_2)$ near x_0 and y_0, is equal to the triple Massey product considered in the proof of Theorem A.

Note that $Def(\mathcal{M})|_{x_0}$ can be identified with the usual class of the first-order deformation of M_{x_0}, associated with $\mathcal{M}$. In particular, in the situation when $\mathcal{M}$ is a universal deformation of M_{x_0} then $Def(\mathcal{M})|_{x_0}$ is nonzero. This is the situation that occurs when we consider families of simple vector bundles (or structure sheaves of points) on elliptic curves and their degenerations, as in [19], [20]. In the case of families of Lagrangians in Fukaya category, we have a similar picture, with algebraic families replaced by analytic families.

2 Trigonometric solutions of the AYBE from symplectic geometry

2.1 A square-tiled surface from Belavin-Drinfeld structures

In this section, starting from an associative Belavin-Drinfeld structure (S, C_1, C_2, A), we construct a punctured Riemann surface Σ together with a non-vanishing holomorphic one-form $\alpha \in \Gamma(C, \Omega^{1,0}_C)$.

Let us begin with a finite set S of n elements, and two permutations $C_1, C_2 \in Aut(S) \cong \mathfrak{S}_n$ such that the subgroup $\langle C_1, C_2 \rangle \subset Aut(S)$ is transitive. Let $\mathbb{T}$ be the square torus $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ and let $\mathbb{T}_0 = \mathbb{T} \setminus \{0\}$ be the punctured torus. Let us also consider the oriented curves, $l_1, l_2 : [0, 1] \to \mathbb{T}$ defined by:

$$l_1(t) = \frac{1 - it}{2} \ , \quad l_2(t) = \frac{t + i}{2}$$

Let $p_0 \in \mathbb{T}_0$ be the point $(1 + i)/2$, which is the unique point in $l_1 \cap l_2$. Consider the n-fold (unramified) covering:

$$\pi : \Sigma_0 \to \mathbb{T}_0$$

corresponding to the subgroup $H \subset \pi_1(\mathbb{T}_0, l_1 \cap l_2) = \langle l_1, l_2 \rangle \cong F_2$ given as the preimage of the stabilizer of a point $s_0 \in S$ under the homomorphism $\rho : \pi_1(\mathbb{T}_0) \to Aut(S)$ defined by

$$\rho(l_1) = C_1 \ , \quad \rho(l_2) = C_2.$$

Thus, we identify the set S with the fiber $\pi^{-1}(p_0)$, so that the action of the generators l_1 and l_2 of the fundamental group $\pi_1(T_0, p_0)$ on the fibre S is given by the permutations $C_1, C_2 : S \to S$. We let $L_i = \pi^{-1}(l_i)$ be the multi-curves in Σ_0 covering the circles l_i, so that $L_1 \cap L_2 = \pi^{-1}(p_0) = S$.

The assumption that the subgroup $\langle C_1, C_2 \rangle \subset Aut(S)$ is transitive guarantees that Σ_0 is connected. The curves L_1 and L_2 are connected if and only if both C_1 and C_2 are transitive permutations. We will always require that Σ_0 is connected. If (S, C_1, C_2) comes from an associative Belavin-Drinfeld data, then C_1, C_2 are required to be transitive permutations, however this condition is not strictly necessary in what follows.

One can lift the flat metric on $\mathbb{T}_0$ to Σ_0. To visualise this metric on Σ_0, let us now give a more geometric construction of the covering map $\pi : \Sigma_0 \to \mathbb{T}_0$. Recall that $\mathbb{T}$ is obtained from the unit square in $[0,1] \times [0,1] \subset \mathbb{R}^2$ by identifying the opposite sides. $\mathbb{T}_0$ is obtained from this by removing the corner point. Now let us take n copies of the unit square (with corners removed) labeled the set $S = \{1, \ldots, n\}$. Given automorphisms $C_1, C_2 \subset \mathfrak{S}_n \cong Aut(S)$, construct a surface Σ_0 as follows: 1) identify the right edge of the i^{th} square with the left edge of j^{th} square if $C_1(i) = j$; 2) identify the bottom edge of the i^{th} square with the top edge of the j^{th} square if $C_2(i) = j$. It is because of this construction Σ_0 is called a *square-tiled surface*. The name was first suggested to Anton Zorich by Alex Eskin [29].

By the Riemann existence theorem ([8, Sec. 4.2.2]), the surface Σ_0 can be completed to a surface $\widehat{\Sigma}_0$ and the covering map extends to a branched covering map:

$$\widehat{\pi} : \widehat{\Sigma}_0 \to \mathbb{T}$$

ramified along the origin $(0,0) \in \mathbb{T}$. The preimage of the origin,

$$\{p_1, p_2, \ldots p_b\} = \widehat{\pi}^{-1}(0)$$

consists of a number of points, which is equal to the number of cycles in the cycle decomposition of the commutator $[C_1, C_2]$ into disjoint union of cycles of varying lengths (from 1 to n). Indeed, the curves L_1 and L_2 divide the surface $\widehat{\Sigma}_0$ into polygons, such that the point p_i is contained in a $(2e(p_i)+2)$-gon, where $e(p_i)$ is the ramification index of the point p_i.

We let b_k denote the number of k-cycles, so that we have $n = \sum_{k=1}^{n} k b_k$. We record the following elementary computation, which follows from the above explicit description of $\widehat{\Sigma}_0$ as a union of b-polygonal regions, or also by the Riemann-Hurwitz formula.

Proposition 2.1.1 *The number of points in $\widehat{\Sigma_0} \setminus \Sigma_0$ is equal to $b = \sum_{k=1}^{n} b_k$. The Euler characteristic of Σ_0 is $\chi(\Sigma_0) = -n$. Consequently, the genus g is determined by the formula*

$$\chi(\Sigma_0) = 2 - 2g - b = -n.$$

In particular, $g = 1$ if and only if C_1 and C_2 commute. □

Finally, we need to incorporate the proper subset $A \subset S$ that appears in an associative Belavin-Drinfeld structure (S, C_1, C_2, A). This data enters in determining a partial compactification of Σ_0.

Namely, recall that A is, by definition, a subset of the set of fixed points of the action of the commutator $[C_1, C_2]$ on S. In terms of the branched covering map $\widehat{\pi} : \widehat{\Sigma}_0 \to \mathbb{T}$, the set of fixed points of the commutator $[C_1, C_2]$ can be identified with the set of points p_i in the preimage $\widehat{\pi}^{-1}(0)$ which have ramification index $e(p_i) = 1$. Using this we can identify A with a subset of points p_i where the map $\widehat{\pi}$ is unramified. To be precise, an element $a \in A$ gives a square with corners

$$\{a, C_2(a), C_1C_2(a), C_2^{-1}C_1C_2(a)\},$$

which determines a point $p_a \in \widehat{\pi}^{-1}(0)$ of ramification index 1 contained in this square. We define $\Sigma = \Sigma_A$ to be the partial compactification $\Sigma_0 \cup \{p_a | a \in A\}$. Note that the covering map $\pi : \Sigma_0 \to \mathbb{T}_0$ extends to a local diffeomorphism:

$$\pi : \Sigma \to \mathbb{T}$$

Hence, the flat metric on $\mathbb{T}$ lifts to a flat metric on Σ so as to make π into a local isometry. From now on, we will consider the square-tiled surface Σ equipped with this flat metric. Note that, for convenience, we always normalize the metric on $\mathbb{T}$ so that the length of the curves l_1 and l_2 are 1.

Equivalently, we write C for the unique Riemann surface structure on Σ making $\pi : \Sigma \to \mathbb{T}$ into a holomorphic map. In this case, we equip C with the one-form $\alpha = \pi^* dz$, the pullback of the standard non-vanishing holomorphic one-form on $\mathbb{T}$.

Example 2.1.2 Figure 1 shows an example of this construction corresponding to $S = \{1, 2, 3, 4\}, C_1 = (1, 4, 2, 3), C_2 = (1, 2, 3, 4)$. The red curve $L_1 \subset \Sigma_0$ and the blue curve $L_2 \subset \Sigma_0$ depict the preimages of the curves $l_1, l_2 \subset \mathbb{T}_0$. The flat metric can be extended over the black labelled point without any singularities. Thus, we can choose A to be either empty or include the unique black labelled point, which corresponds to $\{3\}$ - the unique fixed point of $[C_1, C_2]$. If $A = \{3\}$, then correspondingly, we compactify Σ_0 by filling in the puncture labelled black.

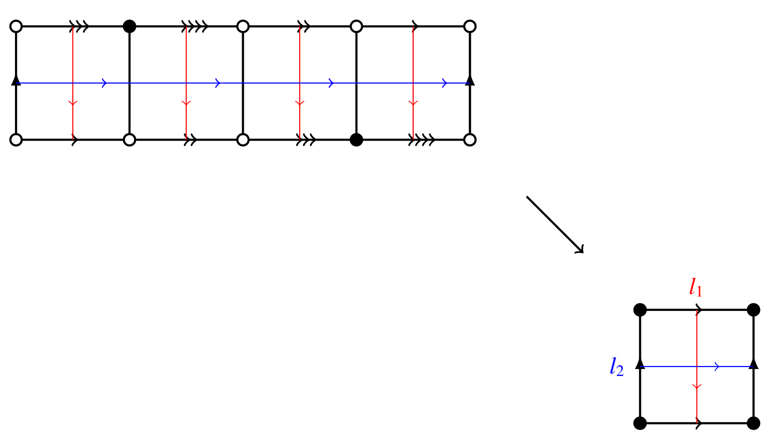

Figure 1: A square-tiled surface

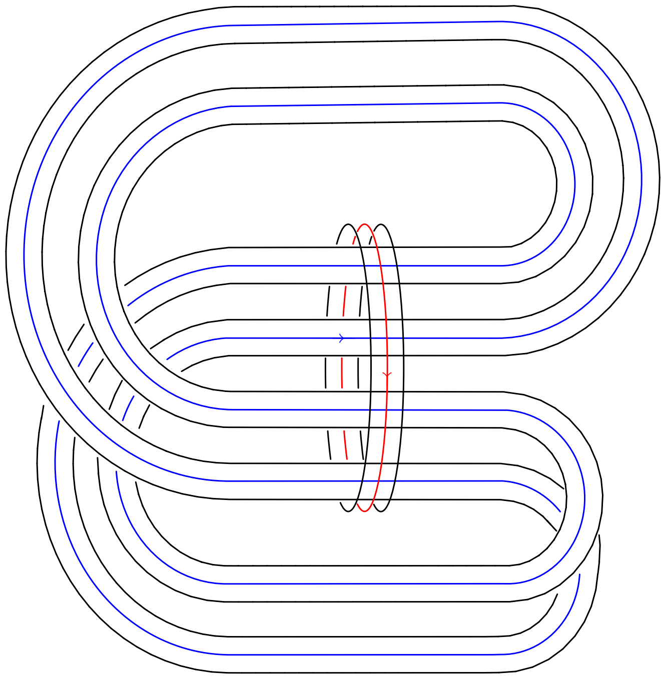

Figure 2: $S = \{1,2,3,4\}, C_1 = (1,2,3,4), C_2 = (1,3,2,4)$

Remark 2.1.3 Topologically the punctured torus $\mathbb{T}_0$ can be seen as the plumbing of two cotangent bundles $T^*l_i \cong T^*S^1$ at one point $l_1 \cap l_2$. Similarly, one can construct the surface Σ_0 as the plumbing of T^*L_1 and T^*L_2 at n-points corresponding to $L_1 \cap L_2$. The Figure 2 illustrates this construction in the case of $S = \{1,2,3,4\}$, $C_1 = (1,2,3,4)$, $C_2 = (1,3,2,4)$.

2.2 A Fukaya category from Belavin-Drinfeld structures

Let (C, α) be the square-tiled surface obtained from an associative Belavin-Drinfeld structure as above. Let Σ be the topological surface underlying C. The square $\Omega = \alpha \otimes \alpha \in \Gamma(C, (\Omega_C^{1,0})^{\otimes 2})$ determines a non-vanishing quadratic form, which gives a flat Riemannian metric $|\Omega|$ on Σ and a horizontal foliation of tangent vectors v with $\Omega(v,v) > 0$. The Riemannian metric determines an area form[4] ω and the horizontal foliation determines a grading structure on Σ, i.e a section of the projectivized tangent bundle of S, which we view as an unoriented line field $l \subset T(\Sigma)$. We note that such line fields form a torsor for $C^\infty(\Sigma, \mathbb{R}P^1)$, and the connected components of this group can be identified with $H^1(\Sigma; \mathbb{Z})$.

To work over $\mathbb{C}$, one works with exact Fukaya categories as in [24]. Thus, we will need to choose a primitive θ for ω, which exists since Σ is non-compact. We choose this so that the Lagrangians L_1 and

[4]Note that our symplectic form ω is not convex at infinity. This is usually required in setting up Floer theory in order to ensure that a maximum principle holds which guarantees that pseudo-holomorphic disks remain in a compact region. However, in dimension 2, this holds for topological reasons. Alternatively, one could modify ω near infinity to make it convex. Either way, the outcome is unchanged and we will simply use the area form ω.

L_2 are exact. One can arrange this as follows: Choose any primitive θ_0 for ω; find a closed 1-form σ_0 such that $\sigma_0([L_i]) = \int_{L_i} \theta_0$, which exists since L_i give independent non-trivial homology classes in $H_1(\Sigma)$; and let $\theta = \theta_0 - \sigma_0$. We also normalize the area form so that the geodesics L_1 and L_2 have length 1.

We can now form a $\mathbb{Z}$-graded triangulated 1-Calabi-Yau $\mathbb{C}$-linear A_∞ category, the Fukaya category $\mathcal{F}(\Sigma)$ of $(\Sigma, d\theta, \Omega)$. The objects of $\mathcal{F}(\Sigma)$ are closed, exact, oriented curves L equipped with grading structures and a rank 1 local system $\xi \to L$. Recall that a grading structure on a curve L means a choice of a homotopy class of a path from TL to the line field $l_{|L}$. If $x \in L \cap L'$ is a transverse intersection point, the grading $|x|$ is given by $\lfloor \alpha/\pi \rfloor + 1$ where α is the net rotation from $T_xL \to l_x \to T_xL'$. This lifts the $\mathbb{Z}_2$-grading on the intersection points given by $\lfloor (L \cdot_x L')/\pi \rfloor + 1$ where $L \cdot_x L'$ is the local algebraic intersection number of L and L' at x associated to orientations of L and L'.

Note also that on a circle there are precisely two spin structures corresponding to connected and disconnected double coverings of the circle. We implicitly fix a spin structure on each closed, exact Lagrangian $L \subset \Sigma$. Changing the spin structure by the action of $H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2$ is equivalent to modifying the monodromy of the local system $\xi \to L$ by the action of $\{\pm 1\} \subset \mathbb{C}^\times$. Therefore, the effect of changing the choice of spin structure on L can be achieved by modifying the $\mathbb{C}^\times$ local system ξ. Spin structures enter in defining orientations of various moduli spaces of holomorphic curves and they play a role in determining the signs in various counts. In the case of Fukaya categories of 2-dimensional surfaces, which is the only situation considered in this paper, there is a combinatorial method given in [23, Sec. 7] that allows us to compute these signs. Throughout, in our explicit computations, we follow this method to determine the signs without giving further explanation.

The morphism spaces in the Fukaya category are given by Floer cochain complexes:

$$CF^*((L_1, \xi_1), (L_2, \xi_2)) = \bigoplus_{x \in L_1 \cap L_2} \hom_{\mathbb{C}}(\xi_1|_x, \xi_2|_x)$$

For brevity, we often suppress the local systems ξ_i from the notation. The A_∞-structure comprises a collection of maps:

$$\mathfrak{m}_k : CF(L_{k-1}, L_k) \otimes \ldots \otimes CF(L_0, L_1) \to CF(L_0, L_k)[2-k]$$

For $p_i \in L_{i-1} \cap L_i$ and $p_0 \in L_0 \cap L_k$, the components of these maps involving $\hom_{\mathbb{C}}(\xi_{i-1}|_{p_i}, \xi_i|_{p_i})$ are defined by counting holomorphic disks with $(k+1)$-boundary punctures such that the boundary components are mapped to $(L_0, L_1, \ldots L_k)$ in the cyclic order. Let us denote the moduli space of such pseudoholomorphic disks u in the homotopy class $[u]$ by $\mathcal{M}(p_k, p_{k-1}, \ldots, p_1, p_0; [u])$. If the index of $[u]$ is fixed to be $2-k$ and the regularity is arranged then Gromov-Floer compactness ensures that this moduli space is a finite set of points, which we can then count (with signs). For $\rho_i \in \hom_{\mathbb{C}}(\xi_{i-1}|_{p_i}, \xi_i|_{p_i})$, we set

$$\text{(2–1)} \qquad \mathfrak{m}_k(\rho_k, \ldots, \rho_1) = \sum_{[u]:\mathrm{ind}([u])=2-k} \#\mathcal{M}(p_k, p_{k-1}, \ldots, p_1, p_0; [u]) \cdot \mathrm{hol}_{\partial u} \in \hom_{\mathbb{C}}(\xi_0|_{p_0}, \xi_k|_{p_0}),$$

where the term $\mathrm{hol}_{\partial u}$ is defined as follows. The boundary component of u mapping to L_i gives isomorphisms $\xi_i|_{p_i} \to \xi_i|_{p_{i+1}}$. Therefore, given elements $\rho_i \in \hom_{\mathbb{C}}(\xi_{i-1}|_{p_i}, \xi_i|_{p_i})$, using the isomorphisms provided by u, we can construct the composition:

$$\text{hol}_{\partial u} = \rho_k \circ \ldots \circ \rho_1 \in \hom_{\mathbb{C}}(\xi_0|_{p_0}, \xi_k|_{p_0}).$$

Note that exactness ensures that the sum in Equation (2–1) is finite, hence is well-defined.

In practice, when we do explicit computations, we will mark points by $\star$ on each Lagrangian circle, and the contribution of a holomorphic disk will be weighted by the holonomy factor each time the boundary of the disk passes through the marked point.

We also note that it was proven by Fukaya [12] that the Fukaya category of compact Lagrangians over $\mathbb{R}$ (equivalently, over any field $\mathbf{k}$ of characteristic 0) has a model with a strictly cyclic A_∞-structure. The existence of such a cyclic structure is important for the applicability of Theorem A to A_∞ algebras that we compute from Fukaya categories below. On the other hand, for the purpose of computation of triple Massey products, we can use any model of the Fukaya category as triple Massey products are homotopy invariant notions. We find it convenient to use the model of the Fukaya category as given in [23, Sec. 7].

2.3 Constructing solutions to the AYBE via Massey products in $\mathcal{F}(\Sigma)$

As was shown in [20], with every associative Belavin-Drinfeld structure (S, C_1, C_2, A) one can associate a trigonometric solution of the AYBE. A slightly different looking definition of an associative Belavin-Drinfeld structure was used in [20]. In the next lemma we show the equivalence of the Definition (0.0.2) with the definition of the associative Belavin-Drinfeld structure in [20].

Let S be a finite set of n elements. We denote a transitive permutation as a map $C : S \to S$ and we write $\Gamma_C := \{(s, C(s))|s \in S\} \subset S \times S$ for its graph.

Lemma 2.3.1 *Let S be a set equipped with a pair of transitive permutations $C_1, C_2 : S \to S$. Then to give a proper subset $A \subset S$, such that (S, C_1, C_2, A) is an associative Belavin-Drinfeld structure, is equivalent to giving a pair of proper subsets $\Gamma_1, \Gamma_2 \subset \Gamma_{C_1}$ such that $(C_2 \times C_2)\Gamma_1 = \Gamma_2$.*

Proof. We set

$$\Gamma_1 = \{(a, C_1(a))|a \in A\}, \quad \Gamma_2 = \{(C_2(a), C_1(C_2(a))|a \in A.\}$$

One immediately sees that the condition

$$(C_2 \times C_2)(\Gamma_1) = \Gamma_2$$

is equivalent to $C_1C_2(a) = C_2C_1(a)$ for every $a \in A$. □

We prefer the form given in Definition 0.0.2 as it makes the symmetry with respect to switching C_1 and C_2 more clear (cf. Proposition 1.4.1). In examples, it may be convenient to identify $S = \{1, \ldots, n\}$ such that $C_1(i) = i + 1$ (modulo n). One then thinks of C_2 as an n-cycle in the symmetric group $\mathfrak{S}_n$.

The commutator $[C_1, C_2] \in \mathfrak{A}_n \subset \mathfrak{S}_n$ plays a special role in the definition as the elements of the set A correspond to a subset of the set of fixed points of the commutator $[C_1, C_2]$. We remark that it can be proven by induction that any element of the alternating group $\mathfrak{A}_n$ arises as the commutator $[C_1, C_2]$ of two n-cycles in $\mathfrak{S}_n$ (see Prop. 4 [13]).

Let A_S denote the algebra of endomorphisms of the $\mathbb{C}$-vector space with the basis $(\mathbf{e}_i)_{i\in S}$, so that $A_S \simeq \mathrm{Mat}_n(\mathbb{C})$, where $n = |S|$. We denote by $e_{ij} \in A_S$ the endomorphism defined by $e_{ij}(\mathbf{e}_k) = \delta_{jk}\mathbf{e}_i$. The solution of the AYBE associated with (S, C_1, C_2, A) is given by

(2–2)

$$\begin{aligned} r(u,v) = {} & \frac{1}{\exp(u)-1}\sum_i e_{ii}\otimes e_{ii} + \frac{1}{1-\exp(-v)}\sum_i e_{ii}\otimes e_{ii} \\ & + \frac{1}{\exp(u)-1}\sum_{0<k<n,i}\exp\Big(\frac{ku}{n}\Big)e_{C_1^k(i),C_1^k(i)}\otimes e_{ii} + \frac{1}{\exp(v)-1}\sum_{0<m<n,i}\exp\Big(\frac{mv}{n}\Big)e_{i,C_2^m(i)}\otimes e_{C_2^m(i),i} \\ & + \sum_{0<k,0<m;a\in A(k,m)}\Big\{\exp\Big(-\frac{ku+mv}{n}\Big)e_{C_2^m(a),a}\otimes e_{C_1^k(a),C_1^kC_2^m(a)} - \exp\Big(\frac{ku+mv}{n}\Big)e_{C_1^k(a),C_1^kC_2^m(a)}\otimes e_{C_2^m(a),a}\Big\}, \end{aligned}$$

where we denote by $A(k,m) \subset A$ the set of all $a \in A$ such that $C_1^iC_2^j(a) \in A$ for all $0 \le i < k, 0 \le j < m$. One can easily check that for $a \in A(k,m)$ one has $C_1^kC_2^m(a) = C_2^mC_1^k(a)$. Note also that $A(k,m)$ can be nonempty only if $k < n$ and $m < n$ (since A is a proper subset of S), so our formula is equivalent to that of [20, Thm. 0.1].

Let us denote by $\mathrm{pr} : \mathrm{Mat}_n(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$ the projection along $\mathbb{C}\cdot 1$. Let $r(u,v)$ be a unitary solution of the AYBE such that the Laurent expansion of r at $u = 0$ has form

$$r(u,v) = \frac{1\otimes 1}{u} + r_0(v) + ur_1(v) + \dots. \tag{2–3}$$

Then one can show that $(\mathrm{pr}\otimes\mathrm{pr})r_0(v)$ is a unitary solution of the CYBE, nondegenerate if $r(u,v)$ was nondegenerate.

One of the main results of [20] is that every nondegenerate unitary solution of the AYBE for $A = \mathrm{Mat}_n(\mathbb{C})$ (where $n > 1$), such that the Laurent expansion of r at $u = 0$ has form (2–3) and $(\mathrm{pr}\otimes\mathrm{pr})r_0(v)$ is a trigonometric solution of the CYBE, is equivalent to one of the solutions (2–2).

We will next show that the above solutions to the AYBE can be recovered from Massey products in $\mathcal{F}(\Sigma)$.

Recall that given a combinatorial data of an associative Belavin-Drinfeld structure (S, C_1, C_2, A), we have constructed a symplectic 2-manifold (Σ, ω) and Lagrangians $L_1, L_2 \subset \Sigma$. Recall also that, $\omega = \omega_g$ is the area form of a flat Riemannian metric g on Σ and the Lagrangians L_1, L_2 are geodesic curves of length 1.

Definition 2.3.2 Given x, $y \in \mathbb{C}$ we define the complex push-off L_1^x of L_1 (resp. L_2^y of L_2) to be the exact Lagrangian L_1 (resp. L_2) equipped with the complex rank 1 local system with monodromy e^x (resp. e^y)

Now, we let $\mathcal{X}$ to be the family of isomorphism classes of objects $\{L_1^x\}$ for $x \in \mathbb{C}$, and similarly, we let $\mathcal{Y}$ to be the family of isomorphism classes of objects $\{L_2^y\}$ for $y \in \mathbb{C}$. For simplicity of notation, we sometimes write x and y for the corresponding objects L_1^x and L_2^y of $\mathcal{F}(\Sigma)$. We remark that since by construction L_1^x and L_2^y are connected, gradable exact Lagrangians in Σ, up to shift there are unique grading structures on L_1^x and L_2^y. We choose the shifts so that $CF(L_1^x, L_2^y)$ is supported in degree 0 for all x, y.

Note that we can apply the discussion from Section 1.5 to deduce that the family of objects in the Fukaya category over the formal disk, associated with an analytic family (L_1^x) (resp. (L_2^y)) over $\mathbb{C}$, is quasi-isomorphic to the twisted object $(L_1, x \cdot \xi_{L_1})$ (resp. $(L_2, y \cdot \xi_{L_2})$) as in Section 1.1. (A sketch of a geometric proof of this also appears as [3, Lemma 4.1].)

Figure 3 shows the simplest example on $\mathbb{T}_0$, where we have drawn four objects (x_1, x_2, y_1, y_2) in the punctured torus, which corresponds to Belavin-Drinfeld data with $S = \{1\}, C_1 = (1), C_2 = (1)$. Note that the underlying exact Lagrangians of x_1 and x_2 (resp. y_1 and y_2) are Hamiltonian isotopic, however the monodromies of the complex local systems on them are different.

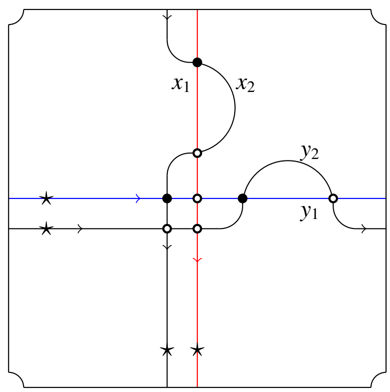

Figure 3: Hamiltonian perturbations of L_1 and L_2 (equipped with orientations and $\mathbb{C}^\times$-local systems)

Let us write $CF(x_1, x_2) = \mathbb{C}s_0 \oplus \mathbb{C}s_1$ and $CF(y_1, y_2) = \mathbb{C}t_0 \oplus \mathbb{C}t_1$. In Figure 3, we denoted degree 0 generators by hollow and degree 1 generators by black dots for these chain complexes. In what follows, the signs come from the orientation of various moduli spaces, which we computed following the prescription in [23, Sec. 7].

We can compute the Floer differential to be:

$$\mathfrak{m}_1(s_0) = -s_1 + e^{x_2 - x_1} s_1 \in CF^1(x_1, x_2)$$
$$\mathfrak{m}_1(t_0) = -t_1 + e^{y_2 - y_1} t_1 \in CF^1(y_1, y_2),$$

where the terms correspond to the two visible lunes in each case. Hence, for $x_1 \neq x_2$ and $y_1 \neq y_2$, we have $HF(x_1, x_2) = HF(y_1, y_2) = 0$.

We also have $CF^{\neq 0}(x, y) = 0$ for all x, y. Therefore, as explained in the introduction, for distinct objects x_1, x_2 and y_1, y_2, the triple Massey product:

$$\mathrm{MP} : CF^0(x_2, y_2) \otimes CF^1(y_1, x_2) \otimes CF^0(x_1, y_1) \to CF^0(x_1, y_2)$$

dualizes to a tensor

$$r^{x_1, x_2}_{y_1, y_2} : CF^0(x_2, y_2) \otimes CF^0(x_1, y_1) \to CF^0(x_1, y_2) \otimes CF^0(x_2, y_1)$$

that satisfies the AYBE.

Our next result (stated as Theorem B in the introduction) is that the obtained solution of the AYBE is precisely the trigonometric solution (2–2) associated with (S, C_1, C_2, A).

Theorem 2.3.3 *Let Σ be the square-tiled surface associated with an associative Belavin-Drinfeld structure (S, C_1, C_2, A). Then the tensor $r^{x_1,x_2}_{y_1,y_2}$ obtained from the triple products in the Fukaya category $\mathcal{F}(\Sigma)$ only depends on $u = x_2 - x_1$, $v = y_2 - y_1$ and is a solution of the AYBE over $\mathbb{C}$ given precisely by the formula (2–2).*

Proof. The proof of this theorem follows from a direct computation of triple Massey products in the Fukaya category $\mathcal{F}(\Sigma)$.

For clarity, we first do the computation for the simplest case, that is when $S = \{p\}$ is a single point and A is empty. Let us label the generators as follows:

$$CF^0(x_2, y_2) = \mathbb{C} \cdot p_{22} \ , \ CF^1(y_1, x_2) = \mathbb{C} \cdot q_{12} \ , \ CF^0(x_1, y_1) = \mathbb{C} \cdot p_{11} \ , \ CF^0(x_1, y_2) = \mathbb{C} \cdot p_{12}$$

Note that geometrically these generators correspond to the corners of the small square in the middle in Figure 3. We are interested in computing the Massey product:

$$\mathrm{MP}(p_{22}, q_{12}, p_{11}) = \mathfrak{m}_3(p_{22}, q_{12}, p_{11}) - \mathfrak{m}_2(h_2, p_{11}) - \mathfrak{m}_2(p_{22}, h_1)$$

where $h_1 \in CF^0(x_1, x_2)$ and $h_2 \in CF^0(y_2, y_1)$ satisfy $\mathfrak{m}_1(h_1) = \mathfrak{m}_2(q_{12}, p_{11})$ and $\mathfrak{m}_1(h_2) = \mathfrak{m}_2(p_{22}, q_{12})$.

From Figure 3, it is straightforward to compute:

$$\mathfrak{m}_2(q_{12}, p_{11}) = e^{x_2 - x_1} \cdot s_1$$
$$\mathfrak{m}_2(p_{22}, q_{12}) = t_1$$

Therefore, we have

$$h_1 = \left(\frac{e^{x_2 - x_1}}{e^{x_2 - x_1} - 1} \right) \cdot s_0$$
$$h_2 = \left(\frac{1}{e^{y_2 - y_1} - 1} \right) \cdot t_0$$

Again, from Figure 3, we can compute

$$\mathfrak{m}_2(p_{22}, s_0) = p_{12}$$
$$\mathfrak{m}_2(t_0, p_{11}) = e^{y_2 - y_1} \cdot p_{12}$$
$$\mathfrak{m}_3(p_{22}, q_{12}, p_{11}) = p_{12}$$

Therefore, letting $u = x_2 - x_1$, $v = y_2 - y_1$, we conclude that

$$\mathrm{MP}(p_{22}, q_{12}, p_{11}) = \left(1 + \frac{e^u}{1 - e^u} + \frac{e^v}{1 - e^v} \right) \cdot p_{12} = \left(\frac{1}{1 - e^u} + \frac{1}{e^{-v} - 1} \right) \cdot p_{12}.$$

Since the points p_{ij}, $i,j=1,2$ can be canonically identified with the intersection point $S=L_1\cap L_2=\{p\}$, we can view this tensor as

$$r^{x_1,x_2}_{y_1,y_2}=\left(\frac{1}{\exp(u)-1}+\frac{1}{1-\exp(-v)}\right)e_{11}\otimes e_{11} \tag{2–4}$$

(note that the dualization formula (0–3) brings in an extra overall sign).

In the case of general S but with $A=\emptyset$, consider the covering map

$$\pi:\Sigma_0\to\mathbb{T}_0$$

and the Lagrangians $X_i=\pi^{-1}(x_i)$ and $Y_i=\pi^{-1}(y_i)$ for $i=1,2$. Let us identify the points of intersection $X_i\cap Y_j$ by the set $(\mathbf{e}_i)_{i\in S}$. Now, as before, we are interested in computing Massey products of the form

$$\mathrm{MP}: HF^0(X_2,Y_2)\otimes HF^1(Y_1,X_2)\otimes HF^0(X_1,Y_1)\to HF^0(X_1,Y_2)$$

Since Massey products are quasi-isomorphism invariants, we can compute each one with a convenient Hamiltonian perturbation. Recall that we have the formula:

$$\mathrm{MP}(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k)=\mathfrak{m}_3(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k)-\mathfrak{m}_2(h_2,\mathbf{e}_k)-\mathfrak{m}_2(\mathbf{e}_i,h_1)$$

where $\mathfrak{m}_1(h_2)=\mathfrak{m}_2(\mathbf{e}_i,\mathbf{e}_j)$ and $\mathfrak{m}_2(h_1)=\mathfrak{m}_2(\mathbf{e}_j,\mathbf{e}_k)$.

We first observe that $\mathfrak{m}_2(\mathbf{e}_i,\mathbf{e}_j)=0$ if $i\neq j$ since there are no triangles that can contribute by construction, and $\mathfrak{m}_3(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k)=0$ unless $i=j$ or $j=k$, since A is empty.

Therefore, the only possibly non-trivial triple Massey products are of the form:

$$\begin{aligned}&\mathrm{MP}(C_1^k(\mathbf{e}_i),\mathbf{e}_i,\mathbf{e}_i)\quad\text{for } k=0,1,\dots(n-1)\\&\mathrm{MP}(\mathbf{e}_i,\mathbf{e}_i,C_2^m(\mathbf{e}_i))\quad\text{for } m=0,1\dots(n-1).\end{aligned}$$

For ease of computation, we arrange that the holonomy contributions of the $\mathbb{C}^\times$-local systems on L_i are divided equally to n parts, each contributing $e^{\frac{u}{n}}$ for L_1 and $e^{\frac{v}{n}}$ for L_2, interlaced between the n intersection points $L_i\cap L_j$. In other words, each time a holomorphic disk has boundary covering one of these regions, there is an associated weight $e^{\pm\frac{u}{n}}$ or $e^{\pm\frac{v}{n}}$, where the sign of the exponent is determined, as before, according to whether the boundary orientation of the holomorphic disk matches that of L_i or not.

The computation of $\mathrm{MP}(\mathbf{e}_i,\mathbf{e}_i,\mathbf{e}_i)$ is done in a completely analogous way to the above computation given for $n=1$, hence we have:

$$\mathrm{MP}(\mathbf{e}_i,\mathbf{e}_i,\mathbf{e}_i)=\left(\frac{1}{1-e^u}+\frac{1}{e^{-v}-1}\right)\mathbf{e}_i$$

Next, we observe that there are two families of rectangles with boundary on (X_1,Y_2,X_2,Y_1) as illustrated in Figure 4.

These contribute to $\mathfrak{m}_3$ products of the form

$$\mathfrak{m}_3(C_1^k(\mathbf{e}_i),\mathbf{e}_i,\mathbf{e}_i)=e^{\frac{ku}{n}}C_1^k(\mathbf{e}_i)$$

for $i=1,\dots,n$ and $k=1,\dots,n-1$ and

$$\mathfrak{m}_3(\mathbf{e}_i,\mathbf{e}_i,C_2^m(\mathbf{e}_i))=e^{\frac{mv}{n}}C_2^m(\mathbf{e}_i)$$

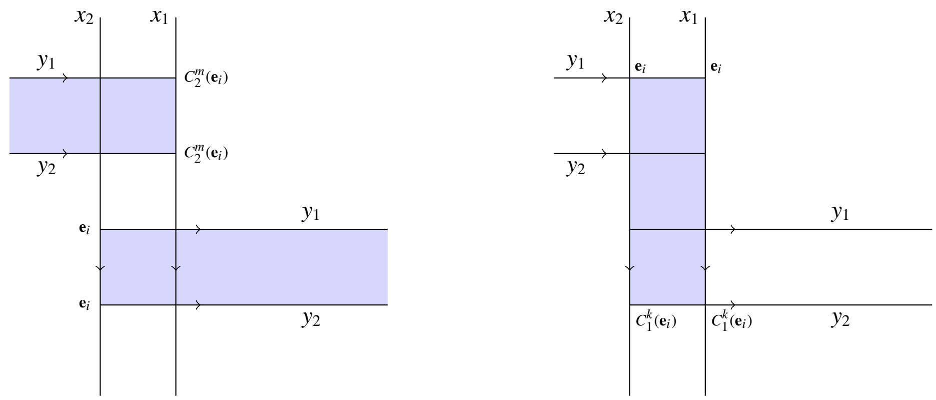

Figure 4: Horizontally (left) and vertically (right) extending rectangles

for $i = 1, \dots, n$ and $m = 1, \dots, n-1$.

Furthermore, as before let $CF(X_1, X_2) = \mathbb{C}s_0 \oplus \mathbb{C}s_1$ and $CF^0(Y_1, Y_2) = \mathbb{C}t_0 \oplus \mathbb{C}t_1$. We compute the products:

$$\begin{aligned}
\mathfrak{m}_2 : CF^1(Y_1, X_2) \otimes CF^0(X_1, Y_1) &\to CF^0(X_1, X_2) \\
\mathfrak{m}_2(\mathbf{e}_i, \mathbf{e}_i) &= e^{\frac{ku}{n}} s_1, \\
\mathfrak{m}_2 : CF^0(X_2, Y_2) \otimes CF^1(Y_1, X_2) &\to CF^0(Y_1, Y_2) \\
\mathfrak{m}_2(\mathbf{e}_i, \mathbf{e}_i) &= t_1,
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{m}_2(\mathbf{e}_i, s_0) &= \mathbf{e}_1, \\
\mathfrak{m}_2(t_0, C_2^m(\mathbf{e}_i)) &= e^{\frac{mv}{n}} \mathbf{e}_1
\end{aligned}$$

Thus, we conclude that

$$\mathrm{MP}(\mathbf{e}_i, C_1^k(\mathbf{e}_i), C_1^k(\mathbf{e}_i)) = e^{\frac{ku}{n}} \left(1 + \frac{e^u}{1 - e^u}\right) \mathbf{e}_i$$

for $i = 1, \dots, n$ and $k = 1, \dots, n-1$ and

$$\mathrm{MP}(\mathbf{e}_i, \mathbf{e}_i, C_2^m(\mathbf{e}_i)) = e^{\frac{mv}{n}} \left(1 + \frac{e^v}{1 - e^v}\right) C_2^m(\mathbf{e}_i)$$

for $i = 1, \dots, n$ and $m = 1, \dots, n-1$.

Dualising to the tensor $r_{y_1,y_2}^{x_1,x_2}$, we get the terms:

$$\begin{aligned}
&\sum_i \left(\frac{1}{\exp(u) - 1} + \frac{1}{1 - \exp(-v)}\right) e_{ii} \otimes e_{ii} \\
&+ \frac{1}{\exp(u) - 1} \sum_{0<k<n,i} \exp\left(\frac{ku}{n}\right) e_{C_1^k(i), C_1^k(i)} \otimes e_{ii} + \frac{1}{\exp(v) - 1} \sum_{0<m<n,i} \exp\left(\frac{mv}{n}\right) e_{i, C_2^m(i)} \otimes e_{C_2^m(i), i}
\end{aligned} \tag{2–5}$$

and we see that this agrees with the stated result in the case A is empty.

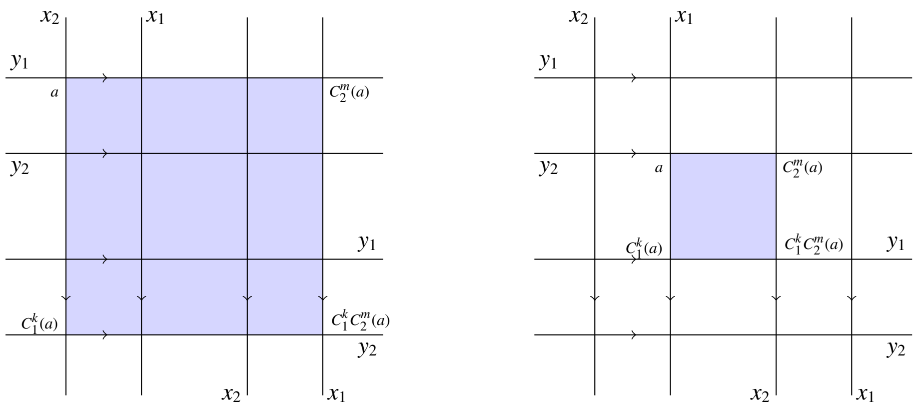

Figure 5: The two rectangles for each $a \in A(m,k)$

Finally, we will compute the contribution of rectangles when A is non-empty. When A is non-empty, the rectangular regions with corners $(a, C_1(a), C_1C_2(a), C_2(a))$ are filled for each $a \in A$. We get two new contributions to the $\mathfrak{m}_3$ product from such regions (See Figure 5). Furthermore, it may happen that union of those regions also give new rectangles. A combinatorial way to encode this is to let $A(k,m) \subset A$ to be the set of all $a \in A$ such that $C_1^iC_2^j(a) \in A$ for all $0 \leq i < k$, $0 \leq j < m$, then for each $a \in A(k,m)$ we have the following contributions because of the filled rectangular region with corners $(a, C_1^k(a), C_1^kC_2^m(a), C_2^m(a))$:

$$\mathfrak{m}_3(C_1^k(a), a, C_2^m(a)) = e^{\frac{ku+mv}{n}} C_1^kC_2^m(a)$$

corresponding to the rectangle drawn on the left of Figure 5, and

$$\mathfrak{m}_3(C_2^m(a), C_1^kC_2^m(a), C_1^k(a)) = -e^{-\frac{ku+mv}{n}} a$$

corresponding to the rectangle drawn on the right of Figure 5.

The signs that appear in the two formulae are affected by the orientations of the Lagrangians and we note that unlike the appearance, there is no typographical error in what we wrote. The sign in the exponentials are determined according to whether the orientation of the Lagrangians agree with the counter-clockwise boundary orientation of the rectangle, and the overall sign is determined according to the orientation of the moduli space which we computed as before using [23, Sec. 7].

Recall also that the dualization formula (0–3) brings in an extra overall sign. Thus, we conclude that in the case of arbitrary A we have in addition the contribution of the following terms to $r(u,v)$, indexed by elements of the subsets $A(k,m)$, $k,m > 0$:

(2–6) $$\sum_{\substack{0<k,0<m;\\ a\in A(k,m)}} \left\{ \exp(-\frac{ku+mv}{n}) e_{C_2^m(a),a} \otimes e_{C_1^k(a), C_1^kC_2^m(a)} - \exp(\frac{ku+mv}{n}) e_{C_1^k(a), C_1^kC_2^m(a)} \otimes e_{C_2^m(a),a} \right\}$$

□

Remark 2.3.4 We would like to mention an alternative to the above computation. It may appear more natural to take complex push-offs of L_1 and L_2 as follows. First, on $\mathbb{T}_0$, let l_1^x (resp. l_2^y) be the geodesic push-off of l_1 (resp. l_2) such that the oriented area of the cylinder bounded by l_1 and l_1^x (resp. l_2 and l_2^y) is Re(x) (resp. Re(y)). We then set $L_1^x = \pi^{-1}(l_1^x)$ (resp. $L_2^y = \pi^{-1}(l_2^y)$) equipped with a U(1)-local system with monodromy $e^{i\,\mathrm{Im}(x)}$ (resp. $e^{i\,\mathrm{Im}(y)}$). The simplest case is shown in Figure 6.

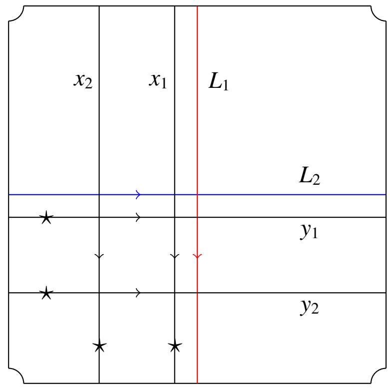

Figure 6: Non-exact push-offs of L_1 and L_2 (equipped with orientations and U(1)-local systems)

However, there is a significant drawback in this approach. Namely, the push-offs L_1^x and L_1^y are no longer exact Lagrangians when $\mathrm{Re}(x), \mathrm{Re}(y) \neq 0$. Hence, we cannot guarantee a priori that the count of holomorphic disks is finite (or convergent). Therefore, in this set-up one has to work over the Novikov field

$$\Lambda = \{\sum_{i\in\mathbb{Z}} a_i q^{t_i} | a_i \in \mathbb{C}, a_i = 0 \text{ for } i \ll 0, t_i \in \mathbb{R}, t_i \to \infty\}$$

and the formula (2–1) should be modified as

$$\mathfrak{m}_k(\rho_k, \dots, \rho_1) = \sum_{[u]:\mathrm{ind}([u])=2-k} \#\mathcal{M}(p_k, p_{k-1}, \dots, p_1, p_0; [u]) \cdot \mathrm{hol}_{\partial u} \cdot q^{\int_u \omega}. \tag{2–7}$$

With this in place, one can compute the corresponding Massey product simply by counting rectangles. In the simplest case, that is when $S = \{p\}$ is a single point and A is empty, computing the tensor $r^{x_1,x_2}_{y_1,y_2}$ boils down to counting rectangles with corners $(p_{12}, p_{22}, q_{12}, p_{11})$ in the counter-clockwise order weighted by their areas. Interestingly, there are indeed infinitely many homotopy classes of rectangles that contribute to this count. The smallest rectangle with corners in $(p_{12}, p_{22}, q_{12}, p_{11})$ has area Re(u)Re(v). Other than this, there are two families of rectangles - namely, those that are horizontally or vertically extending. Writing $x_i = a_i + i\alpha_i$ and $y_i = b_i + i\beta_i$, the horizontally extending ones are weighted by $e^{il(\alpha_2-\alpha_1)}q^{l(a_1-a_2)+(a_1-a_2)(b_1-b_2)}$ for $l = 1, 2, \dots$, and the vertically extending ones are weighted by $e^{il(\beta_2-\beta_1)}q^{l(b_1-b_2)+(a_1-a_2)(b_1-b_2)}$ for $l = 1, 2, \dots$.

The overall contribution of all these rectangles can be computed as:

$$r^{x_1,x_2}_{y_1,y_2}(p_{11} \otimes p_{22}) = -q^{(a_1-a_2)(b_1-b_2)} \left(1 + \sum_{l=1}^{\infty} e^{li(\alpha_2-\alpha_1)} q^{l(a_1-a_2)} + \sum_{l=1}^{\infty} e^{li(\beta_2-\beta_1)} q^{l(b_1-b_2)}\right) (p_{12} \otimes p_{21})$$

Since the points p_{ij}, $i, j = 1, 2$ can be canonically identified with the intersection point $S = L_1 \cap L_2 = \{p\}$, we can view this tensor as

$$r^{x_1,x_2}_{y_1,y_2} = -q^{(a_1-a_2)(b_1-b_2)}\left(1+\sum_{l=1}^{\infty} e^{li(\alpha_2-\alpha_1)}q^{l(a_1-a_2)}+\sum_{l=1}^{\infty} e^{li(\beta_2-\beta_1)}q^{l(b_1-b_2)}\right)(e_{11}\otimes e_{11})$$

Now, we observe that the series expansion has positive radius of convergence equal to 1, hence in particular specializing the Novikov parameter $q = e^{-1}$ leads to the formula:

$$r^{x_1,x_2}_{y_1,y_2} = -e^{-\mathrm{Re}(u)\mathrm{Re}(v)}\left(1+\frac{e^u}{1-e^u}+\frac{e^v}{1-e^v}\right)e_{11}\otimes e_{11} = e^{-\mathrm{Re}(u)\mathrm{Re}(v)}\left(\frac{1}{1-e^u}+\frac{1}{e^{-v}-1}\right)e_{11}\otimes e_{11}$$

This is remarkably in agreement with what we have computed before in formula (2–4) up to the overall constant $e^{-\mathrm{Re}(u)\mathrm{Re}(v)}$ which can be absorbed into the choice of basis. Similar computation can be done in all cases. This gives a hint that in an appropriately defined Fukaya category, the two different ways of pushing-off L_1 and L_2 should lead to quasi-isomorphic objects. (Compare with the discussion in [3, Section 4.1].)

Remark 2.3.5 Since the A_∞-relations hold in the Fukaya category by its general construction, Theorem *B* gives a new proof of the fact that $r(u,v)$ given by (2–2) satisfies the AYBE, which is proved in [20] by a rather tedious calculation. On the other hand, in [20] it was also proven that for $r(u,v)$ given by (2–2),

$$R(u,v) = \left(\frac{(e^{\frac{u}{2}}-e^{\frac{-u}{2}})\cdot(e^{\frac{v}{2}}-e^{\frac{-v}{2}})}{e^{\frac{u}{2}}-e^{\frac{-u}{2}}+e^{\frac{v}{2}}-e^{\frac{-v}{2}}}\right)r(u,v)$$

satisfies the quantum Yang-Baxter equation (for fixed u):

$$R^{12}(v)R^{13}(v+v')R^{23}(v') = R^{23}(v')R^{13}(v+v')R^{12}(v)$$

with the unitarity condition

$$R(u,v)R^{21}(u,-v) = 1\otimes 1.$$

We do not know a conceptual explanation for this. It would be interesting to study this in the setting of Fukaya categories.

We will need the following result in the proof of Theorem C.

Proposition 2.3.6 *Let (S, C_1, C_2, A) be an associative Belavin-Drinfeld structure, such that C_1 and C_2 commute, and let (Σ, L_1, L_2) be the corresponding square-tiled surface with a pair of Lagrangians (where Σ is a punctured torus). Then (L_1, L_2) split generates the Fukaya category $\mathcal{F}(\Sigma)$ of exact, compact (graded) Lagrangians in Σ.*

Proof. We first prove that (L_1, L_2) split generate when $A = \emptyset$ and $\Sigma = \Sigma_0$. Without loss of generality, suppose that $S = \{1,\dots,n\}$, $C_1(i) = i+1$ and that $C_2 = C_1^k$ for some k which is prime to n. We can draw the corresponding square-tiled surface as in Figure 7 (where the case of $n = 5$ and $k = 2$ is drawn). Let $M_1, M_2, \dots, M_n$ be n disjoint Lagrangians corresponding to curves of slope $1/k$, drawn in green in Figure 7.

Note that these Lagrangians have a natural grading structure (since our line field is given by the horizontal foliation). It was proven in [17, Lem. 3.1.1] that the collection $L_1, M_1, M_2, \dots, M_n$ split generates the

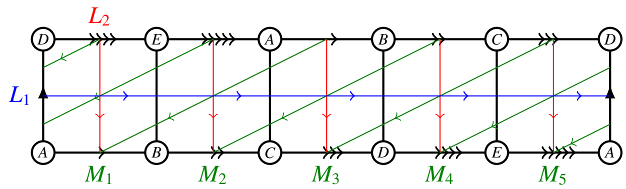

Figure 7: Generators of the exact Fukaya category, $(n,k)=(5,2)$

exact Fukaya category. (This essentially follows from the fact that Dehn twists around $L_1, M_1, M_2, \ldots, M_n$ generate the pure mapping class group of the n-punctured torus.)

Note that we have the following intersections in homology:

$$[M_i]\cdot[L_1]=1$$
$$[M_i]\cdot[L_2]=k$$
$$[L_2]\cdot[L_1]=n$$

In fact, by considering dual curves to M_i, it is easy to see that

$$[L_2]=k[L_1]+[M_1]+[M_2]+\cdots+[M_n]\in H_1(\Sigma_0)$$

We claim that there is an exact triangle of the form:

(2–8)
$$\begin{array}{ccc} M_1\oplus M_2\oplus\ldots\oplus M_n & \longrightarrow & L_1^{\oplus k} \\ & {}_{[1]}\nwarrow & \downarrow \\ & & L_2 \end{array}$$

where the maps $M_i \to L_1^{\oplus k}$ are given by $(c_i, c_i, \ldots, c_i)$, with $c_i \in CF^1(M_i, L_1)$, for each i, being the generator corresponding to the unique intersection point. It is then clear the L_1 and L_2 split generate $\mathcal{F}(\Sigma)$.

The exact triangle is an example of a surgery exact triangle proven in this case by Abouzaid in [1, Lemma 5.4]. It is technically easier to show that the following equivalent statement holds:

$$Cone(\ldots Cone((Cone(M_1\oplus M_2\oplus\ldots\oplus M_n)\to L_1)\to L_1)\ldots\to L_1)\simeq L_2.$$

Indeed, we first do a surgery at each intersection point of L_1 and each M_i and then we perform a new surgery at the n intersection points of the obtained Lagrangian with a new copy of L_1. We do this k times (including the first surgery between L_1 and M_i's) until we arrive at an exact Lagrangian Hamiltonian isotopic to L_2. Note that in each isotopy class of homotopically essential (i.e. not null-homotopic) simple closed curves, there is a unique exact Lagrangian up to Hamiltonian isotopy, so it suffices to check that the end result of all the surgeries, which is an exact Lagrangian, is smoothly isotopic to L_2.

The corresponding picture is drawn in Figures 8 and 9 below for $n=5, k=2$ case, from which it is clear how the general case works.

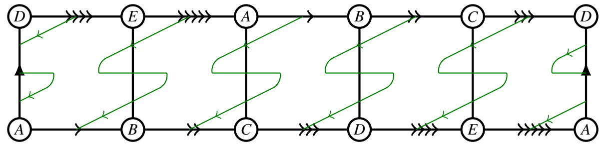

Figure 8: $\tau_{M_1} \circ \tau_{M_2} \circ \tau_{M_3} \circ \tau_{M_4} \circ \tau_{M_5}(L_1)$

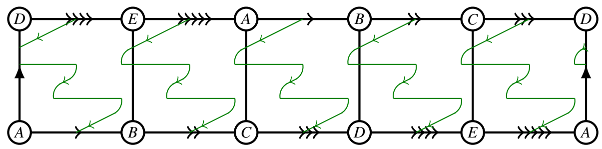

Figure 9: $Cone(Cone((M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_5) \to L_1) \to L_1) \simeq L_2$

Note that by Seidel's exact triangle [26], the first iteration can be identified as

$$Cone((M_1 \oplus M_2 \oplus \ldots \oplus M_n) \to L_1) \simeq \tau_{M_1} \circ \tau_{M_2} \circ \ldots \circ \tau_{M_n}(L_1).$$

When $A \neq \emptyset$, if the puncture between M_i and M_{i+1} is closed, then they become isotopic hence give equivalent objects. (Of course, one has to isotope them with a finger move so as to make both of them exact Lagrangians). The same argument as above, with the understanding that some of the M_i represent equivalent objects, shows the exact triangle (2–8) remains valid. Hence, L_1 and L_2 again split generates $\mathcal{F}(\Sigma)$. □

3 Application to vector bundles over cycles over projective lines

3.1 Simple vector bundles on cycles of projective lines

In this subsection we work over an algebraically closed field k of characteristic $\neq 2$. Let $C = \cup_{j=0}^{n-1} C_j$ be a cycle of n projective lines (also known as the standard n-gon). We identify each C_j with the standard copy $\mathbb{P}^1$ in such a way that the point $\infty \in C_j$ is glued to the point $0 \in C_{j+1}$ (we identify indices with $\mathbb{Z}/n$).

Recall that, up to isomorphism, all simple vector bundles on C are obtained by the following construction (see [5]), which has as an input an integer valued matrix $\mathbf{m} = (m_i^j)_{i=1,\ldots,r; j=0,\ldots,n-1}$ and a nonzero constant $\lambda \in k^*$. The corresponding vector bundle $V = V^\lambda(\mathbf{m})$ is defined by setting

$$V|_{C_j} = V_j = \mathcal{O}_{\mathbb{P}^1}(m_1^j) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(m_r^j)$$

and by making the following identifications $V_j|_\infty \simeq V_{j+1}|_0$: for all j except for $j = n-1$ we use the standard trivializations of the corresponding bundle $\mathcal{O}(m)$ at 0 and at ∞ (given by x_0^m and x_1^m), while for $j = n-1$, we use

$$\lambda \cdot C : V_{n-1}|_\infty \to V_0|_0,$$

where C is the transitive permutation matrix $e_i \mapsto e_{i-1}$ (where the indices are in $\mathbb{Z}/r\mathbb{Z}$. The obtained vector bundle of rank r is simple if and only if a certain condition on $\mathbf{m}$ is satisfied. Namely, let us unroll the matrix $\mathbf{m}$ into an rn-periodic sequence by setting

$$d_{qn+j} = m^j_{-q}, \ \ j, q \in \mathbb{Z}, 0 \le j < n.$$

The conditions are: (1) for every i, i', j one has $|m^j_i - m^j_{i'}| \le 1$; (2) for every q, not divisible by r, the rn-periodic sequence $(d_{qn+j} - d_j)$ is not identically 0 and the occurrences of 1 and -1 in it alternate.

Recall that one of the results of [20] is an explicit computation of the trigonometric solution of the AYBE associated with a pair $(V, \mathcal{O}_p)$, where $V = V^\lambda(\mathbf{m})$ is a simple bundle on C and p is a smooth point. The answer is given by the trigonometric solution corresponding to a certain associative Belavin-Drinfeld structure $\mathrm{ABD}(V, p)$, which we will describe now. Without loss of generality we can assume that $p \in C_0 = C_n$. Let us define the complete order $\prec$ on the set of indices $\mathbb{Z}/r\mathbb{Z} = \{0, 1, \dots, r-1\}$ as follows: $i \prec i'$ if the sequence $(d_{j-in} - d_{j-i'n})_{j=0,1,\dots}$ is nonzero and the first nonzero term in it is negative (the fact that it is a complete order follows from the condition (2) above). We define the transitive permutations C_1 and C_2 on $\mathbb{Z}/r\mathbb{Z}$ by letting C_1 send each non-maximal element with respect to the above complete order to the next element, and by $C_2(i) = i - 1$. Finally, we define a subset $A \subset \mathbb{Z}/r\mathbb{Z}$ to be the set of i such that $i - 1 \prec C_1(i) - 1$ and $m^j_i = m^j_{i'}$ for $0 < j < n$. By [20, Thm. 5.3], in fact C_2 is a power of C_1, and the solution of the AYBE associated with a natural family of deformations of V and p is the solution (2–2) associated with

$$\mathrm{ABD}(V, p) := (\mathbb{Z}/r, C_1, C_2, A).$$

Now the arguments of Section 1.5 imply that the formal solution of the general AYBE associated with the pair (V, p) is equivalent to (2–2), viewed as a formal solution. By Theorem A, this implies that the A_∞-subcategory, split generated by V and $\mathcal{O}_p$, depends only on $\mathrm{ABD}(V, p)$. Here to apply Theorem A (with $R = k$) we need to equip the A_∞-algebra of endomorphisms of $V \oplus \mathcal{O}_p$ with a cyclic structure with respect to a natural pairing coming from the Serre duality. The existence of such a cyclic structure can be proved similarly to [21, Sec. 4.8] (using the assumption that characteristic is not equal to 2). Namely, first, using a 1-spherical twist we can replace $V \oplus \mathcal{O}_p$ with a vector bundle, and then, use Proposition 4.8.2 and Lemma 4.8.4 of [21]. In the characteristic zero case one can instead use the criterion of Kontsevich-Soibelman [15, Thm. 10.2.2] (see [21, Rem. 4.8.3]).

Definition 3.1.1 We say that a vector bundle W on $\mathbb{P}^1$ is of *positive (resp., nonnegative) type* if $W \simeq \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ with all $a_i > 0$ (resp., $a_i \ge 0$). Now let V be a vector bundle on C. We say that V is of *positive (resp., nonnegative) type* if each restriction $V|_{C_i}$ is of positive (resp., nonnegative) type. In the case $n = 1$ we require this property for f^*V, where $f : \mathbb{P}^1 \to C$ is the normalization map.

Recall that a collection of objects (O_i) *split generates* a triangulated category $\mathcal{T}$ if the minimal triangulated subcategory $\mathcal{T}' \subset \mathcal{T}$, closed under direct summands and containing all O_i, is the entire $\mathcal{T}$.

Proposition 3.1.2 *Let V be a simple vector bundle on C of positive type. Then the pair $(\mathcal{O}_C, V)$ split generates* $\mathrm{Perf}(C)$.

Proof. Let us pick smooth points $p_1, \dots, p_n$, one on each component of C. Note that $V(-p_1 - \dots - p_n)$ is of nonnegative type.

Assume first that the rank of V is > 1. Hence, by Lemma 3.1.3(i) below, there exists an injection of $\mathcal{O}_C$ into $V(-p_1 - \dots - p_n)$. Let us consider the composed injective morphism

$$f : \mathcal{O}_C \to \mathcal{O}_C(p_1 + \dots + p_n) \to V,$$

where the first arrow is given by the canonical section of $\mathcal{O}_C(p_1 + \dots + p_n)$ vanishing at the divisor $p_1 + \dots + p_n$. Then the coherent sheaf coker(f) has nonzero torsion at each of the points $p_1, \dots, p_n$. Thus, we have an exact sequence

$$0 \to \bigoplus_{i=1}^{n} \mathcal{T}_i \to \mathrm{coker}(f) \to \mathcal{F} \to 0$$

where $\mathcal{T}_i$ is a nonzero sheaf supported at p_i and $\mathcal{F}$ is locally free near each $p_1, \dots, p_n$. Such a sequence necessarily splits, so each $\mathcal{T}_i$ is a direct summand of coker(f). This shows that the subcategory, split generated by $\mathcal{O}_C$ and V contains $\mathcal{T}_1, \dots, \mathcal{T}_n$. Furthermore, each $\mathcal{T}_i$ has a direct summand of the form $\mathcal{O}_{m_i p_i}$ with some $m_i \geq 1$. It remains to note that the objects $(\mathcal{O}_C, \mathcal{O}_{m_1 p_1}, \dots, \mathcal{O}_{m_n p_n})$ split generate Perf(C). Indeed, this can be checked similarly to [17, Lem. 3.3.1]: starting from $\mathcal{O}_C$ and using the exact sequences of the form

$$0 \to L(-m_i p_i) \to L \to \mathcal{O}_{m_i p_i} \to 0,$$

we derive that all the negative powers of the ample line bundle $\mathcal{O}_C(\sum m_i p_i)$ belong to the subcategory split generated by our objects. The fact that all negative powers of an ample line bundle generate Perf(C) is proved in [18, Thm. 4].

In the case when V is a line bundle, of positive degree on each component, by Lemma 3.1.3(i), we can find a global section $s : \mathcal{O}_C \to V$ which does not vanish at the nodes. Its restriction to every component of C vanishes at some smooth point p_i. Then coker(s) will again have a nonzero torsion part at each p_i, and the above proof goes through. □

Lemma 3.1.3 *(i) Let W be a simple vector bundle on C of nonnegative type. Assume in addition that either W has rank > 1, or has positive degree. Then there exists an injective morphism $\mathcal{O}_C \to W$, which is an embedding as a subbundle near the nodes.*

(ii) Let V be a simple vector bundle of positive type, $p \in C$ a smooth point. Let us denote by $E(V,p)$ the universal extension

$$0 \to \mathrm{Ext}^1(V, \mathcal{O}_C(p))^* \otimes \mathcal{O}_C(p) \to E(V,p) \to V \to 0.$$

Then $E(V,p)$ is the result of applying to V the inverse twist with respect to $\mathcal{O}_C(p)$. In particular, $E(V,p)$ is still a simple vector bundle.

Proof. (i) We use the fact that W has the form $W = V^{\lambda}(\mathbf{m})$, where all $m_i^j \geq 0$. Note that the condition that W is simple and has rank > 1 implies that $m_i^j > 0$ for at least one pair (i,j). To define a global section of W we need to choose a global section $s_i^j \in H^0(C_j, \mathcal{O}(m_i^j)$ for each (i,j) in such a way that they are compatible with the gluing over $C_j \cap C_{j+1}$. We claim that we can make these choices in such a way that each s_i^j is nonzero at 0 and ∞. Indeed, in the case when $m_i^j > 0$ we can arrange s_i^j to have arbitrary values at 0 and ∞, while in the case $m_i^j = 0$, one of these values determine the other. Now looking at the way the gluing is defined for $V^\lambda(\mathbf{m})$ we see that the existence of at least one positive m_i^j guarantees the existence of a global section which is nonzero at all the nodes.

(ii) This follows from the vanishing $\mathrm{Hom}(V, \mathcal{O}_C(p))^* \simeq H^1(V(-p)) = 0$ that holds since $V(-p)$ is of nonnegative type. Note that by Serre duality, any line bundle on C is 1-spherical. □

Now we are going to consider the associative Belavin-Drinfeld structure $\mathrm{ABD}(E(V,p),p)$ associated to $E(V,p)$ and p.

Theorem 3.1.4 *Let V and V' be simple vector bundles on C of positive type. Assume that for some smooth points $p, p' \in C$ one has an isomorphism*

$$\mathrm{ABD}(E(V,p),p) \simeq \mathrm{ABD}(E(V',p'),p')$$

of associative Belavin-Drinfeld structures. Then there exists a Fourier-Mukai autoequivalence Φ of $\mathrm{Perf}(C)$ given by a kernel in $D^b(C \times C)$, such that $\Phi(\mathcal{O}_C) \simeq \mathcal{O}_C$ and $\Phi(V) \simeq V'$.

Proof. By Lemma 3.1.3(ii), the inverse twist with respect to $\mathcal{O}_C(p)$ sends the pair $(\mathcal{O}_C, V)$ to the pair $(O_p, E(V,p)[1])$. Similarly, the twist with respect to $\mathcal{O}_C(p')$ sends $(\mathcal{O}_C, V')$ to $(\mathcal{O}_{p'}, E(V,p)[1])$. By Theorem A, the isomorphism of the corresponding associative Belavin-Drinfeld structures implies that the subcategories, split generated by $(\mathcal{O}_C, V)$ and $(\mathcal{O}_C, V')$ are related by an equivalence Φ in such a way that $\Phi(\mathcal{O}_C) \simeq \mathcal{O}_C$ and $\Phi(V) \simeq V'$. By Proposition 3.1.2, Φ is actually an autoequivalence of $\mathrm{Perf}(C)$, or more precisely, of its A_∞-enhancement. Such an autoequivalence is always given by a kernel on $C \times C$ which could be a complex of quasicoherent sheaves (see [28]). The fact that it belongs to the bounded derived category of coherent sheaves follows from [17, Lem. 3.5.1]. □

3.2 Proof of Theorem C

It is enough to consider the case when V is of positive type. Indeed, starting from an arbitrary bundle we can apply twists at smooth points to replace V with $V(N(p_1 + \ldots + p_n))$, which is of positive type for large N.

By Lemma 3.1.3(ii), the inverse twist with respect to $\mathcal{O}_C(p)$ transforms the pair $(\mathcal{O}_C, V)$ to $(\mathcal{O}_p, E(V,p)[1])$. As was shown in [20], the solution of AYBE, associated with the pair $(E(V,p), \mathcal{O}_p)$, is a trigonometric solution (2–2), corresponding to an associative Belavin-Drinfeld structure (S, C_1, C_2, A) in which $C_2 = C_1^k$ for some k. Hence, by Theorem A, Theorem B and Proposition 2.3.6, the subcategory in $D^b(C)$ split generated by the pair $(\mathcal{O}_C, V)$ is equivalent to the Fukaya category of some square-tiled surface of genus 1, in such a way that $\mathcal{O}_C$ and V correspond to the Lagrangians L_1 and L_2. Note that in establishing this equivalence we apply Theorem A, so we pass to formal solutions of the AYBE, as explained in Section 1.5.

Now we recall that the Dehn twists with respect to the graded Lagrangians $L_1, M_1, \ldots, M_n$ generate the pure mapping class group (see the proof of Proposition 2.3.6). Hence, there exists a composition of these Dehn twists and their inverses that takes L_1 into L_2. Under the above equivalence, this corresponds to a composition Φ of 1-spherical twists and their inverses that takes $\mathcal{O}_C$ into V. □

References

[1] Abouzaid, M., *On the Fukaya category of higher genus surfaces*, Adv. Math., 2008, 217(3), 1192–1235

[2] Aguiar, M., *Infinitesimal Hopf algebras*, New trends in Hopf algebra theory (La Falda, 1999), 1–29. Contemp. Math. 267, AMS, 2000.

[3] Auroux, D., *Special Lagrangian fibrations, wall-crossing, and mirror symmetry*, Surveys in differential geometry, Vol. XIII (Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry), 1–47, Int. Press, Somerville, MA, 2009.

[4] Belavin, A. A., Drinfeld, V. G., *S*olutions of the classical Yang-Baxter equation for simple Lie algebras, Funct. Anal. and its Appl. 16 (1982), 1–29.

[5] Burban, I., Drozd, Yu., Greuel, G.-M., *V*ector bundles on singular projective curves, Applications of Algebraic Geometry to Coding Theory, Physics and Computation (Eilat, Israel, 2001), Kluwer, 2001.

[6] Burban, I., Kreussler, B., *Derived categories of irreducible projective curves of arithmetic genus one* Compos. Math. 142 (2006), no. 5, 1231–1262.

[7] Chari, V., Pressley, A., *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994.

[8] Donaldson, S., *Riemann Surfaces*, Oxford Graduate Texts in Mathematics, 22. Oxford University Press, Oxford, 2011.

[9] Drinfeld, V. G., *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[10] Fukaya, K., *Deformation theory, homological algebra and mirror symmetry* Geometry and physics of branes (Como, 2001), 121–209, Ser. High Energy Phys. Cosmol. Gravit., IOP, Bristol, 2003.

[11] Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K., Lagrangian intersection Floer theory: anomaly and obstruction. Part I. AMS/IP Studies in Advanced Mathematics, 46.1. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[12] Fukaya, K., Cyclic symmetry and adic convergence in Lagrangian Floer theory. Kyoto J. Math. 50 (2010), no. 3, 521–590.

[13] Husemoller, D. H., *Ramified Coverings of Riemann Surfaces*, Duke Math. J. 29 1962 167–174.

[14] Keller, B., *Introduction to A_∞-algebras and modules*, Homology Homotopy Appl. 3 (2001), no. 1, 1–35.

[15] Kontsevich, M., Soibelman, Y., *Notes on A_∞-algebras, A_∞-categories and non-commutative geometry. I*, in *Homological mirror symmetry*, 153–219, Springer, Berlin, 2009.

[16] Lefèvre-Hasegawa, K., *Sur les A_∞-catégories*, Ph.D. thesis, Université Paris 7, 2002.

[17] Lekili, Y., Polishchuk, A., *Arithmetic mirror symmetry for genus 1 curves with n marked points*, Selecta Math. 23 (2017), 1851–1907.

[18] Orlov, D., *Remarks on generators and dimensions of triangulated categories*, Moscow Math. J. 9 (2009), no. 1, 153–159.

[19] Polishchuk, A., *Classical Yang-Baxter equation and the A_∞-constraint*, Adv. Math. 168 (2002), no. 1, 56–95.

[20] Polishchuk, A., *Massey products on cycles of projective lines and trigonometric solutions of the Yang-Baxter equations*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 573–617, Progr. Math., 270, Birkhäuser Boston, Inc., Boston, MA, 2009.

[21] Polishchuk, A., *A_∞-structures associated with pairs of 1-spherical objects and noncommutative orders over curves*, arXiv:1805.11727.

[22] Schedler, T., *Trigonometric solutions of the associative Yang-Baxter equation*, Mat. Res. Lett. 10 (2003), 301–321.

[23] Seidel, P., *Homological mirror symmetry for the genus two curve*, J. Algebraic Geom. 20 (2011), no. 4, 727–769.

[24] Seidel, P., *Fukaya categories and Picard-Lefschetz theory*, European Math. Soc. 2008.

[25] Seidel, P., *Abstract analogues of flux as symplectic invariants*, Mém. Soc. Math. Fr. (N.S.) No. 137 (2014).

[26] Seidel, P., *A long exact sequence for symplectic Floer cohomology*, Topology 42 (2003), no. 5, 1003–1063.

[27] Seidel, P., Thomas, R., *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. 108 (2001), 37–108.

[28] Toën, B., *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. 167 (2007), 615–667.

[29] Zmiaikou, D., *Origamis and permutation groups*, PhD Thesis, Université Paris-Sud 11, 2011.

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