

HOMOLOGICAL MIRROR SYMMETRY FOR THE SYMMETRIC SQUARES OF PUNCTURED SPHERES

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ABSTRACT. For an appropriate choice of a \mathbb{Z} -grading structure, we prove that the wrapped Fukaya category of the symmetric square of a $(k+3)$ -punctured sphere, i.e. the Weinstein manifold given as the complement of $(k+3)$ generic lines in $\mathbb{C}P^2$ is quasi-equivalent to the derived category of coherent sheaves on a singular surface $\mathcal{Z}_{2,k}$ constructed as the boundary of a toric Landau-Ginzburg model $(\mathcal{X}_{2,k}, \mathbf{w}_{2,k})$. We do this by first constructing a quasi-equivalence between certain categorical resolutions of both sides and then localising. We also provide a general homological mirror symmetry conjecture concerning all the higher symmetric powers of punctured spheres. The corresponding toric LG-models $(\mathcal{X}_{n,k}, \mathbf{w}_{n,k})$ are constructed from the combinatorics of curves on the punctured sphere and are related to small toric resolutions of the singularity $x_1 \dots x_{n+1} = v_1 \dots v_k$.

1. INTRODUCTION

1.1. **Motivating examples.** We begin by considering the family

$$\mathcal{Y}_{1,1} := \text{Spec } \mathbb{C}[x, y, V]/(xy - V)$$

over \mathbb{A}_V^1 as a “toric degeneration” of the general fiber to the singular variety $\mathcal{Z}_{1,1} := \{xy = 0\}$ which is a gluing of two toric varieties \mathbb{A}_x^1 and \mathbb{A}_y^1 at their toric boundary 0. The symplectic mirror to this degeneration is well understood [3]. Namely, we consider the cotangent bundle of the circle $T^*\mathbb{S}^1$ as our symplectic manifold and $D_{1,1} := \{z\}$ a marked point on $T^*\mathbb{S}^1$ as a symplectic divisor. Then, one has an equivalence of pre-triangulated categories over $\mathbb{C}[V]$

$$\mathcal{W}(T^*\mathbb{S}^1, D_{1,1}) \simeq D^b\text{Coh}(\mathcal{Y}_{1,1}) \tag{1.1}$$

where the left hand side is the wrapped Fukaya category of $T^*\mathbb{S}^1$ relative to the divisor $D_{1,1}$ (cf. [21] [11] [25]) and the right hand side is the bounded derived category of $\mathcal{Y}_{1,1}$ over \mathbb{A}_V^1 .

The equivalence (1.1) fits into the general framework of Gross-Siebert, Gross-Hacking-Keel [6], Abouzaid-Auroux-Katzarkov [1], where the general fiber is a log-Calabi-Yau variety and the degeneration given is a “large complex structure limit” degeneration of such a log-Calabi-Yau variety to a union of toric varieties. Even though in the case of this specific example both sides can be computed rather easily, in general this is a harder problem. A

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viable strategy to establish this equivalence is to first look at the special fiber and prove the following equivalence

$$\mathcal{W}(V_{1,1}) \simeq D^b\text{Coh}(\mathcal{Z}_{1,1}) \quad (1.2)$$

for $V_{1,1} := T^*\mathbb{S}^1 \setminus D_{1,1}$. Then one can try to deduce the equivalence (1.1) via deformation theory (certain subcategories of those in equivalence (1.2) deform to give rise to the equivalence (1.1)).

Note that $V_{1,1}$ is the pair-of-pants surface. To establish equivalences of the form (1.2), we introduced a method via categorical partial compactifications in [12], [14] (a similar idea was also applied in the log Calabi-Yau setting in [8], [9]). On the A-side we introduce a stop Λ° , that is a symplectic hypersurface on the contact boundary of $T^*\mathbb{S}^1 \setminus D_{1,1}$ and consider the partially wrapped Fukaya category, and on the B-side we partially compactify $\mathcal{Z}_{1,1}$ to $\overline{\mathcal{Z}}_{1,1}$ which is obtained by compactifying \mathbb{A}_y^1 to \mathbb{P}_y^1 by adding a point $y = \infty$. Thus, $\overline{\mathcal{Z}}_{1,1}$ is a union of \mathbb{A}_x^1 and \mathbb{P}_y^1 which intersect transversely at a point. We establish directly an easier equivalence

$$\mathcal{W}(V_{1,1}, \Lambda^\circ) \simeq D^b\text{Coh}(\overline{\mathcal{Z}}_{1,1}) \quad (1.3)$$

and then show that the equivalence of (1.2) is obtained from (1.3) by localization.

The Liouville domain $V_{1,1}$ with the stop Λ° is given by the following picture.

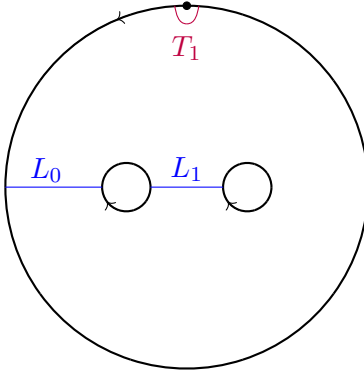


FIGURE 1. Pair-of-pants

The equivalence (1.3) is proved by matching the generators L_0 and L_1 with $\mathcal{O}_{\mathbb{P}_y^1}$ and $\mathcal{O}_{\mathbb{A}_x^1}$, respectively. To deduce the equivalence (1.2), we localize at the subcategory generated by T_1 on the A-side and by the skyscraper sheaf $\mathcal{O}_{y=\infty}$ on the B-side. The reason that the equivalence (1.3) is easier to establish is because the endomorphism algebra of our generators turns out to be formal which is not true before compactification.

The example that we considered above and the proof of the equivalences (1.1), (1.2), (1.3) can be generalized in a straightforward way to the case of $(T^*\mathbb{S}^1, D_{1,k})$ where $D_{1,k}$ is now k points (see [12] for a proof of (1.2), (1.3)) and the family $\mathcal{Y}_{1,k}$ is a k -dimensional family over $\mathbb{A}^k = \text{Spec } \mathbb{C}[V_1, V_2, \dots, V_k]$. The general fiber is again isomorphic to \mathbb{C}^\times and the special fiber is a nodal chain given by the following figure:

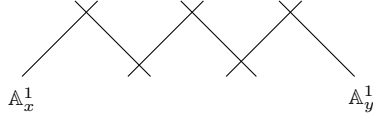


FIGURE 2. Mirror to 7-punctured sphere, $V_{1,5}$

The two components at the ends are \mathbb{A}_x^1 and \mathbb{A}_y^1 , and the “interior components” are isomorphic to \mathbb{P}^1 .

Another way to represent this singular curve is via the image under the moment map induced by the toric action. This results in the following diagram where the rays on the left and right represent \mathbb{A}_x^1 and \mathbb{A}_y^1 and the interior intervals represent 4 components isomorphic to \mathbb{P}^1 .

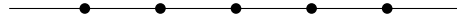


FIGURE 3. Moment map picture of $\mathcal{Z}_{1,5}$ - the mirror to $V_{1,5}$

The A -side in the above example has a natural generalization to higher dimensions. Namely, for $n, k \geq 0$, let $\{x_1, x_2, \dots, x_{n+1}\} \cup \{v_1, \dots, v_k\}$ be $(n + k + 1)$ points in $\mathbb{P}_{\mathbb{C}}^1$ that are mutually distinct. We consider the exact symplectic manifold

$$V_{n,k} = \text{Sym}^n(\mathbb{P}_{\mathbb{C}}^1 \setminus (\{x_1, x_2, \dots, x_{n+1}\} \cup \{v_1, v_2, \dots, v_k\}))$$

with the standard Liouville structure which can alternatively be described as the complement of $(n + k + 1)$ generic hyperplanes in $\mathbb{P}_{\mathbb{C}}^n$. Note that $V_{n,0}$ can be symplectically identified with the cotangent bundle of the n -dimensional torus,

$$\text{Sym}^n(\mathbb{P}_{\mathbb{C}}^1 \setminus (\{x_1, x_2, \dots, x_{n+1}\})) = T^*\mathbb{T}^n,$$

and we have $V_{n,k} = T^*\mathbb{T}^n \setminus D_{n,k}$ with

$$D_{n,k} = \bigcup_{j=1}^k \{v_j\} \times \text{Sym}^{n-1}(\mathbb{P}_{\mathbb{C}}^1 \setminus (\{x_1, x_2, \dots, x_{n+1}\}))$$

In our previous work [14], we studied (partially) wrapped Fukaya categories associated with $V_{n,k}$. In the case $k = 1$, i.e., the case of n -dimensional pairs-of-pants, we proved a homological mirror symmetry statement that provides an equivalence of \mathbb{Z} -graded pre-triangulated categories over a regular commutative ring \mathbf{k} :

$$\mathcal{W}(V_{n,1}) \simeq \text{D}^b \text{Coh}(\mathcal{Z}_{n,1}) \simeq \text{MF}_{\mathbb{G}_m}(\mathcal{Y}_{n,1} \times \mathbb{A}^1, \mathbf{w}_{n,1}) \quad (1.4)$$

where $\mathcal{W}(V_{n,1})$ is the fully wrapped Fukaya category of the n -dimensional pair-of-pants $V_{n,1}$, the middle category is the derived category of coherent sheaves on the singular affine variety

$$\mathcal{Z}_{n,1} = \text{Spec } \mathbf{k}[x_1, \dots, x_{n+1}] / (x_1 x_2 \dots x_{n+1})$$

and the third category is the \mathbb{G}_m -equivariant derived category of matrix factorizations on $\mathcal{Y}_{n,1} \times \mathbb{A}_{U_1}^1 \simeq \mathbb{A}^{n+2}$, where

$$\mathcal{Y}_{n,1} := \text{Spec } \mathbb{C}[x_1, x_2, \dots, x_{n+1}, V_1] / (x_1 x_2 \dots x_{n+1} - V_1) \simeq \mathbb{A}^{n+1}$$

for the potential $\mathbf{w}_{n,1} = U_1 V_1$.

1.2. Formulation of the main result. In this paper, we give an explicit candidate for the B -side for any n and k . To that end, inspired by the work of Abouzaid-Auroux-Katzarkov [1], we introduce a smooth toric variety $\mathcal{Y}_{n,k}$ over $\mathbb{A}^k = \text{Spec } \mathbf{k}[V_1, V_2, \dots, V_k]$ with the general fiber isomorphic to the torus \mathbb{G}_m^n and the special fiber $\mathcal{Z}_{n,k}$ over $V_1 = V_2 = \dots = V_k = 0$ given by a union of toric varieties glued along toric strata (see Sec. 1.3 for the construction). In fact, $\mathcal{Y}_{n,k}$ is a small toric resolution of the singularity $X_1 \dots X_{n+1} = V_1 \dots V_k$, in particular, it is Calabi-Yau (see Sec. 2.4). We also consider a Landau-Ginzburg version by considering the variety

$$\mathcal{X}_{n,k} := \mathcal{Y}_{n,k} \times \text{Spec } \mathbf{k}[U_1, U_2, \dots, U_k]$$

with the potential

$$\mathbf{w}_{n,k} = U_1 V_1 + U_2 V_2 + \dots + U_k V_k$$

There is a universal grading of the Fukaya category of $V_{n,k}$ with values in an abelian group $\mathbb{L} \simeq \mathbb{Z}^{n+k+1}$ which is compatible with a certain natural $\mathbb{Z}/2$ -grading structure. In order to get a relation with the B -side we identify \mathbb{L} with a natural grading group \mathbb{L}_B on the B -side, which is a sublattice in the character group of an algebraic torus acting on $\mathcal{X}_{n,k}$ in a way that rescales $\mathbf{w}_{n,k}$ according to the given character $l_0 \in \mathbb{L}_B$.

The possible \mathbb{Z} -grading structures on $V_{n,k}$ is a torsor over $H^1(V_{n,k}) = \mathbb{Z}^{n+k}$. Among these there is a natural \mathbb{Z} -grading structure η_0 on $V_{n,k}$ which roughly speaking is determined by the property that the primitive chords around the x_i have degree 0, and the primitive chords around v_i have degree 2. This corresponds to the unique subgroup \mathbb{G}_m in the above torus, such that all x_i and V_j have weight 0 (and U_j have weight 1) with respect to this subgroup.

We formulate a conjectural homological mirror symmetry statement that generalizes the above result to arbitrary n, k .

Conjecture 1.2.1. There exists a quasi-equivalence of \mathbb{Z} -graded pre-triangulated categories

$$\mathcal{W}(V_{n,k}, \eta_0) \simeq D^b \text{Coh}(\mathcal{Z}_{n,k}), \quad (1.5)$$

and a quasi-equivalence of \mathbb{L} -graded pre-triangulated categories

$$\mathcal{W}(V_{n,k}) \simeq \mathcal{B}_{n,k}, \quad (1.6)$$

where $\mathcal{B}_{n,k}$ is the category of \mathbb{L}_B -graded equivariant matrix factorizations of $\mathbf{w}_{n,k}$ on $\mathcal{X}_{n,k}$. Moreover, by considering deformations over \mathbb{A}^k , there exists a quasi-equivalence

$$\mathcal{W}(T^* \mathbb{T}^n, D_{n,k}, \eta_0) \simeq D^b \text{Coh}(\mathcal{Y}_{n,k}) \quad (1.7)$$

We give a precise definition of the \mathbb{L} -graded dg-category of equivariant matrix factorizations $\mathcal{B}_{n,k}$ in Sec. 2.3.3. Note that (1.5) follows from (1.6) by choosing a specific \mathbb{Z} -grading and applying localization theorems from [2] on the A-side and from [7] on the B-side. Our main result is the proof of (1.5) and (1.6) when $n = 2$. Along the way, we also give a new proof for $n = 1$.

Theorem 1.2.2. *Equivalence (1.5) holds for $n = 1$ and $n = 2$.*

The equivalence (1.6) of \mathbb{L} -graded categories in the cases $n \leq 2$ should also be within reach. We prove its \mathbb{Z} -graded version, as well as a version for the partially wrapped Fukaya category with one stop (see Theorem 1.4.2 below). To deduce (1.6) one has to develop a setup for localization of \mathbb{L} -graded A_∞ (or dg-)categories.

Figure 4 describes the moment map picture of the singular variety $\mathcal{Z}_{2,k}$ for $k = 4$. This is a two-dimensional analogue of Figure 3. The 3 components on the corners are all isomorphic to \mathbb{A}^2 , 6 components isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$, 3 components isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and 3 components isomorphic to a blow-up of $\mathbb{A}^1 \times \mathbb{P}^1$. The markings C_{ij} denote coordinates on one of the affine charts of each \mathbb{P}^1 along which the components are glued.

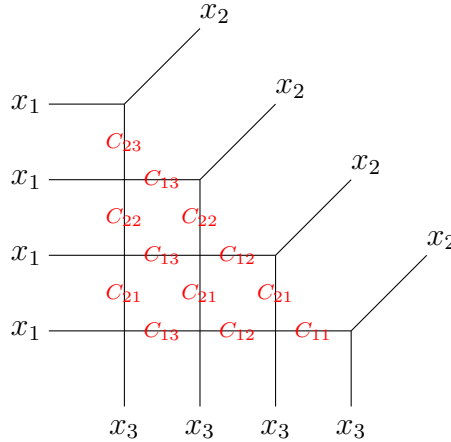


FIGURE 4. Moment map picture of $\mathcal{Z}_{2,4}$ - the mirror to $V_{2,4}$

1.3. Description of the toric variety. We now explicitly describe the toric variety $\mathcal{Y}_{n,k}$. Let us set

$$\Sigma = \mathbb{P}_{\mathbb{C}}^1 \setminus (\{x_1, x_2, \dots, x_{n+1}\} \cup \{v_1, v_2, \dots, v_k\}),$$

the $(n+k+1)$ punctured surface whose boundary components are marked by x_1, \dots, x_{n+1} and v_1, \dots, v_n as in Figure 5.

Here we will describe the toric variety $\mathcal{Y}_{n,k}$ in terms of the dual cones to cones forming the toric fan. These dual cones live in $H \otimes \mathbb{R}$, where $H = H_1(\Sigma)$ is the homology lattice of the punctured surface Σ and are governed by the combinatorics of curves on Σ . We denote by T the algebraic torus with the character lattice H (T will be the torus acting on $\mathcal{Y}_{n,k}$).

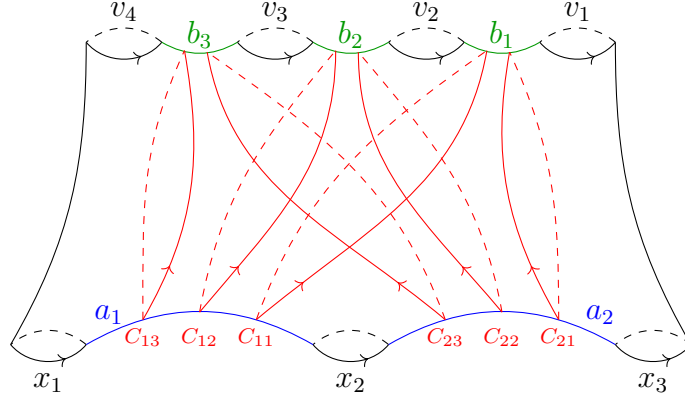


FIGURE 5. The distinguished curves on the surface Σ ; $n = 2$, $k = 4$.

Let us equip each circle x_i with the orientation induced by the orientation of the surface, and we equip each circle v_j with the opposite orientation to the one induced by the orientation of the surface. We still denote by x_i and v_j the corresponding classes in H . As in Figure 5, let us denote by a_i the arc from x_i to x_{i+1} and b_j the arc from v_j to v_{j+1} , and consider the circles c_{ij} which intersect a_i and b_j once. We equip c_{ij} with the orientation induced by the subsurface of Σ to the left of c_{ij} . It is also convenient to set

$$c_{1k} = -x_1 \quad \text{and} \quad c_{n,0} = x_{n+1}.$$

Note that we have the following relation in H :

$$c_{ij} = v_k + v_{k-1} + \dots + v_{j+1} - x_1 - x_2 - \dots - x_i. \quad (1.8)$$

We denote by X_i, V_j, C_{ij} the invertible functions on T corresponding to the elements x_i, v_j, c_{ij} of the lattice H (some of these functions and their inverses will extend to regular functions forming coordinates on affine charts of $\mathcal{Y}_{n,k}$).

We will often use the following relations in our lattice and the corresponding relations between functions:

$$c_{i,j} = c_{i+1,j} + x_{i+1}, \quad C_{i,j} = C_{i+1,j} X_{i+1}, \quad (1.9)$$

$$c_{i,j} = c_{i,j+1} + v_{j+1}, \quad C_{i,j} = C_{i,j+1} V_{j+1}. \quad (1.10)$$

Now, we can define our cones in H . We start with the set $S \subset H$ given by the classes v_1, \dots, v_k . The cones are numbered by nonincreasing functions

$$\phi : [1, n] \rightarrow [1, k].$$

For each such function ϕ , we will define a set of vectors S_ϕ obtained by removing some elements from S and adding some new elements. The vectors of S_ϕ will correspond to coordinates on an affine chart $\mathcal{U}_\phi \subset \mathcal{Y}_{n,k}$. Here is the recipe.

Let $[1, n] = I_1 \sqcup \dots \sqcup I_r$ be the partition of $[1, n]$ into the level sets of the function ϕ . For each interval $I = [i, p] \subset [1, n]$ and an index j , $1 \leq j \leq k$, we consider the subsurface

$\Sigma(I, j)$ bounded by the circles

$$-c_{i,j}, -v_j, c_{p,j-1}, x_{i+1}, \dots, x_p$$

Let us denote by $\partial'\Sigma(I, j)$ the set of classes of all the boundary components of $\Sigma(I, j)$ except for $-v_j$, so that v_j is equal to the sum of the classes in $\partial'\Sigma(I, j)$. Now, for our partition of the arcs I_1, \dots, I_r , we consider the subsurfaces $\Sigma_1, \dots, \Sigma_r$, where $\Sigma_m = \Sigma(I_m, \phi(I_m))$. To get the set S_ϕ from S , we replace each $v_{\phi(I_m)}$ for $m = 1, \dots, r$ with the classes in $\partial'\Sigma_m$. In other words,

$$S_\phi = \{v_j | j \notin \text{im}(\phi)\} \sqcup \{x_{i+1} | \phi(i) = \phi(i+1)\} \sqcup$$

$$\{-c_{i,\phi(i)} | \phi(i-1) > \phi(i) \text{ or } i = 1\} \sqcup \{c_{i,\phi(i)-1} | \phi(i+1) < \phi(i) \text{ or } i = n\}$$

Let $\sigma_\phi^\vee \subset H \otimes \mathbb{R}$ denote the cones generated by S_ϕ , and let $\sigma_\phi \subset H^\vee \otimes \mathbb{R}$ be the dual cones. One can check that the cones σ_ϕ form a smooth toric fan, and we define $\mathcal{Y}_{n,k}$ to be the corresponding smooth toric scheme over the base ring \mathbf{k} . By definition, $\mathcal{Y}_{n,k}$ is covered by the open affine charts \mathcal{U}_ϕ , each isomorphic to the $(n+k)$ -dimensional affine space with the coordinates given by the multiplicative variables corresponding to elements of S_ϕ . Instead of checking that (σ_ϕ) forms a fan we will construct a projective morphism $\mathcal{Y}_{n,k} \rightarrow \mathbb{A}^{n+k+1}$, which proves that $\mathcal{Y}_{n,k}$ is separable (see Corollary 4.3.2).

Finally, we define

$$\mathcal{X}_{n,k} := \mathcal{Y}_{n,k} \times \mathbb{A}_{U_1, \dots, U_k}^k$$

where $\mathbb{A}_{U_1, \dots, U_k}^k$ is the k -dimensional affine space with coordinate functions (U_1, \dots, U_k) .

The vectors v_1, \dots, v_k belong to each of the cones σ_ϕ^\vee , so they give n functions V_1, \dots, V_n on $\mathcal{Y}_{n,k}$. Now, we define the superpotential $\mathbf{w}_{n,k}$ on the smooth toric variety $\mathcal{X}_{n,k}$ by

$$\mathbf{w}_{n,k} := U_1 V_1 + U_2 V_2 + \dots + U_k V_k$$

Furthermore, for each ϕ the vectors v_1, \dots, v_k are obtained by taking sums of disjoint subsets of the bases S_ϕ . Hence, the functions V_1, \dots, V_k form a regular sequence on $\mathcal{Y}_{n,k}$.

We define the singular variety

$$\mathcal{Z}_{n,k} := \{V_1 = V_2 = \dots, V_k = 0\} \subset \mathcal{Y}_{n,k}$$

Let T denote the torus (of dimension $n+k$) acting on $\overline{\mathcal{Y}}_{n,k}$, so that the character lattice of T is the homology lattice H . With respect to this action the weight of the functions X_i , C_{ij} and V_i are the corresponding homology classes x_i , c_{ij} and v_i . The torus $T \times \mathbb{G}_m^k$ acts naturally on $\mathcal{X}_{n,k} = \mathcal{Y}_{n,k} \times \mathbb{A}_{U_1, \dots, U_k}^k$, where the second factor only rescales U_i 's. We restrict this action to the subtorus $\tilde{T} \subset T \times \mathbb{G}_m^k$ given by

$$\tilde{T} = \{(t, \lambda_1, \dots, \lambda_k) \mid v_1(t)\lambda_1 = \dots = v_k(t)\lambda_k\}.$$

We can identify the character lattice of \tilde{T} with

$$\tilde{H} := (H \oplus (\bigoplus_i \mathbb{Z} \cdot u_i)) / (u_i + v_i - u_j - v_j \mid i < j),$$

so that u_i is the weight of the function U_i on $\mathcal{X}_{n,k}$.

By definition, we have the character

$$l_0 := u_1 + v_1 = \dots = u_k + v_k \in \tilde{H},$$

so that the function $\mathbf{w} = \sum_i U_i V_i$ has weight l_0 . Let us consider the sublattice

$$\mathbb{L}_B := 2\tilde{H} + \mathbb{Z}l_0 \subset \tilde{H}.$$

The main category of interest for us is the \mathbb{L}_B -graded category of \tilde{T} -equivariant matrix factorizations of $\mathbf{w}_{n,k}$,

$$\mathcal{B}_{n,k} := \text{MF}_{\tilde{T}}(\mathcal{X}_{n,k}, \mathbf{w}_{n,k})_{\mathbb{L}_B}$$

(see Sec. 2.3.3 for the precise definition). The grading group \mathbb{L}_B can be naturally identified with the grading group \mathbb{L} appearing on the A-side.

Example 1.3.1. Consider the case $n = 2$, $k = 4$. The homology classes of our distinguished curves in this case are, $x_1, x_2, x_3, v_1, v_2, v_3, v_4, c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}$. We have the relations:

$$\begin{aligned} v_4 &= x_1 + c_{13}, \\ v_3 &= c_{12} - c_{13} = c_{22} - c_{23}, \\ v_2 &= c_{11} - c_{12} = c_{21} - c_{22}, \\ v_1 &= x_3 - c_{21}, \\ c_{11} &= c_{21} + x_2, \quad c_{12} = c_{22} + x_2, \quad c_{13} = c_{23} + x_2, \end{aligned}$$

For each such curve, we write the corresponding toric monomial with a capital letter. The cones are numbered by $\phi : [1, 2] \rightarrow [1, 4]$. Let us write them as $\sigma_{\phi(1)\phi(2)} := \sigma_\phi$. We have 10 cones and the corresponding monomials are given as follows:

$$\begin{aligned} \sigma_{44}^\vee &= \langle X_1, X_2, C_{23}, V_3, V_2, V_1 \rangle, & \sigma_{43}^\vee &= \langle X_1, C_{13}, C_{23}^{-1}, C_{22}, V_2, V_1 \rangle, \\ \sigma_{42}^\vee &= \langle X_1, C_{13}, V_3, C_{22}^{-1}, C_{21}, V_1 \rangle, & \sigma_{41}^\vee &= \langle X_1, C_{13}, V_3, V_2, C_{21}^{-1}, X_3 \rangle \\ \sigma_{33}^\vee &= \langle V_4, C_{13}^{-1}, X_2, C_{22}, V_2, V_1 \rangle, & \sigma_{32}^\vee &= \langle V_4, C_{13}^{-1}, C_{12}, C_{22}^{-1}, C_{21}, V_1 \rangle \\ \sigma_{31}^\vee &= \langle V_4, C_{13}^{-1}, C_{12}, V_2, C_{21}^{-1}, X_3 \rangle, & \sigma_{22}^\vee &= \langle V_4, V_3, C_{12}^{-1}, X_2, C_{21}, V_1 \rangle \\ \sigma_{21}^\vee &= \langle V_4, V_3, C_{12}^{-1}, C_{11}, C_{21}^{-1}, X_3 \rangle, & \sigma_{11}^\vee &= \langle V_4, V_3, V_2, C_{11}^{-1}, X_2, X_3 \rangle \end{aligned}$$

Setting $V_1 = V_2 = V_3 = V_4 = 0$ gives $\mathbf{k}[\sigma_{ii}^\vee]/(V_\bullet) \simeq \mathbb{C}[X, Y, Z]$ for all i and $\mathbf{k}[\sigma_{ij}^\vee]/(V_\bullet) \simeq \mathbb{C}[X, Y, Z, W]/(XY, ZW)$ for all $i > j$, which form the affine charts for $\mathcal{Z}_{2,4}$, corresponding to the vertices of Figure 4. Note that these charts are mirror to either the 2-dimensional pair of pants (i.e., the symmetric square of the 4-puncture sphere), or the product of two 1-dimensional pairs of pants. One can see the corresponding 4-punctured spheres or embeddings of two disjoint 3-punctured spheres in the surface Σ bounded by the corresponding curves in Figure 5.

Via homological mirror symmetry, skyscraper sheaves on the components of $\mathcal{Z}_{2,4}$ match with \mathbb{Z} -gradable Lagrangian 2-tori in $V_{2,4}$ equipped with unitary local systems. Such Lagrangian tori can be given as products of two disjoint circles on Σ . We describe this correspondence without proof in Figure 6. Lagrangian tori appearing in the same chamber

are Hamiltonian isotopic ([16]). Any other products among our distinguished curves not appearing in Figure 6 are non-gradable.

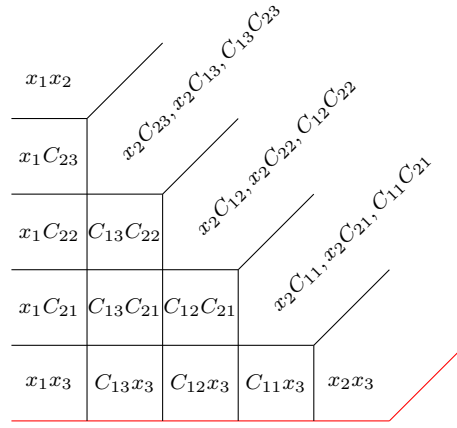


FIGURE 6. Lagrangian tori corresponding to skyscraper sheaves on the irreducible components of $\mathcal{Z}_{2,4}$. The red line indicates the compactification divisor $\overline{\mathcal{Z}}_{2,4} \setminus \mathcal{Z}_{2,4}$

1.4. Compactified LG model. It is easy to see that there is a natural locally closed toric embedding

$$\mathcal{Y}_{n,k} \rightarrow \mathcal{Y}_{n,k+1}$$

whose image is the subscheme $V_1 = 0$ in the union of open charts of $\mathcal{Y}_{n,k+1}$ corresponding to $\phi : [1, n] \rightarrow [1, k+1]$ with $\text{im}(\phi) \subset [2, k+1]$. Under this embedding the functions V_{i+1} on $\mathcal{Y}_{n,k+1}$ restrict to V_i on $\mathcal{Y}_{n,k}$, and function X_{n+1} on $\mathcal{Y}_{n,k}$ gets identified with the restriction of the function $C_{n,1}$ on these open charts of $\mathcal{Y}_{n,k+1}$.

We define $\overline{\mathcal{Y}}_{n,k}$ (resp., $\overline{\mathcal{Z}}_{n,k}$) as the closure of $\mathcal{Y}_{n,k}$ (resp., $\mathcal{Z}_{n,k}$) under this embedding. The functions V_1, \dots, V_k extend regularly to $\overline{\mathcal{Y}}_{n,k}$, so the potential $\mathbf{w}_{n,k}$ extends regularly to $\overline{\mathcal{X}}_{n,k} = \overline{\mathcal{Y}}_{n,k} \times \mathbb{A}^k$. We denote by $\overline{\mathcal{B}}_{n,k}$ the category of \mathbb{L}_B -graded equivariant matrix factorizations of $\mathbf{w}_{n,k}$ on $\overline{\mathcal{X}}_{n,k}$ (recall that \mathbb{L}_B is naturally isomorphic to \mathbb{L}).

We have the following extensions of Conjecture 1.2.1 and Theorem 1.2.2. We equip $V_{n,k}$ with one stop Λ corresponding to a point on the component marked by x_k and consider the corresponding partially wrapped Fukaya category.

Conjecture 1.4.1. There exists a quasi-equivalence of \mathbb{Z} -graded pre-triangulated categories

$$\mathcal{W}(V_{n,k}, \Lambda, \eta_0) \simeq D^b \text{Coh}(\overline{\mathcal{Z}}_{n,k}), \quad (1.11)$$

and a quasi-equivalence of \mathbb{L} -graded pre-triangulated categories

$$\mathcal{W}(V_{n,k}, \Lambda) \simeq \overline{\mathcal{B}}_{n,k}, \quad (1.12)$$

where $\overline{\mathcal{B}}_{n,k}$ is the category of \mathbb{L} -graded matrix factorizations of $\mathbf{w}_{n,k}$ on $\overline{\mathcal{X}}_{n,k}$.

Theorem 1.4.2. *Equivalences (1.11) and (1.12) hold for $n = 1$ and $n = 2$.*

Note that we derive Theorem 1.2.2 from Theorem 1.4.2 by passing to localizations. The Landau-Ginzburg version is also crucial for our approach, as it allows to prove formality of the A_∞ -endomorphism algebra of the generators in $D^b \text{Coh}(\overline{\mathcal{Z}}_{n,k})$ (for $n \leq 2$). Namely, we observe that for a certain \mathbb{Z} -grading the endomorphism algebra of the corresponding matrix factorizations is concentrated in degree zero and derive from this formality of the \mathbb{L} -graded endomorphism algebra.

Conventions. For our main results we work over a ground field \mathbf{k} . In some results \mathbf{k} can be any regular Noetherian ring. Both for Fukaya categories and for matrix factorization categories, we pass to a perfect derived category of the original dg-category.

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2. PRELIMINARIES

2.1. (Multi)gradings of A_∞ -algebras. Let \mathbb{L} be an abelian group equipped with an element $l_0 \in \mathbb{L}$ and a homomorphism $\mathbb{L} \rightarrow \mathbb{Z}/2 : x \mapsto |x|$, such that $|l_0| = 1 \pmod{2}$. We will use this homomorphism to collapse an \mathbb{L} -grading to a $\mathbb{Z}/2$ -grading. The usual notion of an A_∞ -algebra which is a \mathbb{Z} -graded vector space with some operations has a natural \mathbb{L} -graded version.

Definition 2.1.1. An \mathbb{L} -graded A_∞ -algebra is an \mathbb{L} -graded vector space $A = \bigoplus_{l \in \mathbb{L}} A_l$ equipped with operations

$$\mathbf{m}_n : A^{\otimes n} \rightarrow A, \quad n \geq 1,$$

where the \mathbb{L} -degree of \mathbf{m}_n is $(2 - n)l_0$. These operations satisfy the usual A_∞ -axioms

$$\sum_{m,n} (-1)^{|a_1| + \dots + |a_n| - n} \mathbf{m}_{d-m+1}(a_d, \dots, a_{n+m+1}, \mathbf{m}_m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1) = 0.$$

where we use the induced $\mathbb{Z}/2$ -grading on A . Similarly, we can define the notion of an A_∞ -homomorphism between \mathbb{L} -graded A_∞ -algebras.

We will call a homomorphism $f : \mathbb{L} \rightarrow \mathbb{Z}$ *admissible* if $f(l_0) = 1$ and $f(x) \equiv |x| \pmod{2}$. Given an \mathbb{L} -graded A_∞ -algebra A and an admissible homomorphism $f : \mathbb{L} \rightarrow \mathbb{Z}$, we can collapse the \mathbb{L} -grading on A to a \mathbb{Z} -grading and get a usual A_∞ -algebra A/f . Note that an admissible homomorphism f induces a decomposition

$$\mathbb{L} = \mathbb{Z} \cdot l_0 \oplus \mathbb{L}_0,$$

where $\mathbb{L}_0 = \ker(f)$. Then the \mathbb{L} -grading on A can be viewed as an \mathbb{L}_0 -grading on A/f compatible with the \mathbb{Z} -grading, such that all \mathbf{m}_n on A/f preserve the \mathbb{L}_0 -degree. The same is true about A_∞ -homomorphisms (in particular, gauge equivalences) between \mathbb{L} -graded

A_∞ -algebras A and B : they can be viewed as usual A_∞ -homomorphisms between the usual A_∞ -algebras A/f and B/f , preserving the \mathbb{L}_0 -degree.

Note that when \mathbb{L}_0 is finitely generated, an \mathbb{L}_0 -graded vector space can be thought as an algebraic representation of the corresponding commutative algebraic group G , such that \mathbb{L}_0 is the group of characters of G . Thus, in the presence of a splitting $\mathbb{L} = \mathbb{Z} \oplus \mathbb{L}_0$, the above notion of an \mathbb{L} -graded A_∞ -algebra is equivalent to that of a G -equivariant \mathbb{Z} -graded A_∞ -algebra. Since for a reductive group G we can choose all the projections involved in the homological perturbation theory to be G -equivariant, we have the following statement.

Lemma 2.1.2. *Let G be a linearly reductive algebraic group over a field \mathbf{k} . For a dg-algebra A over \mathbf{k} equipped with an algebraic G -action (respecting the dg-algebra structure) there is a G -equivariant version of the homological perturbation producing a G -equivariant minimal A_∞ -structure on $H^*(A)$.*

We have the following obvious formality criterion for \mathbb{L} -graded A_∞ -algebras.

Lemma 2.1.3. *Let $(A, \mathfrak{m}_\bullet)$ be a minimal \mathbb{L} -graded A_∞ -algebra such that for some admissible homomorphism $f : \mathbb{L} \rightarrow \mathbb{Z}$, the algebra A/f is concentrated in degree 0. Then $\mathfrak{m}_n = 0$ for $n > 2$.*

2.2. Multigraded Fukaya categories. Multigradings appear naturally in the context of Fukaya categories: they have been used in [23], [4] and studied extensively by Sheridan [24], [25]. Given a symplectic manifold V , we consider the relative Lagrangian Grassmannian of the tangent bundle TV ,

$$\mathcal{G}V \rightarrow V.$$

The exact sequence associated to this fibration gives the following exact sequence

$$\mathbb{Z} = H_1(\mathcal{G}_p V) \rightarrow H_1(\mathcal{G}V) \rightarrow H_1(V) \rightarrow 0$$

where $\mathcal{G}_p V$ is the Lagrangian Grassmannian of the tangent space at a point $p \in V$, and the identification $\mathbb{Z} = H_1(\mathcal{G}_p V)$ is given by the Maslov index. We set

$$\mathbb{L} := H_1(\mathcal{G}V)$$

and let l_0 be the image of $1 \in \mathbb{Z}$ under the homomorphism $H_1(\mathcal{G}_p V) \rightarrow H_1(\mathcal{G}V)$.

In addition, we choose a homomorphism

$$\mathbb{L} = H_1(\mathcal{G}V) \rightarrow \mathbb{Z}/2$$

which sends l_0 to 1.

In [24] [25], a canonical choice of such a homomorphism is given by the pairing with the first Stiefel-Whitney class $w_1(\mathcal{L}) \in H^1(\mathcal{G}V; \mathbb{Z}/2)$ of the universal bundle $\mathcal{L} \rightarrow \mathcal{G}V$ whose fiber over a point is the Lagrangian subspace at that point. This corresponds to the canonical $\mathbb{Z}/2$ -grading structure on Fukaya categories corresponding to the fibrewise double cover of $\mathcal{G}V$ given by the oriented Lagrangian Grassmannian of the tangent bundle TV .

To allow for other choices of $\mathbb{Z}/2$ -gradings, we consider pairings with

$$w_1(\mathcal{L}) + p^* \sigma \in H^1(\mathcal{G}V; \mathbb{Z}/2)$$

for $\sigma \in H^1(V; \mathbb{Z}/2)$, where $p : \mathcal{G}V \rightarrow V$ is the natural projection. By a slight abuse of notation, we denote the corresponding homomorphism by

$$\sigma : \mathbb{L} \rightarrow \mathbb{Z}/2$$

Note that as we obtained this by pulling back a class from V , this homomorphism still satisfies $\sigma(l_0) = 1$.

Sheridan explains in [24, Section 3] that the gluing formulae for index theory of Cauchy-Riemann operators allow one to define an \mathbb{L} -graded Fukaya category and in [25, Appendix B.3] provides careful elaborations of the arguments by Seidel in [22, Sections 11 & 12]. The key topological observation that makes this work is that if there is a holomorphic disk contributing to some A_∞ -product, then its boundary is nullhomologous. The proof given in [25, Appendix B.3] applies in the same way if we use the more general $\mathbb{Z}/2$ -gradings corresponding to $\sigma : \mathbb{L} \rightarrow \mathbb{Z}/2$ after modifying the definition in [25, Definition B.4] so that $\mathbb{Z}/2$ degree of a generator is computed with respect to the $\mathbb{Z}/2$ grading associated to σ .

The objects of this Fukaya category are \mathbb{L} -graded Lagrangian submanifolds of V . This means that, for a Lagrangian submanifold L , we require a lift of its Gauss map $L \rightarrow \mathcal{G}V$ to the universal abelian cover $\tilde{\mathcal{G}}V \rightarrow \mathcal{G}V$ and a Pin structure on L . If $H_1(L) \rightarrow H_1(V)$ is trivial, then such a lift exists (and is unique up to translation by \mathbb{L} for connected L). A Pin structure on L exists if and only if the second Stiefel-Whitney class vanishes, and the space of Pin structures on L is a torsor over $H^1(L; \mathbb{Z}_2)$. In this paper, we work only with contractible Lagrangians, so they will always be gradable (uniquely up to translation by \mathbb{L}) and will have canonical Pin structures on them.

In summary, for each $\mathbb{Z}/2$ -grading structure σ on V , which can be prescribed by exhibiting a homomorphism $\sigma : \mathbb{L} \rightarrow \mathbb{Z}_2$ with $\sigma(l_0) = 1$, we can construct a \mathbb{L} -graded Fukaya category of V , where $\mathbb{L} = H_1(\mathcal{G}V)$. There is an effective $H^1(V; \mathbb{Z}_2)$ worth of choices for the $\mathbb{Z}/2$ -grading structures, but once we fix such a $\mathbb{Z}/2$ -grading, there is a canonical \mathbb{L} -graded Fukaya category.

Remark 2.2.1. i) Vanishing of $2c_1(V)$ guarantees that the morphism $\mathbb{Z} = H_1(\mathcal{G}_p V) \rightarrow H_1(\mathcal{G}V)$ is injective, or equivalently $l_0 \in \mathbb{L}$ is non-torsion.

ii) Recall that the usual way of equipping Fukaya categories with a \mathbb{Z} -grading goes via similar construction (see [20]) however instead of using the universal abelian cover of $\mathcal{G}V$, one considers a fibrewise universal cover of $\mathcal{G}V$. Such a cover exists if and only if $2c_1(V) = 0$. Moreover, if non-empty, the space of such covers is an $H^1(V)$ -torsor.

2.3. Categories of graded matrix factorizations. Let X be a smooth space over \mathbf{k} (possibly an algebraic stack) with a regular function W . We are going to recall the definition of \mathbb{Z} -graded categories of matrix factorizations following [17] and explain how to generalize it to get multigraded versions of these categories.

2.3.1. \mathbb{Z} -graded categories of G -equivariant matrix factorizations. Assume that we have an action of a reductive group G (over \mathbf{k}) on X , and the potential W on X satisfies

$$W(gx) = l_0(g)W(x).$$

for some character $l_0 : G \rightarrow \mathbb{G}_m$. A (G, l_0) -equivariant matrix factorization of W consists of G -equivariant vector bundles E_0 and E_1 and G -equivariant homomorphisms

$$d_1 : E_1 \rightarrow E_0, \quad d_0 : E_0 \rightarrow E_1 \otimes l_0,$$

such that $d_1 d_0 = d_0 d_1 = W$.

For a pair $(E, d_E), (F, d_F)$ of (G, l_0) -equivariant matrix factorizations we define a (G, l_0) -equivariant matrix factorization $\underline{\text{Hom}} = \underline{\text{Hom}}((E, d_E), (F, d_F))$ of 0, by setting ¹

$$\underline{\text{Hom}}_0 := E_0^\vee \otimes F_0 \oplus E_1^\vee \otimes F_1, \quad \underline{\text{Hom}}_1 := E_0^\vee \otimes F_1 \oplus E_1^\vee \otimes F_0 \otimes l_0^{-1}.$$

Now for any (G, l_0) -equivariant matrix factorization (H, d_H) of 0, we can consider a \mathbb{Z} -graded complex of G -equivariant vector bundles

$$\text{com}(H) = [\dots H_1 \rightarrow H_0 \rightarrow H_1 \otimes l_0 \rightarrow H_0 \otimes l_0 \dots]$$

with H_0 in degree 0. Note that it is equipped with an isomorphism

$$\alpha_H : \text{com}(H)[2] \rightarrow \text{com}(H) \otimes l_0. \quad (2.1)$$

For a pair $E = (E, d_E), F = (F, d_F)$ as above, the \mathbb{Z} -graded morphism space is defined by

$$\text{hom}_{\mathbb{Z}}(E, F) := R\Gamma(X, \text{com}(\underline{\text{Hom}}(E, F)))^G,$$

where we use some functorial multiplicative resolutions to compute $R\Gamma$. ² We denote by $\text{MF}_{G, l_0}(X, W)_{\mathbb{Z}}$ the perfect derived category of the \mathbb{Z} -graded dg-category of (G, l_0) -equivariant matrix factorizations of W . Note that if W is not a zero then l_0 is uniquely determined by W , so we will sometimes omit l_0 from the notation.

Note that due to isomorphism (2.1), for any G -equivariant matrix factorization E we have

$$E \otimes l_0 \simeq E[2].$$

Assume that W is not a zero divisor and consider the hypersurface $X_0 \subset X$ given by $W = 0$. We denote by $D_{\text{sing}}([X_0/G])$ the singularity category of the stack $[X_0/G]$, i.e., the quotient of the bounded derived category of G -equivariant sheaves on X_0 by the subcategory perfect complexes. Assuming that the stack $[X/G]$ is sufficiently nice (of finite cohomological dimension and with a resolution property) one has a natural equivalence (see [17, Thm. 3.14])

$$\text{MF}_G(X, W)_{\mathbb{Z}} \simeq D_{\text{sing}}([X_0/G]). \quad (2.2)$$

¹This construction can be seen as a combination of two operations on matrix factorizations, the tensor product and the duality (see [18, Sec. 1]).

²Equivalently, one can first consider the naive dg-category, without deriving the functor of global sections, and then pass to the quotient by the subcategory of locally contractible objects (see [17]).

2.3.2. \mathbb{G}_m -equivariant matrix factorizations of a potential of degree 1. An important case is when $G = \mathbb{G}_m$ and $l_0 : \mathbb{G}_m \rightarrow \mathbb{G}_m$ is the identity character, so the potential W satisfies

$$W(\lambda \cdot x) = \lambda \cdot W(x),$$

where $\lambda \in \mathbb{G}_m$, $x \in X$. In this case the above definition gives a definition of the dg category $\mathrm{MF}_{\mathbb{G}_m}(X, W) := \mathrm{MF}_{\mathbb{G}_m, l_0}(X, W)_{\mathbb{Z}}$ of \mathbb{G}_m -equivariant matrix factorizations of W .

Remark 2.3.1. 1. It can be shown that the above category $\mathrm{MF}_{\mathbb{G}_m}(X, W)$ is equivalent to the category of B -branes defined as in Segal's paper [19] for the *doubled action* of \mathbb{G}_m on X .

2. If W is a potential of weight $d > 0$ with respect to the \mathbb{G}_m -action, we can pass to the stack $\mathcal{X} := [X/\mu_d]$ and equip it with the action of $\mathbb{G}_m/\mu_d \simeq \mathbb{G}_m$. Then W descends to a function on \mathcal{X} of weight 1 with respect to the \mathbb{G}_m -action. Then the above definition can be applied to the stack \mathcal{X} with the descended potential.

In the case when W is not a zero divisor and the stack $[X/\mathbb{G}_m]$ is sufficiently nice, we get from (2.2) an equivalence

$$\mathrm{MF}_{\mathbb{G}_m}(X, W) \simeq D_{\mathrm{sing}}([X_0/\mathbb{G}_m]). \quad (2.3)$$

Now assume that V_1, \dots, V_k is a regular sequence of global functions on a smooth scheme Y , and let $Z \subset Y$ be the zero locus $Z(V_1, \dots, V_k)$. Then we can consider the scheme $X = Y \times \mathbb{A}_{U_1, \dots, U_k}^k$ with the potential $W = U_1 V_1 + \dots + U_k V_k$. We equip X with the natural \mathbb{G}_m -action, trivial on Y and such that each U_i has weight 1, and consider the category of matrix factorizations $\mathrm{MF}_{\mathbb{G}_m}(X, W)$. Then Işık's theorem [7] states that there is an equivalence

$$D^b \mathrm{Coh}(Z) \simeq \mathrm{MF}_{\mathbb{G}_m}(X, W). \quad (2.4)$$

2.3.3. *Multigraded categories of matrix factorizations.* Now we observe that in the context of Sec. 2.3.1, we can also define a multigraded dg-category of matrix factorizations.

Namely, let L denote the group of characters of G . We assume that $l_0 \notin 2L$ and consider the sublattice

$$L(l_0) := \mathbb{Z} \cdot l_0 + 2L \subset L.$$

Note that we have a natural surjective homomorphism

$$L(l_0) \rightarrow \mathbb{Z}/2$$

sending l_0 to 1 mod 2, and sending $2L$ to 0.

For any (G, l_0) -equivariant matrix factorization (H, d_H) of 0 we can define an $L(l_0)$ -graded complex of G -equivariant bundles $\mathrm{com}_{L(l_0)}(H)$ by

$$\mathrm{com}_{L(l_0)}(H)_{2l} := H_0 \otimes l, \quad \mathrm{com}_{L(l_0)}(H)_{2l+l_0} := H_1 \otimes l \otimes l_0,$$

with the differential of degree l_0 induced by d_H .

Now for a pair $E = (E, d_E)$, $F = (F, d_F)$ of (G, l_0) -equivariant matrix factorizations we define the $L(l_0)$ -graded space of morphisms

$$\mathrm{hom}_{L(l_0)}(E, F) := R\Gamma(X, \mathrm{com}_{L(l_0)}(\underline{\mathrm{Hom}}(E, F)))^G,$$

where on the right we take the totalization of the $\mathbb{Z} \times L(l_0)$ -graded complex with respect to the homomorphism $\mathbb{Z} \times L(l_0) \rightarrow L(l_0) : (n, l) \mapsto nl_0 + l$. We denote by $\mathrm{MF}_{G, l_0}(X, W)_{L(l_0)}$ the corresponding $L(l_0)$ -graded dg-category (sometimes we will omit l_0 from the notation).

Note that we have a natural shift operation $(E, d_E) \mapsto (E[1], d_{E[1]})$, where

$$E[1]_0 = E_1 \otimes l_0, \quad E[1]_1 = E_0 \otimes l_0,$$

and $d_{E[1]}$ is induced by $-d_E$. This shift operation is compatible with the shift of degree by l_0 on the spaces of morphisms.

Remark 2.3.2. It is clear from the definition that if a global function f on X has weight l with respect to the G -action, then the multiplication by f gives an endomorphism of degree $2l$ of any object of $\mathrm{MF}_{G, l_0}(X, W)_{L(l_0)}$. Thus, the natural G -weights get doubled when passing to the endomorphisms in this category.

Lemma 2.3.3. (i) *There is an equivalence of \mathbb{Z} -graded dg-categories*

$$\mathrm{MF}_{G, l_0}(X, W)_{\mathbb{Z}l_0} \simeq \mathrm{MF}_{G, l_0}(X, W)_{\mathbb{Z}},$$

where on the left we consider the subcategory corresponding to the subcomplexes of hom-spaces that have grading in $\mathbb{Z}l_0 \subset L(l_0)$ (and identify the grading group $\mathbb{Z}l_0$ with \mathbb{Z}).

(ii) *Assume that G is a split algebraic torus: $G \simeq \mathbb{G}_m^n$. Then for a subgroup $\mathbb{G}_m \subset G$, such that l_0 restricts to the identity character of \mathbb{G}_m , the natural forgetful functor*

$$\mathrm{MF}_{G, l_0}(X, W)_{L(l_0)} \rightarrow \mathrm{MF}_{\mathbb{G}_m}(X, W)$$

acts as identity on morphism spaces by collapsing the $L(l_0)$ -grading to the \mathbb{Z} -grading via the homomorphism $L(l_0) \hookrightarrow L \rightarrow \mathbb{Z}$ dual to the embedding $\mathbb{G}_m \rightarrow G$.

Proof. Part (i) follows immediately from the definitions. For part (ii), we first observe that the homomorphism $p_{\mathbb{Z}} : L \rightarrow \mathbb{Z}$ induces a splitting

$$L = \mathbb{Z}l_0 \oplus K,$$

where $K = \ker(p_{\mathbb{Z}})$. From this we get an induced splitting

$$L(l_0) = \mathbb{Z}l_0 \oplus 2K.$$

Next, let $(H, d_H) = \underline{\mathrm{Hom}}(E, F)$, for some (G, l_0) -equivariant matrix factorizations E and F . Then we observe that

$$\mathrm{com}_{L(l_0)}(H) = \bigoplus_{k \in K} \mathrm{com}(H) \otimes k.$$

Due to our assumption on G , this implies that

$$R\Gamma(X, \mathrm{com}_{L(l_0)}(H))^G \simeq R\Gamma(X, \mathrm{com}(H))^{\mathbb{G}_m}$$

as claimed. □

From Lemma 2.3.3(i) we see that one can recover all morphisms in $\mathrm{MF}_{G, l_0}(X, W)_{L(l_0)}$ from those in $\mathrm{MF}_{G, l_0}(X, W)_{\mathbb{Z}}$ using the isomorphism of morphism spaces in $\mathrm{MF}_{G, l_0}(X, W)_{L(l_0)}$,

$$\mathrm{Hom}^{2l+ml_0}(E, F) \simeq \mathrm{Hom}^{ml_0}(E, F \otimes l).$$

Assume now that W is not a zero divisor. Then using the equivalence (2.2) we get an equivalence

$$\mathrm{MF}_{G,l_0}(X, W)_{\mathbb{Z}l_0} \simeq \mathrm{MF}_{G,l_0}(X, W)_{\mathbb{Z}} \simeq D_{\mathrm{sing}}([X_0/G]).$$

Thus, given a G -equivariant coherent sheaf F on X_0 , we get from the corresponding object of the G -equivariant singularity category an object of $\mathrm{MF}_{G,l_0}(X, W)_{L(l_0)}$.

2.3.4. Grothendieck-Serre duality and a vanishing criterion for morphisms. Assume now that $G = \mathbb{G}_m$ and l_0 is the identity character. Recall that for a pair of \mathbb{G}_m -equivariant matrix factorizations of W , E and F , the cohomology of the morphism space are given by

$$\mathrm{Hom}^*(E, F) \simeq H^*(X, \mathrm{com}(\underline{\mathrm{Hom}}(E, F)))^{\mathbb{G}_m}.$$

Now assume that we have a \mathbb{G}_m -equivariant proper morphism $\pi : X \rightarrow S$, where $S = \mathrm{Spec}(A)$ is a regular Noetherian affine scheme with a \mathbb{G}_m -action. Then we can define a \mathbb{G}_m -equivariant sheaf on S by

$$R\mathrm{Hom}(E, F)_S := R\pi_* \mathrm{com}(\underline{\mathrm{Hom}}(E, F)).$$

In addition, we will need the following construction from [17, Sec. 1.1]. For a bounded complex (C^\bullet, δ) of \mathbb{G}_m -equivariant vector bundles on X , we define a \mathbb{G}_m -equivariant matrix factorization $\mathrm{mf}(C)$ of 0, by setting

$$\mathrm{mf}(C)_0 = \bigoplus_n C^{2n} \otimes l_0^{-n}, \quad \mathrm{mf}(C)_1 = \bigoplus_n C^{2n-1} \otimes l_0^{-n},$$

with the differential induced by δ . It is easy to see that there is a natural isomorphism of \mathbb{G}_m -equivariant complexes

$$\mathrm{com}(\mathrm{mf}(C)) = \bigoplus_n C \otimes l_0^n[-2n].$$

Lemma 2.3.4. (i) *In the above situation, for \mathbb{G}_m -equivariant matrix factorizations E and F , one has an isomorphism*

$$R\mathrm{Hom}(E, F)_S^\vee \simeq R\mathrm{Hom}(F, E \otimes \omega_{X/S}[n])_S,$$

where n is the relative dimension of π , and $M^\vee = R\mathrm{Hom}(M, \mathcal{O}_S)$, for $M \in D_{\mathbb{G}_m}(S)$.

(ii) *In the same situation assume that $\underline{\mathrm{Hom}}(F, E) \simeq \mathrm{mf}(i_*H)$ for a closed \mathbb{G}_m -invariant subscheme $i : Z \hookrightarrow X$ and $H \in D_{\mathbb{G}_m}(Z)$, such that $i^*\omega_{X/S} \simeq \mathcal{O}_Z \otimes l_0^m$ for some $m \in \mathbb{Z}$. Then the vanishing of $\mathrm{Hom}^*(E, F)$ implies the vanishing of $\mathrm{Hom}^*(F, E)$.*

Proof. (i) Since $\mathrm{com}(\underline{\mathrm{Hom}}(F, E \otimes \omega_{X/S})) \simeq \mathrm{com}(\underline{\mathrm{Hom}}(E, F))^\vee \otimes \omega_{X/S}$, this follows immediately from the usual (\mathbb{G}_m -equivariant) Grothendieck-Serre duality applied to $\mathrm{com}(\underline{\mathrm{Hom}}(E, F))$.

(ii) Set $\mathcal{H} := R\mathrm{Hom}(E, F)_S$. By assumption, we have $R\Gamma(S, \mathcal{H})^{\mathbb{G}_m} = 0$. But we also have

$$\mathcal{H} \otimes l_0^n \simeq \mathcal{H}[2n],$$

hence, we get $R\Gamma(S, \mathcal{H} \otimes l_0^n)^{\mathbb{G}_m} = 0$ for every $n \in \mathbb{Z}$. Since S is affine, this implies that $\mathcal{H} = 0$. Therefore, by part (i), we deduce that

$$R\pi_* \mathrm{com}(\underline{\mathrm{Hom}}(F, E \otimes \omega_{X/S}[n])) \simeq R\mathrm{Hom}(F, E \otimes \omega_{X/S}[n])_S = 0.$$

By assumption, we have an isomorphism

$$\mathrm{com}(\underline{\mathrm{Hom}}(F, E)) \otimes \omega_{X/S} \simeq \bigoplus_n i_* H \otimes l_0^n[-2n] \otimes \omega_{X/S} \simeq \bigoplus_n i_* H \otimes l_0^{n+m}[-2n] \simeq \mathrm{com}(\underline{\mathrm{Hom}}(F, E)) \otimes l_0^m.$$

Therefore, we deduce the vanishing of $R\pi_* \mathrm{com}(\underline{\mathrm{Hom}}(F, E))$, and hence, of $\mathrm{Hom}^*(F, E)$. \square

2.3.5. Generation. We will use the standard generation result for the singularity categories (see [15, Thm. 3.5]). However, we need to adapt this result to the multigraded categories we work with.

Let us assume that we are in the setup of Sec. 2.3.3. Assume in addition that G is a split algebraic torus, and let us fix a subgroup $\mathbb{G}_m \subset G$, such that $l_0|_{\mathbb{G}_m}$ is the identity character χ .

Lemma 2.3.5. *Let $Z \subset X_0$ be a G -invariant closed subscheme containing the singular locus of X_0 , and let $(E_i)_{i \in I}$ be a collection of G -equivariant coherent sheaves on Z , which we view as objects of $D_{\mathrm{sing}}([X_0/G])$ and as objects of $\mathrm{MF}_G(X, W)_{L(l_0)}$. Assume that $(E_i \otimes \chi^n)_{i \in I, n \in \mathbb{Z}}$ generate the $D^b \mathrm{Coh}(Z/\mathbb{G}_m)$. Then (E_i) generate $\mathrm{MF}_G(X, W)_{L(l_0)}$.*

Proof. The category $\mathrm{MF} := \mathrm{MF}_G(X, W)_{L(l_0)}$ has a natural quasicoherent extension $\mathrm{MF}^{\mathrm{qcoh}}$, such that the objects of MF are compact. Thus, to prove that (E_i) generate MF , it is enough to check that for $X \in \mathrm{MF}^{\mathrm{qcoh}}$, the vanishing $\mathrm{Hom}(E_i, X) = 0$ for all i implies that $X = 0$. It is enough to know the same assertion after collapsing the $L(l_0)$ -grading to the \mathbb{Z} -grading via some homomorphism $f : L(l_0) \rightarrow \mathbb{Z}$ sending l_0 to 1. Hence, by Lemma 2.3.3(ii), we can replace G with our fixed subgroup \mathbb{G}_m . Thus, we are reduced to showing that (E_i) generate $\mathrm{MF}_{\mathbb{G}_m}(X, W)$. Note that $E_i \otimes l_0^n \simeq E_i[2n]$ in this category. Hence, using equivalence with the singularity category (2.3), it is enough to check that $(E_i \otimes \chi^n)$ generate $D_{\mathrm{sing}}([X_0/\mathbb{G}_m])$. By the equivariant version of [15, Thm. 3.5], it is enough to check that the subcategory of $D^b \mathrm{Coh}(X_0/\mathbb{G}_m)$ generated by $(E_i \otimes \chi^n)$ contains $i_* D^b \mathrm{Coh}(\mathrm{Sing} X_0)$, where $i : \mathrm{Sing} X_0 \hookrightarrow X_0$ is the singular locus. But this follows from our assumption on (E_i) . \square

2.3.6. Elementary computations with Koszul matrix factorizations. Let us denote by $\{a, b\}$ the Koszul matrix factorization of $a \cdot b$, where a and b are possibly sections of line bundles,

$$E_1 = \mathcal{O}(-b) \xrightarrow{b} E_0 = \mathcal{O} \xrightarrow{a} \mathcal{O}(a) = E_1(ab)$$

where a (resp., b) is a section of a line bundle $\mathcal{O}(a)$ (resp., $\mathcal{O}(b)$). In the computations below we view coherent sheaves as matrix factorizations of 0 via the functor $\mathrm{mf}(\cdot)$ described in Sec. 2.3.4.

Lemma 2.3.6. (i) *Assume (a, b) is a regular sequence. Then one has natural quasiisomorphisms*

$$\underline{\mathrm{End}}(\{a, b\}) \simeq \{a, b\}^\vee|_{b=0} \simeq \mathcal{O}/(a, b).$$

(ii) *Assume (a, b, c) is a regular sequence. Then one has natural quasiisomorphisms*

$$\underline{\mathrm{Hom}}(\{bc, a\}, \{ca, b\}) \simeq \{bc, a\}^\vee|_{b=0} \simeq \mathcal{O}(a)/(a, b)[-1].$$

The generating morphism comes from the exact triangle

$$\mathcal{O}(-a)/(b) \xrightarrow{a} \mathcal{O}/(ab) \rightarrow \mathcal{O}/(a) \rightarrow \mathcal{O}(-a)/(b)[1].$$

(iii) Assume (a, b, c) is a regular sequence. Then one has natural quasiisomorphisms

$$\underline{\mathrm{Hom}}(\{c, ab\}, \{ca, b\}) \simeq \{c, ab\}^\vee|_{b=0} \simeq \mathcal{O}/(b, c),$$

$$\underline{\mathrm{Hom}}(\{ca, b\}, \{c, ab\}) \simeq \{ca, b\}^\vee|_{ab=0} \simeq \mathcal{O}(-a)/(b, c).$$

The generating morphism for the former is the natural projection $\mathcal{O}/(ab) \rightarrow \mathcal{O}/(b)$, while the generating morphism for the latter corresponds to the natural embedding

$$\mathcal{O}/(b) \xrightarrow{a} \mathcal{O}(a)/(ab).$$

Proof. We use the standard quasiisomorphism of matrix factorizations of zero,

$$\underline{\mathrm{Hom}}((E, d_E), \{a, b\}) \simeq (E, d_E)^\vee \otimes \{a, b\} \simeq (E, d_E)^\vee|_{b=0}$$

(see [18, Prop. 1.6.3(ii)]).

In (i) this leads to

$$\mathcal{O}(-a)/(b) \xrightarrow{a} \mathcal{O}/(b) \xrightarrow{0} \dots$$

which is quasiisomorphic to $\mathcal{O}/(a, b)$.

In (ii) we get

$$\mathcal{O}(-bc)/(b) \xrightarrow{0} \mathcal{O}/(b) \xrightarrow{a} \mathcal{O}(a)/(b)$$

which is quasiisomorphic to $\mathcal{O}(a)/(a, b)[-1]$.

In (iii), for the first $\underline{\mathrm{Hom}}$ we get

$$\mathcal{O}(-c)/(b) \xrightarrow{c} \mathcal{O}/(b) \xrightarrow{0} \dots$$

which is quasiisomorphic to $\mathcal{O}/(b, c)$.

Finally, for second $\underline{\mathrm{Hom}}$ in (iii) we get

$$\mathcal{O}(-ca)/(ab) \xrightarrow{ca} \mathcal{O}/(ab) \xrightarrow{b} \mathcal{O}(b)/(ab) \rightarrow \dots$$

We observe that

$$\ker(ca : \mathcal{O}/(ab) \rightarrow \mathcal{O}(ca)/(ab)) = \mathrm{im}(b : \mathcal{O}(-b)/(ab) \rightarrow \mathcal{O}/(ab)),$$

$$\ker(b : \mathcal{O}/(ab) \rightarrow \mathcal{O}(b)/ab) = \mathrm{im}(a : \mathcal{O}(-a)/(ab) \rightarrow \mathcal{O}/(ab)).$$

Hence, the above matrix factorization of 0 is quasiisomorphic to

$$\mathrm{coker}(\mathcal{O}(-ca)/(ca) \xrightarrow{ca} a\mathcal{O}(-a)/(ab)) \simeq \mathrm{coker}(\mathcal{O}(-ca)/(b) \xrightarrow{c} \mathcal{O}(-a)/(b)) \simeq \mathcal{O}(-a)/(b, c).$$

□

2.4. Toric geometry of $\mathcal{Y}_{n,k}$. Below we use the notation from the description of the dual cones of $\mathcal{Y}_{n,k}$ from Sec. 1.3. Let us denote by $\langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}$ the linear combinations of vectors in S_ϕ with nonnegative integer coefficients.

Lemma 2.4.1. *Fix a nonincreasing map $\phi : [1, n] \rightarrow [1, k]$.*

- (i) *Each subset $S_\phi \subset H$ is a basis of H .*
- (ii) *One has $c_{i,j} \in \langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}$ whenever $j < \phi(i)$ and $-c_{i,j} \in \langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}$ whenever $j \geq \phi(i)$.*
- (iii) *For each $i = 1, \dots, k$ and each $j = 1, \dots, n+1$, the vectors v_i and x_j are in $\langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}$.*

Proof. (i) This follows immediately from the fact that the natural map of lattices

$$H_1(\Sigma_1) \oplus \dots \oplus H_1(\Sigma_r) \oplus \bigoplus_{i \notin \text{im}(\phi)} \mathbb{Z} \cdot v_i \rightarrow H$$

is an isomorphism.

(ii) The relations $c_{i,j-1} = c_{i,j} + v_j$ and $-c_{i,j+1} = -c_{i,j} + v_{j+1}$ reduce this to

$$c_{i,\phi(i)-1} \in \langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}, \quad -c_{i,\phi(i)} \in \langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}.$$

Let us prove the first inclusion—the proof of the second is similar. We use descending induction on i . If $\phi(i+1) < \phi(i)$ or $i = n$ then $c_{i,\phi(i)-1} \in S_\phi$ by the definition. If $\phi(i+1) = \phi(i)$ then $x_{i+1} \in S_\phi$ and we can write $c_{i,\phi(i)-1} = c_{i+1,\phi(i+1)-1} + x_{i+1}$, and the assertion follows from the induction assumption.

(iii) The fact that $v_i \in \langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}$ follows from the first definition of S_ϕ (to get S_ϕ from S we keep all v_i for $i \notin \text{im}(\phi)$ and replace each v_i for $i \in \text{im}(\phi)$ with a collection of vectors with the sum v_i).

Next, let us prove that $x_i \in \langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}$. For $i = 1$ this follows from the equality $x_1 = -c_{1,k}$. For $i > 1$ we use the equality $x_i = c_{i-1,j} - c_{i,j}$. If $\phi(i) < \phi(i-1)$ then we set $j = \phi(i)$ and note that $-c_{i,j} \in \langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}$ and $c_{i-1,j} \in \langle S_\phi \rangle_{\mathbb{Z}_{\geq 0}}$ since $j < \phi(i-1)$. In the case $\phi(i) = \phi(i-1)$ we have $x_i \in S_\phi$ by the definition. \square

Let $\mathcal{S}_{n,k}$ denote the toric hypersurface $X_1 \dots X_{n+1} = V_1 \dots V_k$ in the affine space with coordinates $(X_1, \dots, X_{n+1}, V_1, \dots, V_k)$. We have a natural toric birational morphism

$$\pi : \mathcal{Y}_{n,k} \rightarrow \mathcal{S}_{n,k},$$

which is a resolution of singularities since $\mathcal{Y}_{n,k}$ is smooth.

Note that $\mathcal{S}_{n,k}$ is the affine toric variety associated with the cone $C \subset \mathbb{R}H^\vee$ spanned by the vectors $\omega_{ij} = e_i + f_j$, $i = 1, \dots, n+1$, $j = 1, \dots, k$. Here we use the presentation

$$H = (\mathbb{Z}x_1 \oplus \dots \oplus \mathbb{Z}x_{n+1} \oplus \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_k) / (x_1 + \dots + x_{n+1} - v_1 - \dots - v_k),$$

which allows to identify $\mathbb{R}H^\vee$ with the hyperplane $\alpha_1 + \dots + \alpha_{n+1} = \beta_1 + \dots + \beta_k$ in $\mathbb{R}^{n+1} \times \mathbb{R}^k$, and we denote by (e_1, \dots, e_{n+1}) and (f_1, \dots, f_k) the standard bases in \mathbb{R}^{n+1} and \mathbb{R}^k .

Note that the vectors ω_{ij} are precisely the vertices of the polytope

$$\Pi := \Delta_n \times \Delta_{k-1} \subset \mathbb{R}^{n+1} \times \mathbb{R}^k,$$

where $\Delta_m \subset \mathbb{R}^{m+1}$ is the standard simplex spanned by the basis vectors. Thus, C is the cone generated by this polytope. Now taking cones over simplices of a standard triangulation of Π (see e.g., [5, Ch. 7, Sec. 3D]) one gets a toric fan, and the corresponding toric variety is a small resolution of $\mathcal{S}_{n,k}$ (since we do not add new 1-dimensional cones). We claim that this precisely leads to our variety $\mathcal{Y}_{n,k}$.

Proposition 2.4.2. *The map π is the small resolution, associated with the standard triangulation of Π corresponding to the usual ordering of the vertices of Δ_n and the reverse ordering of the vertices of Δ_{k-1} .*

Proof. We need to identify the dual simplices to σ_ϕ^\vee , where ϕ runs through nonincreasing maps $[1, n] \rightarrow [1, k]$, with the simplices of the standard triangulation of Π described in [5, Ch. 7, Sec. 3D]. Note that due to our ordering of the vertices of Δ_{k-1} , the latter simplices are numbered by lattice paths from $(1, k)$ to $(n+1, 1)$, i.e., paths obtained by starting at the point $(1, k)$ and moving either to the right or down along the grid until reaching the point $(n+1, 1)$. Alternatively, they can be numbered by shuffles w of the word $A_1 A_2 \dots A_n B_1 B_2 \dots B_{k-1}$ (preserving the order of A_i 's and of B_j 's), where A_i stands for the i th horizontal move and B_j stands for the j th vertical move.

The simplex Δ_L of the triangulation associated with the lattice path L is given inside Π by the collection of inequalities: $\alpha_1 + \dots + \alpha_i \leq \beta_k + \dots + \beta_j$ whenever A_i precedes B_j in the corresponding shuffled word w_L , and $\alpha_1 + \dots + \alpha_i \geq \beta_k + \dots + \beta_j$ whenever B_j precedes A_i .

We have a natural bijection between lattice paths and nonincreasing functions $\phi : [1, n] \rightarrow [1, k]$: for a lattice path L we let ϕ_L to be the unique function such that $(i + 1/2, \phi_L(i))$ lies on the path L for each $i = 1, \dots, n$. Now it is easy to see that A_i precedes B_j in the shuffled word w_L if and only if $\phi_L(i) \geq j$. Thus, due to formula (1.8) the simplex Δ_L is given inside Π by the inequalities $\langle v, c_{i,j-1} \rangle \geq 0$ whenever $\phi_L(i) \geq j$, and $\langle v, -c_{i,j-1} \rangle \geq 0$ whenever $\phi_L(i) < j$. Hence, the same inequalities give the cone σ_L generated by Δ_L inside the cone C .

It remains to observe that the cone C is dual to the cone generated by all vectors x_i and v_j in H . Hence, Lemma 2.4.1 implies that the cone σ_L is dual to the cone $\sigma_{\phi_L}^\vee$. Thus, $\mathcal{Y}_{n,k}$ is precisely the toric variety associated with the fan with the maximal cones σ_L . \square

Corollary 2.4.3. *The canonical line bundle $\omega_{\mathcal{Y}_{n,k}}$ is trivial.*

3. A-SIDE

3.1. Fukaya categories.

3.1.1. *Generalities.* In [14], we studied the partially wrapped Fukaya category of $\text{Sym}^n(\Sigma)$ equipped with a stop of the form $Z \times \text{Sym}^{n-1}(\Sigma)$, where Σ is a genus 0 surface with boundary and Z is either one point or two points lying on a single component of $\partial\Sigma$. Our main tool was Auroux's determination of generators for this category [2] and the combinatorial description of their endomorphism algebras via strand algebras from [10]. In the case where Z is two points we explicitly wrote out the endomorphism algebra of a particular generator of these categories.

Here, we will concentrate in the case where Z is a single point marked at the outer boundary component as in Figure 7. We will use the same Lagrangians as in [14]. Namely, they are given by products of n disjoint arcs among the Lagrangians drawn on Σ in Figure 7 except that as we only consider the case Z is a single point, the last arc L_{n+k} is not needed. Thus, our generators are of the form

$$L_S = L_{i_1} \times L_{i_2} \times \dots \times L_{i_n}$$

where S is a subset of $[0, n+k-1]$ of size n . The computation given in [14, Thm 3.2.5] readily gives the endomorphism algebras we need, and the corresponding A_∞ -algebra is formal.

In order to avoid repetition, we have chosen to skip over many of the details; the reader is referred to [14, Sections 2 & 3] for a complete background on these calculations. We also note that in [14], Σ was a $(k+1)$ -punctured surface of genus 0. Here, we let Σ to be the genus 0 surface with $(n+k+1)$ -punctures. This slight modification in notation turns out to be more natural. Finally, we note that in [14] we worked with \mathbb{Z} -graded categories (for all possible grading structures), whereas here we consider the \mathbb{L} -graded categories, which in a sense is equivalent to working with all the \mathbb{Z} -grading structures compatible with a given $\mathbb{Z}/2$ -grading structure. The latter structure is characterized by the fact that all endomorphisms of our generators have even grading (this choice is made in order to get an equivalence with the B-side).

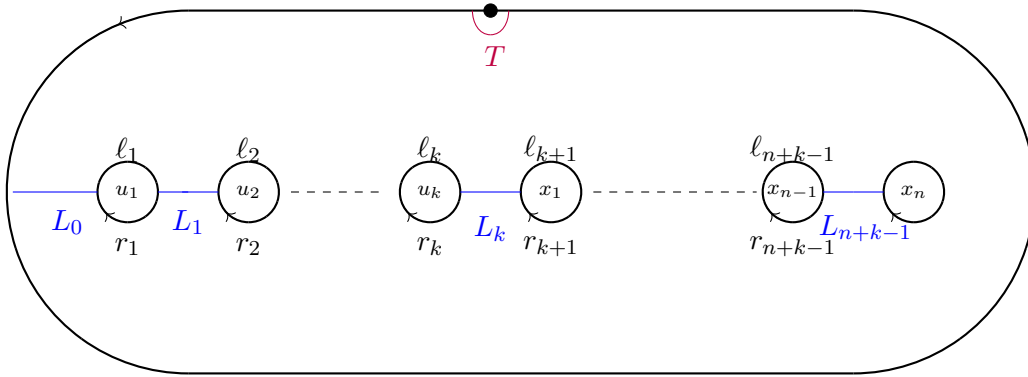


FIGURE 7. Surface Σ and generating Lagrangians

3.1.2. *The universal grading group and its splitting.* Let us set $V = \text{Sym}^n(\Sigma)$. Note that there is an isomorphism between $H_1(V)$ and $H_1(\Sigma) = \mathbb{Z}^{n+k}$ and $c_1(V) = 0$. Thus, the grading group $\mathbb{L} = H_1(\mathcal{G}V)$, where $\mathcal{G}V$ is the bundle of Lagrangian subspaces in TV , fits into an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{L} \rightarrow H_1(\Sigma) \rightarrow 0 \quad (3.1)$$

In Lemma 3.1.1 below, we will see that there is a canonical splitting of this sequence. Using this splitting, we can lift the boundary loops $u_1, \dots, u_k, x_1, \dots, x_n$ to homology

classes in $\mathbb{L} = H_1(\mathcal{G}V)$ (denoted in the same way) and we choose the unique $\mathbb{Z}/2$ -grading structure

$$\sigma : \mathbb{L} \rightarrow \mathbb{Z}/2$$

which sends $u_1, \dots, u_k, x_1, \dots, x_n$ to zero (and $\sigma(l_0) = 1$). This particular choice of $\mathbb{Z}/2$ -grading structure is made in order to match the B -side computations.

Following [24], the grading of a morphism in a partially wrapped Fukaya category can be defined as follows. Suppose L and L' are two Lagrangians in V equipped with grading structures, i.e., lifts of their Gauss maps $L, L' \rightarrow \mathcal{G}V$ to $\tilde{\mathcal{G}}V$, the universal abelian cover of $\mathcal{G}V$. Note that the group \mathbb{L} acts on $\tilde{\mathcal{G}}V$ by deck transformations. This gives a simply transitive action of \mathbb{L} , on the set of grading structures on a Lagrangian L , which we denote as $L \mapsto L(\ell)$, where $\ell \in \mathbb{L}$. By definition, one has $\text{hom}^{\ell_1}(L, L'(\ell_2)) = \text{hom}^{\ell_1 + \ell_2}(L, L')$, so it is enough to characterize which morphisms have degree $0 \in \mathbb{L}$.

We have the following commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{G}}V & \longrightarrow & \tilde{V} \\ \downarrow & & \downarrow \\ \mathcal{G}V & \longrightarrow & V \end{array}$$

where \tilde{V} is the universal abelian cover of V . Note that the top arrow $\tilde{\mathcal{G}}V \rightarrow \tilde{V}$ also can be identified as the (unique) fibre-wise universal cover of the Lagrangian Grassmannian $\mathcal{G}\tilde{V}$. Thus, the symplectic manifold \tilde{V} is equipped with a (unique) \mathbb{Z} -grading structure.

Now assume that $\gamma \in \text{hom}(L, L')$ is a Reeb chord. The grading structures on L, L' determine lifts \tilde{L} and \tilde{L}' of L and L' to \tilde{V} . Furthermore, \tilde{L} and \tilde{L}' come equipped with \mathbb{Z} -grading structures, so morphisms between \tilde{L} and \tilde{L}' in the partially wrapped Fukaya category of \tilde{V} are equipped with \mathbb{Z} -grading. The path $\gamma : L \rightarrow L'$ lifts to a unique path $\tilde{\gamma} : \tilde{L} \rightarrow \tilde{L}'$ where \tilde{L}' is a lift of L' to \tilde{V} , possibly different from \tilde{L}' . Now we say that γ has degree $0 \in \mathbb{L}$ if $\tilde{L}' = \tilde{L}$ and the \mathbb{Z} -degree of the morphism given by $\tilde{\gamma}$ is 0.

Note that the classes of the boundary loops $v_1, \dots, v_k, x_1, \dots, x_n$ (see Figure 7) form a basis of $H_1(\Sigma)$. Let us set u_i to be the circle associated oriented with the opposite orientation to v_i so that its boundary orientation agrees with the boundary orientation of the surface. We are going to use these classes to define a splitting of the sequence (3.1).

Lemma 3.1.1. *There exist canonical lifts of the classes of the boundary loops u_1, \dots, u_k , and x_1, \dots, x_n to \mathbb{L} (which we will denote by the same letters), such that the \mathbb{L} -degree of any nonzero endomorphism $u_i \in \text{End}(L_S)$ (resp., $x_i \in \text{End}(L_S)$) in the partially wrapped Fukaya category of $V = \text{Sym}^n(\Sigma)$ is given by this lift, i.e. the \mathbb{L} -degree is independent of S .*

Proof. Recall that away from the big diagonal $\Delta \subset \text{Sym}^n(\Sigma)$, the Reeb flow along the contact boundary is given by the product of the Reeb flow on the surface and this agrees with the circular flow around the boundary components of Σ in the direction induced by the boundary orientations. Therefore, to an orbit γ going once around a boundary component

of Σ , we can associate a Reeb orbit of the form

$$\tilde{\gamma} = p_1 \times \dots \times p_{n-1} \times \gamma$$

in $\text{Sym}^n(\Sigma) \setminus \Delta$, where p_1, \dots, p_{n-1} are fixed points.

Given a boundary component γ of Σ , other than the one marked with a stop, and an arc L_i intersecting it, for any subset $S = \{i, i_1, \dots, i_{n-1}\}$, we construct as above the Reeb orbit $\tilde{\gamma} = p_1 \times \dots \times p_{n-1} \times \gamma$ with $\tilde{\gamma}(0) = \tilde{\gamma}(1) \in L_S$, by fixing some points $p_1 \in L_{i_1}, \dots, p_{n-1} \in L_{i_{n-1}}$. We need to check that the grading in \mathbb{L} of the corresponding morphism in $\text{hom}(L_S, L_S)$ does not depend on S . Suppose we are given some other S' containing i , $S' = \{i, i'_1, \dots, i'_{n-1}\}$ and some fixed points $p'_1 \in L_{i'_1}, \dots, p'_{n-1} \in L_{i'_{n-1}}$, so that we have the corresponding Reeb chord $\tilde{\gamma}' = p'_1 \times \dots \times p'_{n-1} \times \gamma$. Now we note that the definition of the grading of $\tilde{\gamma}$ does not change if we replace L_S by its open subset $U_S = L_i \times U_{i_1} \times \dots \times U_{i_{n-1}}$, where U_{i_r} is an open neighborhood of p_{i_r} in L_{i_r} . Similarly, we can replace $L_{S'}$ by an open subset $U_{S'}$. Now we can deform $(U_S, \tilde{\gamma})$ to $(U_{S'}, \tilde{\gamma}')$ continuously by moving the fixed points p_1, \dots, p_{n-1} and their neighborhoods in the arcs $L_{i_1}, \dots, L_{i_{n-1}}$ to the points p'_1, \dots, p'_{n-1} and their neighborhoods in the arcs $L_{i'_1}, \dots, L_{i'_{n-1}}$. Clearly, such a continuous deformation does not change the calculation of the grading.

Next, given another arc L_j with $j = i \pm 1$, intersecting γ , and indices i_1, \dots, i_{n-1} distinct from i and j , we have to check that the grading of the chord from $L_i \times L_{i_1} \times \dots \times L_{i_{n-1}}$ to itself, induced by γ , is equal to the grading of the similar chord from $L_j \times L_{i_1} \times \dots \times L_{i_{n-1}}$. For this we use a similar trick: we can replace L_i and L_j by neighborhoods of the point of intersection with γ and deform one into another.

Finally, for any indices i_1, \dots, i_{n-2} , we can check that the gradings of the two chords from $L_i \times L_j \times L_{i_1} \times \dots \times L_{i_{n-2}}$ to itself, induced by γ (which we can stick either in the first or in the second factor) are the same: we replace L_i and L_j by neighborhoods of the points of intersection with γ and move them along γ until they swap. \square

Lemma 3.1.1 allows us to determine the \mathbb{L} -grading on the endomorphism algebra of our generators. Indeed, the \mathbb{L} -grading structure can be computed as in [14, Thm. 3.2.5(iii)], which shows that up to shifts of the generators, the grading is uniquely determined by the degrees of u_r and x_r in \mathbb{L} .

3.2. Endomorphism algebra of the generating Lagrangians: case $n = 1$.

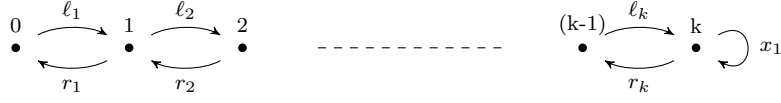
The generating objects are

$$L_0, L_1, \dots, L_k$$

We can express the endomorphism algebra

$$\mathcal{A}_{1,k} = \bigoplus_{i,j=0}^k \text{hom}(L_i, L_j)$$

via the following quiver



with relations

$$\begin{aligned}\ell_{i+1}\ell_i &= 0 = r_i r_{i+1} \\ x_1 \ell_k &= 0 = r_k x_1\end{aligned}$$

Let us work with the \mathbb{Z} -graded category for which $\deg(\ell_i) = \deg(r_i) = 1$ for $i \leq k$ and $\deg(x_1) = 0$. To remove the stop, we need to localize by the subcategory generated by an arc T around the stop. It is easy to see that T can be represented by the twisted complex

$$L_0[1] \xrightarrow{\ell_1} L_1[1] \xrightarrow{\ell_2} \dots \xrightarrow{\ell_{k-1}} L_{k-1}[1] \xrightarrow{\ell_k} L_k[1] \xrightarrow{x_1} L_k \xrightarrow{r_k} L_{k-1} \dots \xrightarrow{r_2} L_1 \xrightarrow{r_1} L_0. \quad (3.2)$$

3.3. Endomorphism algebra of the generating Lagrangians: case $n = 2$.

The generating objects are

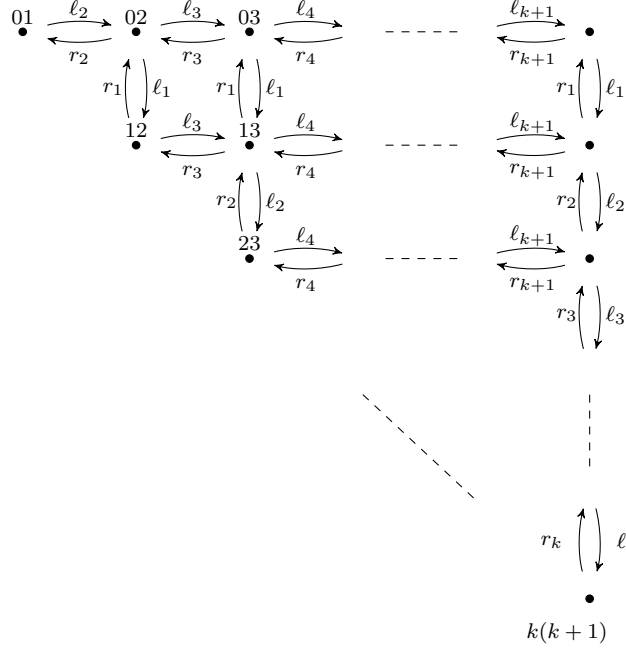
$$L_{i,j} := L_i \times L_j \text{ for } i, j \in \{0, 1, \dots, k+1\} \text{ with } i \neq j.$$

The endomorphism algebra is defined by

$$\mathcal{A}_{2,k} = \bigoplus_{S, S'} \text{hom}(L_S, L_{S'})$$

where the sum is over distinct two element subsets of $\{0, 1, \dots, k+1\}$. Let us write $e_{ij} \in \text{end}(L_S)$ for the idempotent corresponding to $S = \{i, j\}$.

$\mathcal{A}_{2,k}$ can be described via the following quiver over $\mathbf{k}[u_1, u_2, \dots, u_k, x_1, x_2]$



with relations, using the convention that $x_1 = u_{k+1}, x_2 = u_{k+2}$,

$$\begin{aligned}
u_r e_{ij} &= 0 \text{ if } r \neq i, i+1, j, j+1, \\
\ell_{i+1} \ell_i &= 0 = r_i r_{i+1} \text{ for all } i, \\
r_i r_{i+j} &= r_{i+j} r_i \text{ for all } i \text{ and for all } j \geq 2, \\
\ell_i \ell_{i+j} &= \ell_{i+j} \ell_i \text{ for all } i \text{ and for all } j \geq 2, \\
\ell_i r_{i+j} &= r_{i+j} \ell_i \text{ for all } i \text{ and for all } j \geq 2, \\
r_i \ell_{i+j} &= \ell_{i+j} r_i \text{ for all } i \text{ and for all } j \geq 2, \\
r_i \ell_i &= u_i = \ell_i r_i \text{ for } i = 1, \dots, k+1, \\
x_2 \ell_{k+1} &= 0 = r_{k+1} x_2.
\end{aligned}$$

To remove the stop, we need to localize by the subcategory generated by

$$T \times L_i, \text{ for } i = 0, \dots, k+1$$

where T is the arc around the stop. We will identify resolutions for these objects in terms of the generators $(L_{i,j})$. Let us work in the \mathbb{Z} -graded Fukaya category, where we use the gradings of $(L_{i,j})$ such that $\deg(\ell_i) = \deg(r_i) = 1$ for $i \leq k$ and $\deg(\ell_{k+1}) = \deg(r_{k+1}) = \deg(x_2) = 0$.

Proposition 3.3.1. *For $i \leq k$, $T \times L_i$ is represented by the twisted complex*

$$\begin{aligned}
L_{0,i}[1] &\xrightarrow{\ell_1} \dots \xrightarrow{\ell_{i-1}} L_{i-1,i}[1] \xrightarrow{\ell_i \times \ell_{i+1}} L_{i,i+1}[2] \xrightarrow{\ell_{i+2}} \dots \xrightarrow{\ell_k} L_{i,k}[2] \xrightarrow{\ell_{k+1}} L_{i,k+1}[1] \xrightarrow{x_2} \\
L_{i,k+1} &\xrightarrow{r_{k+1}} L_{i,k}[-1] \xrightarrow{r_k} \dots \xrightarrow{r_{i+2}} L_{i,i+1}[-1] \xrightarrow{r_i \times r_{i+1}} L_{i-1,i} \xrightarrow{r_{i-1}} \dots \xrightarrow{r_1} L_{0,i},
\end{aligned} \tag{3.3}$$

while $T \times L_{k+1}$ is represented by

$$L_{0,k+1}[1] \xrightarrow{\ell_1} \dots \xrightarrow{\ell_k} L_{k,k+1}[1] \xrightarrow{x_1} L_{k,k+1} \xrightarrow{r_k} \dots \xrightarrow{r_1} L_{0,k+1}. \quad (3.4)$$

Proof. By considering the $n = 1$ case, we already know that T can be represented by the twisted complex

$$L_0[1] \xrightarrow{\ell_1} L_1[1] \xrightarrow{\ell_2} \dots \xrightarrow{\ell_k} L_k[1] \xrightarrow{\ell_{k+1}} L_{k+1}[1] \xrightarrow{x_2} L_{k+1} \xrightarrow{r_{k+1}} L_k \dots \xrightarrow{r_2} L_1 \xrightarrow{r_1} L_0$$

Since this description involves L_i , we cannot directly multiply this complex with L_i to get a description of $T \times L_i$. We instead will argue in two steps by using the arcs U and D drawn in Figure 8. This applies for $i \leq k$ and for $i = k + 1$ we use the arc E which plays a similar role. Using these, we see that T can be represented by

$$\begin{aligned} L_0[1] \xrightarrow{\ell_1} \dots L_{i-2}[1] \rightarrow U \rightarrow L_{i+2}[1] \xrightarrow{\ell_{i+3}} \dots \xrightarrow{\ell_{k+1}} L_{k+1}[1] \xrightarrow{x_2} L_{k+1} \xrightarrow{r_{k+1}} \\ L_{i+2}[-1] \rightarrow D \rightarrow L_{i-2} \dots \xrightarrow{r_1} L_0 \end{aligned} \quad (3.5)$$

for $i \leq k$ and

$$L_0[1] \xrightarrow{\ell_1} \dots \xrightarrow{\ell_{k-1}} L_{k-1}[1] \rightarrow E \rightarrow L_{k-1} \xrightarrow{r_{k-1}} \dots \xrightarrow{r_1} L_0 \quad (3.6)$$

for $i = k + 1$.

Since none of these complexes involve L_i , we can multiply every term by L_i to get a twisted complex describing $T \times L_i$.

Thus, it remains to describe $U \times L_i$, $D \times L_i$ for $i \leq k$ and $E \times L_{k+1}$ in terms of our generators. We will prove that $U \times L_i$ is given by the cone

$$L_{i-1} \times L_i \xrightarrow{\ell_i \times \ell_{i+1}} L_i \times L_{i+1}$$

The other cases are similar. The existence of this triangle can be seen as follows: First slide L_{i-1} over L_i to \tilde{U} . As this is a Hamiltonian isotopy, this gives an equivalence $L_{i-1} \times L_i \simeq \tilde{U} \times L_i$. Next, observe that sliding \tilde{U} over L_{i+1} gives U , hence we have an exact C is given as a cone $\tilde{U} \rightarrow L_{i+1}$. Multiplying this with L_i , we conclude that $L_{i-1} \times L_i \simeq \tilde{U} \times L_i \rightarrow L_i \times L_{i+1}$ describes the object $U \times L_i$.

Finally, since none of the slides involved lower part of the surface, the morphism in the cone description has to be $\ell_i \times \ell_{i+1}$ as any other morphism from $L_{i-1} \times L_i \rightarrow L_i \times L_{i+1}$ goes around the holes. Indeed, one can see that this exact triangle in \mathbb{L} -graded setting which necessitates that the connecting morphism to be $\ell_i \times \ell_{i+1}$. Similarly, note that there exists no morphism from $L_{i-2} \times L_i$ to $L_i \times L_{i+1}$ (resp. $L_{i-1} \times L_i$ to $L_i \times L_{i+2}$) and there exists a unique morphism from $L_{i-2} \times L_i$ to $L_{i-1} \times L_i$ (resp. $L_i \times L_{i+1} \rightarrow L_i \times L_{i+2}$), namely $\ell_{i-1} \times \text{id}_{L_i}$ (resp. $\text{id}_{L_i} \times \ell_{i+2}$), which does not go around the holes. Hence, for reasons of \mathbb{L} -grading, it follows that the morphisms in the twisted complex representation of the $T \times L_i$ in terms of our generators must be those given in Equations 3.3 and 3.4. \square

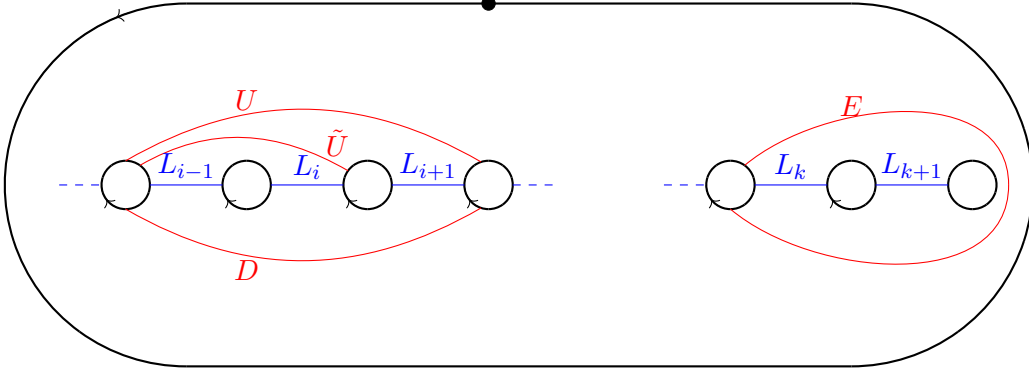


FIGURE 8. Auxiliary Lagrangians

4. B-SIDE: CASE $n = 1$

4.1. **Compactified LG model.** The variety $\overline{\mathcal{Y}}_{1,k}$ is covered by $k+1$ affine charts $\mathcal{U}_0, \dots, \mathcal{U}_k$, each isomorphic to the affine space, where the coordinates on each chart are given as follows

$$\begin{aligned} \mathcal{U}_0 &: V_k, \dots, V_2, V_1, X_2^{-1}, \\ \mathcal{U}_1 &: V_k, \dots, V_2, C_1^{-1}, X_2, \\ &\dots \\ \mathcal{U}_{k-1} &: V_k, C_{k-1}^{-1}, C_{k-2}, V_{k-2}, \dots, V_1, \\ \mathcal{U}_k &: X_1, C_{k-1}, V_{k-1}, \dots, V_1. \end{aligned}$$

where we abbreviate C_{1i} as C_i . We have

$$V_k = X_1 C_{k-1}, \quad V_{k-1} = C_{k-1}^{-1} C_{k-2}, \quad \dots, \quad V_1 = C_1^{-1} X_2,$$

and we consider $\overline{\mathcal{X}}_{1,k} = \overline{\mathcal{Y}}_{1,k} \times \mathbb{A}^k$ with the potential

$$\mathbf{w}_{1,k} = U_1 V_1 + \dots + U_k V_k.$$

It is convenient to set $C_k = X_1^{-1}$, $C_0 = X_2$.

The subvariety $\overline{\mathcal{Z}}_{1,k}$ given as the vanishing locus of (V_1, \dots, V_k) is the nodal chain of rational curves $\overline{R}_0 \cup R_1 \cup \dots \cup R_{k-1} \cup R_k$, where $R_k = \mathbb{A}^1$ with coordinate X_1 ; $R_i = \mathbb{P}^1$ for $0 < i < k$ with C_i and C_i^{-1} being the coordinates on its affine charts; $\overline{R}_0 = \mathbb{P}^1$ with X_2 and X_2^{-1} being the coordinates on its affine charts. Note that the intersection point $R_i \cap R_{i-1}$ lies on the nodal union of the affine lines $\mathbb{A}_{C_i}^1 \subset R_i$ and $\mathbb{A}_{C_{i-1}}^1 \subset R_{i-1}$.

The critical locus $\text{crit}(\mathbf{w}_{1,k}) \subset \overline{\mathcal{X}}_{1,k}$ is the union of our nodal chain $\overline{\mathcal{Z}}_{1,k}$ with k affine lines L_1, \dots, L_k , such that L_i intersects the nodal chain at the i th node.

Lemma 4.1.1. (i) For every $i = 0, \dots, k-1$, there is a morphism $p_i : \overline{\mathcal{Y}}_{1,k} \rightarrow \mathbb{P}^1$ such that $p_i^* z = C_i$ for $i > 0$ and $p_0^* z = X_2$, where z is a coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$. Furthermore, the induced map

$$f : \overline{\mathcal{Y}}_{1,k} \xrightarrow{X_1, V_1, \dots, V_k, p_0, \dots, p_{k-1}} \mathbb{A}_{X_1, V_1, \dots, V_k}^{k+1} \times (\mathbb{P}^1)^k$$

is a closed embedding.

(ii) For $i = 1, \dots, k-1$, we have sections of $p_i^* \mathcal{O}(-1)$, defined in an open neighborhood of R_i ,

$$s_1(i) = \frac{C_{i+1}^{-1}}{p_i^* z_0} = \frac{V_{i+1}}{p_i^* z_1}, \quad s_2(i) = \frac{C_{i-1}}{p_i^* z_1} = \frac{V_i}{p_i^* z_0} \quad (4.1)$$

(where $(z_0 : z_1)$ are homogeneous coordinates on \mathbb{P}^1 such that $z = z_1/z_0$), such that R_i is the vanishing locus of

$$V_1, \dots, V_{i-1}, s_1(i), s_2(i), V_{i+2}, \dots, V_k.$$

Similarly, we have a section of $p_0^* \mathcal{O}(-1)$, defined in a neighborhood of \overline{R}_0

$$s_1(0) = \frac{C_1^{-1}}{p_0^* z_0} = \frac{V_1}{p_0^* z_1},$$

such that \overline{R}_k is the vanishing locus $s_1(0), V_2, \dots, V_k$. Finally, locally near R_k , it is given as the vanishing locus of $V_1, \dots, V_{k-1}, C_{k-1}$.

Proof. (i) Let $\mathbb{A}_0 = (z_0 \neq 0)$ and $\mathbb{A}_1 = (z_1 \neq 0)$ be the standard open covering of \mathbb{P}^1 (where $(z_0 : z_1)$ are the homogeneous coordinates such that $z = z_1/z_0$). Then we define p_i as the unique map such that

$$p_i^{-1}(\mathbb{A}_0) = \mathcal{U}_{i+1} \cup \dots \cup \mathcal{U}_k, \quad p_i^{-1}(\mathbb{A}_1) = \mathcal{U}_0 \cup \dots \cup \mathcal{U}_i,$$

with $p_i^* z = C_i$ on $\mathcal{U}_{i+1} \cup \dots \cup \mathcal{U}_k$ and $p_i^* z^{-1} = C_i^{-1}$ on $\mathcal{U}_0 \cup \dots \cup \mathcal{U}_i$. The fact that C_i (resp., C_i^{-1}) is indeed well defined on \mathcal{U}_j with $j > i$ (resp., $j \leq i$) follows from the identity $C_j = C_{j+1} V_{j+1}$.

For a sequence (i_1, \dots, i_k) of 0's and 1's let us set

$$\mathbb{A}_{i_1, \dots, i_k} := \mathbb{A}_{i_1} \times \dots \times \mathbb{A}_{i_k} \subset (\mathbb{P}^1)^k.$$

To prove the second assertion we note that $f^{-1}(\mathbb{A}^{k+1} \times \mathbb{A}_{i_1, \dots, i_k})$ is nonempty only when $(i_1, \dots, i_k) = ((0)^i, (1)^{k-i})$ for some $0 \leq i \leq k$, and in this case

$$f^{-1}(\mathbb{A}^{k+1} \times \mathbb{A}_{(0)^i, (1)^{k-i}}) = \mathcal{U}_i.$$

It is easy to check that the induced map

$$\mathcal{U}_i \xrightarrow{X_1, V_1, \dots, V_k, C_0, \dots, C_{i-1}, C_i^{-1}, \dots, C_{k-1}} \mathbb{A}^{k+1} \times \mathbb{A}_{(0)^i, (1)^{k-i}}$$

is a closed embedding.

(ii) The fact that $s_1(i)$ (resp., $s_2(i)$) is well defined follows from the identity $C_{i+1}^{-1} C_i = V_{i+1}$ (resp., $C_i^{-1} C_{i-1} = V_i$). The fact that R_i coincides with the claimed vanishing locus can be checked locally. \square

Let us set $\overline{R}_i = R_i$ for $i > 0$, and consider the objects $(P_i)_{i=0, \dots, k}$ of the multigraded category $\overline{\mathcal{B}}_{1,k}$ of \widetilde{T} -equivariant matrix factorization of $\mathbf{w}_{1,k}$ on $\overline{\mathcal{X}}_{1,k}$, associated with the objects $\mathcal{O}_{\overline{R}_i \times \mathbb{A}^k}$ of the corresponding singularity category. Note that the character group \widetilde{H} of the torus \widetilde{T} is a free abelian group with the basis $x_1, u_1, \dots, u_k, l_0$. Hence, the grading group $\mathbb{L}_B = 2\widetilde{H} + \mathbb{Z}l_0$ is a free abelian group with the basis $2x_1, 2u_1, \dots, 2u_k, l_0$, so that

the grading of X_1 is $2x_1$ and the grading of U_i is $2u_i$. We have $v_i = l_0 - u_i$, so the relations $v_i = -c_i + c_{i-1}$ allow to compute the grading c_i of C_i (recall that $c_k = -x_1$).

Proposition 4.1.2. *The only nonzero morphisms between (P_i) in $\overline{\mathcal{B}}_{1,k}$ are the following:*

$$\text{End}(P_k) = k[X_1, U_k]/(X_1 U_k), \quad \text{End}(P_{k-1}) = k[U_k, U_{k-1}]/(U_k U_{k-1}), \quad \dots, \quad \text{End}(P_1) = k[U_2, U_1]/(U_2 U_1), \\ \text{End}(P_0) = k[U_1],$$

$$\text{Hom}(P_i, P_{i-1}) = k[U_i] \cdot a_i, \quad \text{Hom}(P_{i-1}, P_i) = k[U_i] \cdot b_i, \quad \text{for } i = 1, \dots, k,$$

where

$$\deg(a_i) = l_0 - 2c_{i-1}, \quad \deg(b_i) = l_0 + 2c_i, \\ a_i b_i = b_i a_i = U_i.$$

The proof uses presentations of \overline{R}_i as complete intersections and Lemma 2.3.6. We will give the details of similar calculations in the case $n = 2$.

There is a unique projection $|\cdot| : \mathbb{L}_B \rightarrow \mathbb{Z}$ sending l_0 to 1 and x_1 and u_i to 0. In the obtained \mathbb{Z} -graded category the \mathbb{Z} -gradings are given by

$$|X_1| = |U_1| = \dots = |U_k| = 0.$$

With this choice of \mathbb{Z} -grading we have $|a_i| = 1 - 2(k + 1 - j)$, and the endomorphism algebra

$$\text{End}(P_k \oplus P_{k-1}[-1] \oplus P_{k-2}[-4] \oplus \dots \oplus P_0[-k^2])$$

will be concentrated in degree 0.

By Lemma 2.1.3, this implies formality of the \mathbb{L}_B -graded dg-endomorphism algebra of $\bigoplus P_i$.

The following simple general result will help us to prove generation.

Lemma 4.1.3. *Suppose a scheme X is the union of two closed subschemes: $X = Y \cup Z$. Set*

$$U = X \setminus Y = Z \setminus Y \cap Z.$$

Assume we have a collection of coherent sheaves $(F_i)_{i \in I}$ on Y , generating $D^b \text{Coh}(Y)$ and a collection of coherent sheaves $(G_j)_{j \in J}$ on X such that $(G_j|_U)_{j \in J}$ generate $D^b \text{Coh}(U)$. Then $(F_i)_{i \in I}$ and $(G_j)_{j \in J}$ together generate $D^b \text{Coh}(X)$. The similar assertion holds for categories of equivariant sheaves.

Proof. Indeed, in this case (F_i) generate the subcategory in $D^b \text{Coh}(X)$ consisting of objects with cohomology supported on Y . But latter subcategory is precisely the kernel of the restriction functor

$$D^b \text{Coh}(X) \rightarrow D^b \text{Coh}(U).$$

The assertion follows from this. □

Lemma 4.1.4. *Let us work over a regular base ring \mathbf{k} . For any subgroup $\mathbb{G}_m \subset T$, the twists by χ^n of the equivariant sheaves $\mathcal{O}_{\overline{R}_0}, \dots, \mathcal{O}_{R_k}$ (resp., $\mathcal{O}_{R_1}, \dots, \mathcal{O}_{R_k}, \mathcal{O}_{\overline{\mathcal{Z}}_{1,k}}$) generate $D^b \text{Coh}(\overline{\mathcal{Z}}_{1,k}/\mathbb{G}_m)$.*

Proof. This can be proved by induction on k . By the induction assumption, the objects $\mathcal{O}_{R_1}, \dots, \mathcal{O}_{R_k}$ and their twists generate the subcategory of equivariant sheaves supported on $R_1 \cup \dots \cup R_k$. Hence, by Lemma 4.1.3, it remains to check that the twists of the restriction of \mathcal{O}_{R_0} (resp., $\mathcal{O}_{\overline{\mathcal{Z}}_{1,k}}$) to $\overline{R}_0 \setminus R_1$ generate $D^b \text{Coh}(\overline{R}_0 \setminus R_1)$. But this restriction is the structure sheaf, and $\overline{R}_0 \setminus R_1$ is the affine line, so this is true. \square

Proposition 4.1.5. *The objects P_0, \dots, P_k generate the category $\overline{\mathcal{B}}_{1,k}$.*

Proof. By Lemma 4.1.4 with the base ring being the polynomial ring in U_1, \dots, U_k , we get that the twists by χ^n of the equivariant coherent sheaves $(\mathcal{O}_{\overline{R}_i \times \mathbb{A}^k})$ generate $D^b \text{Coh}(\overline{\mathcal{Z}}_{1,k}/\mathbb{G}_m \times \mathbb{A}^k)$. It is easy to see that the critical locus $\text{crit}(\mathbf{w}_{1,k})$ is contained in $\overline{\mathcal{Z}}_{1,k} \times \mathbb{A}^k$, so the assertion follows from Lemma 2.3.5. \square

4.2. Localization for $n = 1$. The complement $\overline{\mathcal{Z}}_{1,k} \setminus \mathcal{Z}_{1,k}$ consists of one point p , namely the infinite point of $\overline{R}_0 \simeq \mathbb{P}^1$. Furthermore, the critical locus of $\mathbf{w}_{1,k}$ intersects the complement $\overline{\mathcal{X}}_{1,k} \setminus \mathcal{X}_{1,k}$ exactly at the point p .

Let us denote by $\mathcal{B}_{1,k}^{\mathbb{Z}}$ (resp., $\overline{\mathcal{B}}_{1,k}^{\mathbb{Z}}$) the category of \mathbb{Z} -graded matrix factorizations on $\mathbf{w}_{1,k}$ on $\mathcal{X}_{1,k}$ (resp., $\overline{\mathcal{X}}_{1,k}$), corresponding to the grading

$$|U_i| = 2, \quad |X_i| = |C_i| = |V_i| = 0.$$

By Isik's theorem (see (2.4)), we have equivalences

$$\mathcal{B}_{1,k}^{\mathbb{Z}} \simeq D^b \text{Coh}(\mathcal{Z}_{1,k}), \quad \overline{\mathcal{B}}_{1,k}^{\mathbb{Z}} \simeq D^b \text{Coh}(\overline{\mathcal{Z}}_{1,k}),$$

compatible with the obvious restriction functors.

It is well known that the restriction functor induces an equivalence of $D^b \text{Coh}(\mathcal{Z}_{1,k})$ with the quotient of $D^b \text{Coh}(\overline{\mathcal{Z}}_{1,k})$ by the subcategory of sheaves supported at $\overline{\mathcal{Z}}_{1,k} \setminus \mathcal{Z}_{1,k} = \{p\}$. Since the restriction functor has an obvious dg-enhancement, this is also true at the level of dg-categories. Hence, we can identify $\mathcal{B}_{1,k}^{\mathbb{Z}} \simeq D^b \text{Coh}(\mathcal{Z}_{1,k})$ with the quotient of $\overline{\mathcal{B}}_{1,k}^{\mathbb{Z}}$ by the subcategory of matrix factorizations supported at p . The latter category is generated by the matrix factorization E associated with the object $\mathcal{O}_{p \times \mathbb{A}^k}$ of the singularity category.

We have the following resolution of E in terms of the generating objects P_i . For an interval $[i, j]$, where $j < k$ and for integers m_i, \dots, m_j , let us consider the object

$$\mathcal{O}_{R_{[i,j]}}(m_i, \dots, m_j) = \bigotimes_{r=i}^j p_r^* \mathcal{O}(m_r)|_{R_{[i,j]}},$$

where $R_{[i,j]} = \overline{R}_i \cup \dots \cup R_j$. Similarly, for $i < k$ we set

$$\mathcal{O}_{R_{[i,k]}}(m_i, \dots, m_{k-1}) = \bigotimes_{r=i}^{k-1} p_r^* \mathcal{O}(m_r)|_{R_{[i,k]}}.$$

Lemma 4.2.1. *One has the following exact triangles*

$$\mathcal{O}_{R_{[0,i]}}(1, \dots, 1, 0) \xrightarrow{p_i^* z_1} \mathcal{O}_{R_{[0,i+1]}}(1, \dots, 1, 0) \rightarrow \mathcal{O}_{R_{i+1}} \rightarrow \dots, \quad \text{for } i < k-1,$$

$$\begin{aligned}
& \mathcal{O}_{R_{[0,k-1]}}(1, \dots, 1, 0) \xrightarrow{p_{k-1}^* z_1} \mathcal{O}_{R_{[0,k]}}(1, 1, \dots, 1) \rightarrow \mathcal{O}_{R_k} \rightarrow \dots, \\
& \mathcal{O}_{R_{i+1}} \xrightarrow{p_{i+1}^* z_0} \mathcal{O}_{R_{[0,i+1]}}(0, 1, \dots, 1) \rightarrow \mathcal{O}_{R_{[0,i]}}(0, 1, \dots, 1) \rightarrow \dots, \text{ for } 0 < i < k-1, \\
& \mathcal{O}_{R_k} \xrightarrow{X_1} \mathcal{O}_{R_{[0,k]}}(0, 1, \dots, 1) \rightarrow \mathcal{O}_{R_{[0,k-1]}}(0, 1, \dots, 1) \rightarrow \dots, \\
& \mathcal{O}_{R_1} \xrightarrow{p_1^* z_0} \mathcal{O}_{R_{[0,1]}}(0, 1) \rightarrow \mathcal{O}_{R_0} \rightarrow \dots,
\end{aligned}$$

which express $\mathcal{O}_{R_{[0,k]}}(1, \dots, 1)$ and $\mathcal{O}_{R_{[0,k]}}(0, 1, \dots, 1)$ in terms of the generators. Finally, there is an exact triangle

$$\mathcal{O}_{R_{[0,k]}}(0, 1, \dots, 1) \xrightarrow{X_2^{-1}} \mathcal{O}_{R_{[0,k]}}(1, \dots, 1) \rightarrow \mathcal{O}_p \rightarrow \dots$$

Proof. These exact triangles come from the natural exact sequences of coherent sheaves extending the surjections of the form $L|_C \rightarrow L|_{C'}$, where L is a line bundle, C is a union of components of the nodal curve, C' is either a smaller union of components, or the point p . \square

4.3. Toric maps $\mathcal{Y}_{n,k} \rightarrow \mathcal{Y}_{n,1}$. Here we will prove that $\mathcal{Y}_{n,k}$ is projective over \mathbb{A}^{n+k+1} , and hence it is a toric variety over \mathbf{k} . We reduce this to the case $n = 1$ using the following observation.

Proposition 4.3.1. *For every $a = 1, \dots, n$, there is a morphism*

$$f_a : \mathcal{Y}_{n,k} \rightarrow \mathcal{Y}_{1,k}$$

such that $f_a^{-1}\mathcal{U}_i = \cup_{\phi: \phi(a)=i} \mathcal{U}_\phi$, and

$$f_a^* V_i = V_i, \quad f_a^* C_i = C_{ai}, \quad f_a^* X_1 = X_1 \dots X_a, \quad f_a^* X_2 = X_{a+1} \dots X_{n+1}.$$

Furthermore, the morphism

$$\mathcal{Y}_{n,k} \xrightarrow{(f_1, \dots, f_n), X_2, \dots, X_n} (\mathcal{Y}_{1,k} \times_{\mathbb{A}_{V_1, \dots, V_k}^k} \dots \times_{\mathbb{A}_{V_1, \dots, V_k}^k} \mathcal{Y}_{1,k}) \times_{\mathbb{A}_{X_2, \dots, X_n}^{n-1}}$$

is a closed embedding.

Proof. The proof is a straightforward calculation in the affine charts. \square

Corollary 4.3.2. *The morphism*

$$\mathcal{Y}_{n,k} \xrightarrow{V_1, \dots, V_k, X_1, \dots, X_{n+1}} \mathbb{A}^{n+k+1}$$

is projective.

Proof. Combining Proposition 4.3.1 with Lemma 4.1.1 we see that the composition

$$\mathcal{Y}_{n,k} \rightarrow \mathbb{A}_{V_1, \dots, V_k}^k \times \mathbb{A}_{X_1, \dots, X_{n+1}}^{n+1} \xrightarrow{\text{id} \times \phi} \mathbb{A}_{V_1, \dots, V_k}^k \times \mathbb{A}^{3n-1}$$

is projective, where

$$\phi(X_1, \dots, X_{n+1}) = (X_1, X_2 \dots X_{n+1}; X_1 X_2, X_3 \dots X_{n+1}; \dots; X_1 \dots X_n, X_{n+1}; X_2, \dots, X_{n-1}).$$

Since ϕ is a closed embedding, the morphism $\mathcal{Y}_{n,k} \rightarrow \mathbb{A}^{k+n+1}$ is also projective. \square

5. B-SIDE: CASE $n = 2$

5.1. **Varieties $\mathcal{Z}_{2,k}$ and $\overline{\mathcal{Z}}_{2,k}$.** The affine charts \mathcal{U}_{ij} covering $\mathcal{Y}_{2,k}$ are numbered by pairs $i \geq j$ in $[1, k]$, where $i = \phi(1)$, $j = \phi(2)$. For $i > j$ the coordinates on \mathcal{U}_{ij} are

$$V_k, \dots, V_{i+1}, C_{1,i}^{-1}, C_{1,i-1}, V_{i-1}, \dots, V_{j+1}, C_{2,j}^{-1}, C_{2,j-1}, V_{j-1}, \dots, V_1,$$

while for $i = j$, the coordinates on \mathcal{U}_{ii} are

$$V_k, \dots, V_{i+1}, C_{1,i}^{-1}, X_2, C_{2,i-1}, V_{i-1}, \dots, V_0$$

(recall that $C_{1k}^{-1} = X_1$, $C_{2,0} = X_3$).

The variety $\mathcal{Z}_{2,k}$ (given as the vanishing locus of V_1, \dots, V_k) is 2-dimensional and is a union of smooth irreducible components R_{ij} , with $k \geq i \geq j \geq 0$. Here for $i < k$, R_{ii} is the closure of the locus $C_{1,i}^{-1} = 0$ in $\mathcal{U}_{ii} \cap \mathcal{Z}_{2,k}$, while R_{kk} is the locus $C_{2,k-1} = 0$ in $\mathcal{U}_{kk} \cap \mathcal{Z}_{2,k}$. For $k > i > j$, R_{ij} is the closure of the locus $C_{1,i}^{-1} = C_{2,j}^{-1} = 0$ in $\mathcal{U}_{ij} \cap \mathcal{Z}_{2,k}$, while for $j < k - 1$, R_{kj} is the closure of the locus $C_{1,k-1} = C_{2,j}^{-1} = 0$ in $\mathcal{U}_{kj} \cap \mathcal{Z}_{2,k}$, and $R_{k,k-1}$ is the closure of the locus $X_2 = 0$ in $\mathcal{U}_{kk} \cap \mathcal{Z}_{2,k}$.

Note that R_{kk} is the affine plane with coordinates X_1, X_2 ; R_{k0} is the affine plane with coordinates X_1, X_3 ; and R_{00} is the affine plane with coordinates X_2, X_3 . The components R_{kj} are isomorphic to $\mathbb{P}^1 \times \mathbb{A}_{X_1}^1$, the components R_{i0} are isomorphic to $\mathbb{P}^1 \times \mathbb{A}_{X_3}^1$, the components R_{ii} are isomorphic to the blow up of the point $(0, 0)$ in $\mathbb{P}^1 \times \mathbb{A}_{X_2}^1$, and the remaining components are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. In the case $k = 4$, the variety $\mathcal{Z}_{2,4}$ is schematically depicted in Figure 4, with R_{44} being the component at the top corner, R_{40} at the bottom left corner, and R_{00} at the bottom right corner.

In the compactification $\overline{\mathcal{Y}}_{2,k}$ we have extra charts $\mathcal{U}_{i,0}$, where $i = 0, \dots, k$. For $i > 1$ the coordinates on $\mathcal{U}_{i,0}$ are

$$V_k, \dots, V_{i+1}, C_{1,i}^{-1}, C_{1,i-1}, V_{i-1}, \dots, V_1, X_3^{-1},$$

while the coordinates on $\mathcal{U}_{1,0}$ are

$$V_k, \dots, V_2, C_{1,1}^{-1}, X_2 X_3, X_3^{-1},$$

and the coordinates on $\mathcal{U}_{0,0}$ are

$$V_k, \dots, V_1, X_2^{-1} X_3^{-1}, X_2.$$

The compactified variety $\overline{\mathcal{Z}}_{2,k}$ is the union of irreducible components \overline{R}_{ij} , $k \geq i \geq j \geq 0$, where $\overline{R}_{ij} = R_{ij}$ for $j > 0$, while R_{i0} get compactified as follows. The affine plane R_{k0} gets compactified to $\mathbb{P}^1 \times \mathbb{A}_{X_1}^1$. For $0 < i < k$, the component $R_{i0} = \mathbb{P}^1 \times \mathbb{A}_{X_3}^1$ gets compactified to $\mathbb{P}^1 \times \mathbb{P}^1$. The component $R_{00} = \mathbb{A}_{X_2, X_3}^2$ gets compactified to the blow up of a point in $\mathbb{P}^1 \times \mathbb{A}_{X_2}^1$.

Thus, the complement of $\mathcal{Z}_{2,k}$ in $\overline{\mathcal{Z}}_{2,k}$ is the nodal chain

$$D = D'_0 \cup D_0 \cup D_1 \cup \dots \cup D_k,$$

where $D'_0 = D_k = \mathbb{A}^1$, $D_i = \mathbb{P}^1$ for $i = 0, \dots, k - 1$; $D_i = \overline{R}_{i0} \setminus R_{i0}$ for $i > 0$, $D'_0 \cup D_0 = \overline{R}_{00} \setminus R_{00}$.

We have the following extension of Proposition 4.3.1 with a straightforward proof.

Lemma 5.1.1. (i) *There are two natural morphisms*

$$f_1 : \overline{\mathcal{Y}}_{2,k} \rightarrow \mathcal{Y}_{1,k}, \quad f_2 : \overline{\mathcal{Y}}_{2,k} \rightarrow \overline{\mathcal{Y}}_{1,k}$$

such that $f_1^{-1}\mathcal{U}_i = \cup_j \mathcal{U}_{ij}$, $f_2^{-1}\mathcal{U}_j = \cup_i \mathcal{U}_{ij}$, and

$$f_1^*X_1 = X_1, \quad f_1^*X_2 = X_2X_3, \quad f_1^*C_i = C_{1i}, \quad f_1^*V_i = V_i.$$

$$f_2^*X_1 = X_1X_2, \quad f_2^*X_2 = X_3, \quad f_2^*C_i = C_{2i}, \quad f_2^*V_i = V_i,$$

Furthermore, the morphism

$$\overline{\mathcal{Y}}_{2,k} \xrightarrow{(f_1, f_2), X_2} (\mathcal{Y}_{1,k} \times_{\mathbb{A}_{V_1, \dots, V_k}^k} \overline{\mathcal{Y}}_{1,k}) \times \mathbb{A}_{X_2}^1$$

is a closed embedding.

(ii) *The schematic preimage $f_2^{-1}(\overline{R}_j)$ is equal to the union (with the reduced scheme structure)*

$$\overline{R}_{*j} := \cup_i \overline{R}_{ij}$$

(these are “rows” in Figure 4).

In the following result we determine equations that present \overline{R}_{ij} as a complete intersection in its open neighborhood in $\overline{\mathcal{Y}}_{2,k}$. Note that for $\overline{R}_{*j} = f_2^{-1}(\overline{R}_j)$ we have such a description due to Lemma 4.1.1.

Lemma 5.1.2. (i) *For $k > i > j > 0$, such that $i - j > 1$, R_{ij} is the vanishing locus of*

$$V_k, \dots, V_{i+2}, f_1^*s_1(i), f_1^*s_2(i), V_{i-1}, \dots, V_{j+2}, f_2^*s_1(j), f_2^*s_2(j), V_{j-1}, \dots, V_1,$$

where $s_1(i)$ and $s_2(i)$ are defined by (4.1).

(ii) *For $1 \leq i < k$, there exists a section s_i of $f_2^*p_{i-1}^*\mathcal{O}(-1) \otimes f_1^*p_i^*\mathcal{O}(-1)$, defined in a neighborhood of $\overline{R}_{i,i-1}$, given by*

$$s_i := \frac{f_1^*s_2(i)}{f_2^*p_{i-1}^*z_1} = \frac{f_2^*s_1(i-1)}{f_1^*p_i^*z_0} = \frac{X_2}{f_2^*p_{i-1}^*z_0 \cdot f_1^*p_i^*z_1}.$$

For $1 < i < k$, $R_{i,i-1}$ is given as the vanishing locus of

$$V_k, \dots, V_{i+2}, f_1^*s_1(i), s_i, f_2^*s_2(i-1), V_{i-2}, \dots, V_1,$$

while $\overline{R}_{1,0}$ is the vanishing locus of

$$V_k, \dots, V_3, f_1^*s_1(1), s_1.$$

On the other hand, there exists a section s_k of $f_2^*p_1^*\mathcal{O}(-1)$ defined locally near $R_{k,k-1}$, given by

$$s_k = \frac{X_2}{f_2^*p_{k-1}^*z_0} = \frac{C_{1,k-1}}{f_2^*p_{k-1}^*z_1},$$

such that $R_{k,k-1}$ is the vanishing locus of

$$s_k, f_2^*s_2(k-1), V_{k-2}, \dots, V_1.$$

(iii) *For $0 < j < k - 1$, \overline{R}_{kj} is cut out in its neighborhood as the vanishing locus of*

$$C_{1,k-1}, V_{k-1}, \dots, V_{j+2}, f_2^*s_1(j), f_2^*s_2(j), V_{j-1}, \dots, V_1.$$

For $1 < i < k$, \overline{R}_{i0} is the vanishing locus of

$$V_k, \dots, V_{i+2}, f_1^* s_1(i), f_1^* s_2(i), V_{i-1}, \dots, V_2, f_2^* s_1(0).$$

(iv) \overline{R}_{k0} is cut out in its neighborhood as the vanishing locus of

$$C_{1,k-1}, V_{k-1}, \dots, V_2, f_2^* s_1(0).$$

Proof. (i) Note that for $i > j$, we have

$$\overline{R}_{ij} = f_1^{-1} R_i \cap f_2^{-1} \overline{R}_j.$$

Thus, our assertion follows from Lemma 4.1.1(ii), where we use the fact that

$$V_{i+1} = s_1(i) \cdot p_i^* z_1, \quad V_i = s_2(i) \cdot p_i^* z_0,$$

so that V_i and V_{i+1} vanish on the zero locus of $s_1(i)$ and $s_2(i)$.

(ii) The existence of s_j follows from the claimed identities, which themselves follow from

$$f_1^* s_2(j) \cdot f_1^* p_j^* z_0 = V_j = f_2^* s_1(j-1) \cdot f_2^* p_{j-1}^* z_1,$$

which holds by the definition of $s_1(j-1)$ and $s_2(j)$, and from

$$f_2^* s_1(j-1) \cdot f_2^* p_{j-1}^* z_0 \cdot f_1^* p_j^* \left(\frac{z_1}{z_0} \right) = C_{2,j}^{-1} \cdot C_{1,j} = X_2.$$

For the definition of s_k we use the identity

$$C_{1,k-1} \cdot \frac{f_2^* p_{k-1}^* z_0}{f_2^* p_{k-1}^* z_1} = C_{1,k-1} \cdot C_{2,k-1}^{-1} = X_2.$$

It is clear that the claimed k sections vanish on $R_{j,j-1}$. To prove that the subscheme they cut out is exactly $R_{j,j-1}$, we recall that by Lemma 4.1.1(ii), we know that for $1 < j < k$, $\overline{R}_{j-1,j}$ is the vanishing locus of

$$V_1, \dots, V_{j+2}, f_1^* s_1(j), f_1^* s_2(j), f_2^* s_1(j-1), f_2^* s_2(j-1), V_{j-2}, \dots, V_k.$$

It remains to observe that $f_1^* s_2(j)$ and $f_2^* s_1(j-1)$ both vanish on the zero locus of s_j .

In the case of $R_{k,k-1}$, from Lemma 4.1.1(ii) we get the description of $R_{k,k-1}$ as the vanishing locus of

$$C_{1,k-1}, f_2^* s_1(k-1), f_2^* s_2(k-1), V_{k-2}, \dots, V_1.$$

The fact that $C_{1,k-1}$ and $f_2^* s_1(k-1)$ vanish on the zero locus of s_k follows from the factorizations

$$C_{1,k-1} = s_k \cdot f_2^* p_{k-1}^* z_1, \quad f_2^* s_1(k-1) = X_1 \cdot s_k.$$

(iii), (iv) The proofs are analogous, so we omit them. \square

5.2. Generating matrix factorizations. For each $k \geq i > j \geq 0$, let P_{ij} denote the matrix factorization of $\mathbf{w}_{2,k}$ on $\overline{\mathcal{X}}_{2,k}$ corresponding to the object (denoted in the same way),

$$P_{ij} := \mathcal{O}_{\overline{R}_{ij} \times \mathbb{A}^k} \otimes f_1^* p_i^* \mathcal{O}(-1)$$

of the singularity category, where our convention is that $p_k^* \mathcal{O}(-1)$ is trivial. Here we equip $f_1^* p_i^* \mathcal{O}(-1)$ with the T -equivariant structure by letting $f_1^* p_i^* z_0$ to have weight 0, and $f_1^* p_i^* z_1$ to have weight c_{1i} .

We also define Q_j to be the matrix factorization corresponding to the object of the singularity category

$$Q_j := \mathcal{O}_{\overline{R}_{*j} \times \mathbb{A}^k}.$$

Lemma 5.2.1. *Let us work over a regular base ring. Then for any subgroup $\mathbb{G}_m \subset T$, the twists by χ^n of the equivariant sheaves*

$$(\mathcal{O}_{\overline{R}_{ij}} \otimes f_1^* p_i^* \mathcal{O}(-1))_{0 \geq i > j \geq k} \quad \text{and} \quad (\mathcal{O}_{\overline{R}_{*j}})_{0 \leq j \leq k}$$

generate $D^b \text{Coh}(\overline{\mathcal{Z}}_{2,k}/\mathbb{G}_m)$.

Proof. Let us prove by descending induction on $j_0 \geq 0$ that the twists of the sheaves $(\mathcal{O}_{\overline{R}_{ij}} \otimes f_1^* p_i^* \mathcal{O}(-1))_{j \geq j_0}$, $(\mathcal{O}_{\overline{R}_{*j}})_{j \geq j_0}$ generate the subcategory of equivariant sheaves supported on

$$\overline{R}_{* \geq j_0} := \overline{R}_{*k} \cup \dots \cup \overline{R}_{*,j_0}.$$

The base $j_0 = k$ is clear as $\overline{R}_{*k} = R_{kk} \simeq \mathbb{A}^2$, so the twists of $\mathcal{O}_{R_{kk}}$ generates the needed subcategory. Assume that $j_0 < k$ and the assertion holds for $j_0 + 1$. By Lemma 4.1.3, it is enough to check generation after restricting to $\overline{R}_{*j_0} \setminus \overline{R}_{*,j_0+1}$. Now we use the further restriction

$$D^b \text{Coh}(\overline{R}_{*j_0} \setminus \overline{R}_{*,j_0+1}) \rightarrow D^b \text{Coh}(\overline{R}_{j_0 j_0} \setminus (\overline{R}_{j_0+1, j_0+1} \cup \overline{R}_{j_0+1, j_0})).$$

We note that

$$\overline{R}_{j_0 j_0} \setminus (\overline{R}_{j_0+1, j_0+1} \cup \overline{R}_{j_0+1, j_0}) \simeq \mathbb{A}^2,$$

so its derived category is generated by the twists of the structure sheaf obtained as the restriction of $\mathcal{O}_{\overline{R}_{*k_0}}$. Hence, it remains to show that the subcategory of $D^b \text{Coh}(\overline{R}_{*j_0} \setminus \overline{R}_{*,j_0+1})$ consisting of sheaves supported on $(\overline{R}_{kj_0} \cup \dots \cup \overline{R}_{j_0+1, j_0}) \setminus \overline{R}_{*,j_0+1}$ is generated by the restrictions of our objects. But we have

$$(\overline{R}_{kj_0} \cup \dots \cup \overline{R}_{j_0+1, j_0}) \setminus \overline{R}_{*,j_0+1} \simeq (R_k \cup \dots \cup R_{j_0+1}) \times \mathbb{A}^1,$$

and our assertion follows from Lemma 4.1.4 (after tensoring with the generators with the line bundle $p_{k-1}^* \mathcal{O}(-1) \otimes \dots \otimes p_{j_0+1}^* \mathcal{O}(-1)$). \square

Proposition 5.2.2. *The objects $(P_{ij})_{i > j}$ and (Q_j) generate $\text{MF}(\overline{\mathcal{X}}_{2,k}, \mathbf{w}_{2,k})$.*

Proof. It is easy to see that the critical locus $\text{crit}(\mathbf{w}_{2,k})$ is contained in $\overline{\mathcal{Z}}_{2,k} \times \mathbb{A}^k$. Thus, as in Proposition 4.1.5, the statement follows from Lemma 5.2.1 and Lemma 2.3.5. \square

5.3. Computation of morphisms. For the computation of morphisms, we consider first the \mathbb{Z} -graded category $\mathrm{MF}_{\mathbb{G}_m}(\overline{\mathcal{X}}_{2,k}, \mathbf{w}_{2,k})$, where the subgroup $\mathbb{G}_m \subset \widetilde{T}$ corresponds to the homomorphism $\widetilde{H} \rightarrow \mathbb{Z}$ sending $x_1, x_2, u_1, \dots, u_k$ to zero and l_0 to 1. This corresponds to the following \mathbb{Z} -grading of the variables:

$$|X_1| = |X_2| = |U_i| = 0, \quad |V_i| = 2, \quad |C_{1i}| = |C_{2i}| = 2(k-i), \quad |X_3| = 2k. \quad (5.1)$$

We will consider the case of an arbitrary grading later in Sec. 5.3.5.

5.3.1. Endomorphisms. Let us consider the critical locus $\mathrm{crit}(\mathbf{w}_{2,k}) \subset \overline{\mathcal{X}}_{2,k}$. It is easy to see that its projection to $\overline{\mathcal{Y}}_{2,k}$ factors through $\overline{\mathcal{Z}}_{2,k}$, so we have a natural projection

$$p : \mathrm{crit}(\mathbf{w}_{2,k}) \rightarrow \overline{\mathcal{Z}}_{2,k}.$$

Recall that $\overline{\mathcal{Z}}_{2,k}$ is the union of the irreducible components \overline{R}_{ij} . Let us introduce the notation for some other strata in $\overline{\mathcal{Z}}_{2,k}$. For $k \geq i \geq j \geq 1$, set

$$L_{ij}^h := R_{i,j-1} \cap R_{i,j},$$

Similarly, for $k \geq i \geq j \geq 1$, we set

$$L_{ij}^v := \overline{R}_{i,j-1} \cap \overline{R}_{i-1,j-1}.$$

Also, for $i = 1, \dots, k$, set

$$L_i^d := R_{i-1,i-1} \cap R_{ii}.$$

Finally, for $k \geq i > j \geq 1$, we set $p_{ij} := L_{ij}^h \cap L_{ij}^v$. A straightforward calculation gives the following description of the critical locus of $\mathbf{w}_{2,k}$.

Lemma 5.3.1. *The locus $\mathrm{crit}(\mathbf{w}_{2,k})$ is the union (with the reduced scheme structure) of the following smooth two dimensional components:*

- \overline{R}_{ij} for $k \geq i \geq j \geq 0$;
- $L_{ij}^h \times \mathbb{A}_{U_j}^1$ for $k \geq i \geq j \geq 1$;
- $L_{ij}^v \times \mathbb{A}_{U_i}^1$ for $k \geq i \geq j \geq 1$;
- $L_i^d \times \mathbb{A}_{U_i}^1$ for $1 \leq i \leq k$;
- $p_{ij} \times \mathbb{A}_{U_i, U_j}^2$ for $k \geq i > j \geq 1$.

The crucial step in calculating the morphism spaces is the following local result.

Lemma 5.3.2. *For every $k \geq i > j \geq 0$, one has a natural quasiisomorphism*

$$\underline{\mathrm{End}}(P_{ij}) \simeq \mathcal{O}_{p^{-1}\overline{R}_{ij}}.$$

Let us set $L_{*j} = \cup_i L_{ij}^h \cup L_j^d$. Then for $0 < j < k$ we have

$$\underline{\mathrm{End}}(Q_j) = \mathcal{O}_{\overline{R}_{*j} \cup L_{*j} \times \mathbb{A}_{U_j}^1 \cup L_{*j+1} \times \mathbb{A}_{U_{j+1}}^1},$$

while

$$\underline{\mathrm{End}}(Q_k) = \mathcal{O}_{p^{-1}R_{00}} = \mathcal{O}_{R_{00} \cup L_{*k} \times \mathbb{A}_{U_k}^1},$$

$$\underline{\mathrm{End}}(Q_0) = \mathcal{O}_{\overline{R}_{*0} \cup L_{*1} \times \mathbb{A}_{U_1}^1}.$$

Proof. We will combine the description of \overline{R}_{ij} as complete intersections from Lemma 5.1.2 with the computation of Lemma 2.3.6(i). In the case of Q_j we use the description of $\overline{R}_{*j} = f_2^{-1}(\overline{R}_j)$ as a complete intersection. Below we provide details of the calculation (in later calculations we will skip over some similar steps).

Case of $\underline{\text{End}}(P_{ij})$, for $k > i > j > 0$.

Assume first that $i - j > 1$. Then we can decompose W as

$$W = \sum_{r \neq i, i+1, j, j+1} V_r U_r + f_1^* s_1(i) \cdot f_1^* p_i^* z_1 \cdot U_{i+1} + f_1^* s_2(i) \cdot f_1^* p_i^* z_0 \cdot U_i + f_2^* s_1(j) \cdot f_2^* p_j^* z_1 \cdot U_{j+1} + f_2^* s_2(j) \cdot f_2^* p_j^* z_0 \cdot U_j. \quad (5.2)$$

The description of \overline{R}_{ij} as a complete intersection implies that $P_{ij} \otimes f_1^* p_i^* \mathcal{O}(1)$ can be presented as the tensor product of the following Koszul matrix factorizations:

- for $r \neq i, i+1, j, j+1$, $\{U_r, V_r\}$;
- $\{f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* s_1(i)\}$, $\{f_1^* p_i^* z_0 \cdot U_i, f_1^* s_2(i)\}$;
- $\{f_2^* p_j^* z_1 \cdot U_{j+1}, f_2^* s_1(j)\}$, $\{f_2^* p_j^* z_0 \cdot U_j, f_2^* s_2(j)\}$.

Hence, $\underline{\text{End}}(P_{ij})$ is the tensor product of endomorphism sheaves of these Koszul matrix factorizations. Applying Lemma 2.3.6(i), we get on $\overline{R}_{ij} \times \mathbb{A}^k$

$$\underline{\text{End}}(P_{ij}) \simeq$$

$$\mathcal{O}/((U_r \mid r \neq i, i+1, j, j+1), f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* p_i^* z_0 \cdot U_i, f_2^* p_j^* z_1 \cdot U_{j+1}, f_2^* p_j^* z_0 \cdot U_j) = \mathcal{O}_{p^{-1}\overline{R}_{ij}}.$$

In the case $j = i - 1$, $P_{i, i-1} \otimes f_1^* p_i^* \mathcal{O}(1)$ is the tensor product of the Koszul matrix factorizations,

- $\{U_r, V_r\}$, for $r \neq i - 1, i, i + 1$;
- $\{f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* s_1(i)\}$, $\{f_1^* p_i^* z_0 \cdot f_2^* p_{i-1}^* z_1 \cdot U_i, s_i\}$, $\{f_2^* p_{i-1}^* z_0 \cdot U_{i-1}, f_2^* s_2(i - 1)\}$.

Hence, we get an isomorphism on $\overline{R}_{i, i-1} \times \mathbb{A}^k$,

$$\underline{\text{End}}(P_{i, i-1}) \simeq$$

$$\mathcal{O}/((U_r \mid r \neq i - 1, i, i + 1), f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* p_i^* z_0 \cdot f_2^* p_{i-1}^* z_1 \cdot U_i, f_2^* p_{i-1}^* z_0 \cdot U_{i-1}) \simeq \mathcal{O}_{p^{-1}\overline{R}_{i, i-1}}.$$

Case of $\underline{\text{End}}(P_{i0})$, for $0 < i < k$.

Assuming that $i > 1$ we use the presentation of P_{i0} as the tensor product of

- $\{U_r, V_r\}$, for $r \neq 0, i, i + 1$;
- $\{f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* s_1(i)\}$, $\{f_1^* p_i^* z_0 \cdot U_i, f_1^* s_2(i)\}$, $\{f_2^* p_0^* z_1 \cdot U_1, f_2^* s_1(0)\}$.

As before, from this we deduce an isomorphism on $\overline{R}_{i0} \times \mathbb{A}^k$,

$$\underline{\text{End}}(P_{i0}) \simeq \mathcal{O}/((U_r \mid r \neq 0, i, i + 1), f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* p_i^* z_0 \cdot U_i, f_2^* p_0^* z_1 \cdot U_1) \simeq \mathcal{O}_{p^{-1}\overline{R}_{i0}}.$$

In the case $i = 1$, we use the presentation of $P_{1,0}$ as the tensor product of

- $\{U_r, V_r\}$, for $r > 1$;
- $\{f_1^* p_1^* z_1 \cdot U_2, f_1^* s_1(1)\}$, $\{f_1^* p_1^* z_0 \cdot f_2^* p_0^* z_1 \cdot U_1, s_1\}$.

From this we get an isomorphism on $\overline{R}_{1,0} \times \mathbb{A}^k$,

$$\underline{\text{End}}(P_{1,0}) \simeq \mathcal{O}/((U_r \mid r > 1), f_1^* p_1^* z_1 \cdot U_2, f_1^* p_1^* z_0 \cdot f_2^* p_0^* z_1 \cdot U_1) \simeq \mathcal{O}_{p^{-1}\overline{R}_{1,0}}.$$

Case of $\underline{\text{End}}(P_{kj})$.

Assume first that $0 < j < k - 1$. Then in a neighborhood of $R_{kj} \times \mathbb{A}^k$, P_{kj} is obtained as the tensor product of

- $\{U_r, V_r\}$, for $r \neq j, j + 1, k$;
- $\{X_1 U_k, C_{1,k-1}\}$;
- $\{f_2^* p_j^* z_1 \cdot U_{j+1}, f_2^* s_1(j)\}$, $\{f_2^* p_j^* z_0 \cdot U_j, f_2^* s_2(j)\}$.

This leads to an isomorphism on $R_{kj} \times \mathbb{A}^k$,

$$\underline{\text{End}}(P_{kj}) \simeq \mathcal{O}/((U_r \mid r \neq j, j + 1, k), X_1 U_k, f_2^* p_j^* z_1 \cdot U_{j+1}, f_2^* p_j^* z_0 \cdot U_j) \simeq \mathcal{O}_{p^{-1}R_{kj}}.$$

In the case $j = k - 1$, we present $P_{k,k-1}$ as the tensor product of

- $\{U_r, V_r\}$, for $r < k - 1$;
- $\{X_1 U_k \cdot f_2^* p_{k-1}^* z_1, s_k\}$;
- $\{f_2^* p_{k-1}^* z_0 \cdot U_{k-1}, f_2^* s_2(k - 1)\}$,

which leads to an isomorphism on $R_{k,k-1} \times \mathbb{A}^k$,

$$\underline{\text{End}}(P_{k,k-1}) \simeq \mathcal{O}/((U_r \mid r < k - 1), X_1 U_k \cdot f_2^* p_{k-1}^* z_1, f_2^* p_{k-1}^* z_0 \cdot U_{k-1}) \simeq \mathcal{O}_{p^{-1}R_{k,k-1}}.$$

In the case $j = 0$, in a neighborhood of $\overline{R}_{k0} \times \mathbb{A}^k$ we can present P_{k0} as the tensor product of

- $\{U_r, V_r\}$, for $r \neq 1, k$;
- $\{X_1 U_k, C_{1,k-1}\}$;
- $\{f_2^* p_0^* z_1 \cdot U_1, f_2^* s_1(0)\}$,

so we get an isomorphism on $\overline{R}_{k0} \times \mathbb{A}^k$,

$$\underline{\text{End}}(P_{k0}) \simeq \mathcal{O}/((U_r \mid r \neq 1, k), X_1 U_k, f_2^* p_0^* z_1 \cdot U_1) \simeq \mathcal{O}_{p^{-1}\overline{R}_{k0}}.$$

Case of $\underline{\text{End}}(Q_j)$.

For $0 < j < k$, we present Q_j as the tensor product of

- $\{U_r, V_r\}$, for $r \neq j, j + 1$;
- $\{f_2^* p_j^* z_1 \cdot U_{j+1}, f_2^* s_1(j)\}$, $\{f_2^* p_j^* z_0 \cdot U_j, f_2^* s_2(j)\}$.

From this we get an isomorphism on $R_{*j} \times \mathbb{A}^k$,

$$\underline{\text{End}}(Q_j) \simeq \mathcal{O}/((U_r \mid r \neq j, j + 1), f_2^* p_j^* z_1 \cdot U_{j+1}, f_2^* p_j^* z_0 \cdot U_j) \simeq \mathcal{O}_Z,$$

where $Z = \overline{R}_{*j} \cup L_{*j} \times \mathbb{A}_{U_j}^1 \cup L_{*j+1} \times \mathbb{A}_{U_{j+1}}^1$.

For $j = k$, in a neighborhood of $R_{kk} \times \mathbb{A}^k$, we can present Q_k as the tensor product of

- $\{U_r, V_r\}$, for $r < k$;
- $\{X_1 X_2 \cdot U_k, C_{2,k-1}\}$,

which leads to an isomorphism on $R_{kk} \times \mathbb{A}^k$,

$$\underline{\text{End}}(Q_k) \simeq \mathcal{O}/((U_r \mid r < k), X_1 X_2 \cdot U_k) \simeq \mathcal{O}_{R_{kk} \cup L_{*1} \times \mathbb{A}_{U_1}^1}.$$

For $j = 0$, we present Q_0 as the tensor product of

- $\{U_r, V_r\}$, for $r \neq 1$;
- $\{f_2^* p_0^* z_1 \cdot U_1, f_2^* s_1(0)\}$,

which leads to an isomorphism on $\overline{R}_{*0} \times \mathbb{A}^k$,

$$\underline{\text{End}}(Q_0) \simeq \mathcal{O}/((U_r \mid r \neq 1), f_2^* p_0^* z_1 \cdot U_1) \simeq \mathcal{O}_{\overline{R}_{*0} \cup L_{*1} \times \mathbb{A}_{U_1}^1}.$$

□

Lemma 5.3.3. *Let us set $U_{k+1} := X_1$.*

(i) *For $k \geq i > j > 0$, such that $i - j > 1$, the natural embedding $k[U_i, U_{i+1}, U_j, U_{j+1}] \subset \mathcal{O}(\overline{\mathcal{X}}_{2,k})$ induces an isomorphism*

$$\text{End}(P_{ij}) \simeq k[U_i, U_{i+1}, U_j, U_{j+1}]/(U_i U_{i+1}, U_j U_{j+1}).$$

(ii) *For $k \geq i > 1$, there is a similar isomorphism*

$$\text{End}(P_{i0}) \simeq k[U_1, U_i, U_{i+1}]/(U_i U_{i+1}).$$

(iii) *For $i > 1$, there is a similar isomorphism*

$$\text{End}(P_{i,i-1}) \simeq k[U_{i-1}, U_i, U_{i+1}]/(U_{i-1} U_i U_{i+1}).$$

(iv) *There is a similar isomorphism*

$$\text{End}(P_{1,0}) \simeq k[U_1, U_2].$$

Proof. By Lemma 5.3.2, we have to calculate $H^*(\mathcal{O}_{p^{-1}\overline{R}_{ij}})$. Let us consider the natural projection

$$\pi : \overline{\mathcal{X}}_{2,k} \rightarrow \mathbb{A}_{U_1, \dots, U_k, U_{k+1}}^{k+1}.$$

For part (i), it is enough to show the natural isomorphism

$$R\pi_* \mathcal{O}_{p^{-1}\overline{R}_{ij}} \simeq \mathcal{O}/((U_r)_{r \neq i, i+1, j, j+1}, U_i U_{i+1}, U_j U_{j+1}),$$

and similarly for parts (ii)–(iv).

Suppose that $k > i, j > 0$ and $i - j > 1$. Then

$$p^{-1}R_{ij} \simeq C_1(U_i, U_{i+1}) \times C_2(U_j, U_{j+1}),$$

where $C_1(U_i, U_{i+1})$ and $C_2(U_j, U_{j+1})$ are nodal chains with 3 components:

$$C_1(U_i, U_{i+1}) = \mathbb{A}_{U_i}^1 \cup \mathbb{P}^1 \cup \mathbb{A}_{U_{i+1}}^1, \quad C_2(U_j, U_{j+1}) = \mathbb{A}_{U_j}^1 \cup \mathbb{P}^1 \cup \mathbb{A}_{U_{j+1}}^1,$$

The morphism $C_1(U_i, U_{i+1}) \times C_2(U_j, U_{j+1}) \rightarrow \mathbb{A}^k$ is the product of the natural maps (contracting the \mathbb{P}^1 components)

$$C_1(U_i, U_{i+1}) \rightarrow A_{U_i, U_{i+1}}^2, \quad C_2(U_j, U_{j+1}) \rightarrow \mathbb{A}_{U_j, U_{j+1}}^2$$

composed with the coordinate embedding $\mathbb{A}_{U_i, U_{i+1}, U_j, U_{j+1}}^4 \hookrightarrow \mathbb{A}^{k+1}$. This immediately implies the assertion.

In the case $i = k$ and $0 < j < k - 1$, we use a similar decomposition of $p^{-1}R_{kj}$, where $C_1(U_k, U_{k+1})$ is a nodal chain with the 2 components: $C_1(U_k, U_{k+1}) = \mathbb{A}_{U_k}^1 \cup \mathbb{A}_{U_{k+1}}^1$. In the case $1 < i < k$ and $j = 0$, we use a similar decomposition

$$p^{-1}\bar{R}_{i0} \simeq C_1(U_i, U_{i+1}) \times C_2(U_1),$$

where $C_2(U_1)$ is a nodal chain with 2 components: $C_2(U_1) = \mathbb{A}_{U_1}^1 \cup \mathbb{P}^1$. In the case $i = k$ and $j = 0$ we use

$$p^{-1}\bar{R}_{k0} \simeq C_1(U_1, U_{k+1}) \times C_2(U_1).$$

On the other hand, for $1 < i < k$,

$$p^{-1}R_{i,i-1} = Z \cup Z', \quad \text{where}$$

$$Z \simeq C_1(U_{i-1}) \times C_2(U_i, U_{i+1}), \quad Z' \simeq C_1(U_{i-1}, U_i) \times C'_2(U_{i+1}),$$

$$Z \cap Z' \simeq C_1(U_{i-1}) \times C'_2(U_{i+1}),$$

where $C_1(U_{i-1}) = \mathbb{A}_{U_{i-1}}^1 \cup \mathbb{P}^1$ and $C'_2(U_{i+1}) = \mathbb{P}^1 \cup \mathbb{A}_{U_{i+1}}^1$. Now the exact triangle

$$R\pi_*\mathcal{O}_{p^{-1}R_{i,i-1}} \rightarrow R\pi_*\mathcal{O}_Z \oplus R\pi_*\mathcal{O}_{Z'} \rightarrow R\pi_*\mathcal{O}_{Z \cap Z'} \rightarrow \dots$$

easily gives the result.

Similarly for $i = 1$ we use

$$p^{-1}R_{1,0} = Z \cup Z', \quad \text{where}$$

$$Z \simeq C_1(U_1) \times C_2(U_0), \quad Z' \simeq C_1(U_0, U_1) \times \mathbb{P}^1,$$

$$Z \cap Z' \simeq C_1(U_1) \times \mathbb{P}^1.$$

For $i = k$ we use

$$p^{-1}R_{k,k-1} = Z \cup Z', \quad \text{where}$$

$$Z \simeq \mathbb{A}_{U_{k+1}}^1 \times C_2(U_{k-1}, U_k), \quad Z' \simeq C_1(U_k, U_{k+1}) \times C'_2(U_{k-1}),$$

$$Z \cap Z' \simeq \mathbb{A}_{U_{k+1}}^1 \times C'_2(U_{k-1}).$$

□

Similarly we compute endomorphisms of Q_j .

Lemma 5.3.4. *For $0 < j < k$, the embedding $k[X_1, X_2, U_j, U_{j+1}] \subset \mathcal{O}(\bar{\mathcal{X}}_{2,k})$ induces an isomorphism*

$$\text{End}(Q_j) \simeq k[X_1, X_2, U_j, U_{j+1}]/(X_1X_2, U_jU_{j+1}).$$

Similarly,

$$\text{End}(Q_k) \simeq k[X_1, X_2, U_k]/(U_kX_1X_2),$$

$$\text{End}(Q_0) \simeq k[U_1, X_1, X_2]/(X_1X_2).$$

It follows from Lemmas 5.3.3 and 5.3.4 that for of our generators $G = P_{ij}$ or $G = Q_j$ the algebra $\text{End}(G)$ is a quotient of the algebra $A = \mathbf{k}[U_1, \dots, U_k, X_1, X_2]$. Let us set for any pair of our generators G, G' ,

$$\text{End}(G, G') := \text{End}(G) \otimes_A \text{End}(G').$$

The morphism spaces $\text{Hom}(G, G')$ and $\text{Hom}(G', G)$ are $\text{End}(G, G')$ -modules, and we will show that they are either 0 or free of rank 1 over $\text{End}(G, G')$.

5.3.2. *Morphisms between P_{ij} 's.* Looking at the supports we see that $\text{Hom}(P_{ij}, P_{i'j'}) = 0$ unless $|i' - i| \leq 1$ and $|j' - j| \leq 1$. The nonzero morphisms are computed in the Lemma below.

Lemma 5.3.5. (i) *For $i > j$, we have generators*

$$\alpha_{ij} \in \text{Hom}^{-1-2(k-j)}(P_{i,j+1}, P_{i,j}), \quad \beta_{ij} \in \text{Hom}^{1+2(k-j)}(P_{i,j}, P_{i,j+1})$$

such that $\alpha_{ij}\beta_{ij} = \beta_{ij}\alpha_{ij} = U_{j+1}$, and such that the spaces

$\text{Hom}^*(P_{i,j+1}, P_{i,j}) = \text{End}(P_{i,j}, P_{i,j+1}) \cdot \alpha_{ij}$, $\text{Hom}^*(P_{i,j}, P_{i,j+1}) = \text{End}(P_{i,j}, P_{i,j+1}) \cdot \beta_{ij}$ are free $\text{End}(P_{i,j}, P_{i,j+1})$ -modules. The generator α_{ij} (resp., β_{ij}) comes from the exact triangle

$$\mathcal{O}_{R_{i,j}}(-2 - 2(k-j)) \xrightarrow{c_{2,j}} \mathcal{O}_{R_{i,j} \cup R_{i,j+1}} \rightarrow \mathcal{O}_{R_{i,j+1}} \xrightarrow{\alpha_{ij}} \mathcal{O}_{R_{i,j}}[1](-2 - 2(k-j))$$

(resp., from the triangle

$$\mathcal{O}_{R_{i,j+1}}(2(k-j)) \xrightarrow{c_{2,j+1}^{-1}} \mathcal{O}_{R_{i,j} \cup R_{i,j+1}} \rightarrow \mathcal{O}_{R_{i,j}} \xrightarrow{\beta_{ij}} \mathcal{O}_{R_{i,j+1}}[1](2(k-j)),$$

well defined near $R_{ij} \cap R_{i,j+1}$.

(ii) *Similarly, for $i > j$, we have generators*

$$\gamma_{ij} \in \text{Hom}^{-1}(P_{i+1,j}, P_{i,j}), \quad \delta_{ij} \in \text{Hom}^1(P_{i,j}, P_{i+1,j})$$

such that $\gamma_{ij}\delta_{ij} = \delta_{ij}\gamma_{ij} = U_{i+1}$, and such that the spaces

$\text{Hom}^*(P_{i+1,j}, P_{i,j}) = \text{End}(P_{i,j}, P_{i+1,j}) \cdot \gamma_{ij}$, $\text{Hom}^*(P_{i,j}, P_{i+1,j}) = \text{End}(P_{i,j}, P_{i+1,j}) \cdot \delta_{ij}$ are free $\text{End}(P_{i,j}, P_{i+1,j})$ -modules. For $i < k$, the generator γ_{ij} (resp., δ_{ij}) has the form

$$\gamma_{ij} = \gamma'_{ij} \cdot \frac{f_1^* p_{i+1}^* z_1}{f_1^* p_i^* z_0},$$

$$(\text{ resp., } \delta_{ij} = \delta'_{ij} \cdot \frac{f_1^* p_i^* z_0}{f_1^* p_{i+1}^* z_1}),$$

where γ'_{ij} (resp., δ'_{ij}) comes from the exact triangle

$$\mathcal{O}_{R_{i,j}}(-2 - 2(k-i)) \xrightarrow{c_{1,i}} \mathcal{O}_{R_{i,j} \cup R_{i+1,j}} \rightarrow \mathcal{O}_{R_{i+1,j}} \xrightarrow{\gamma'_{ij}} \mathcal{O}_{R_{i,j}}[1](-2 - 2(k-i))$$

(resp., from the triangle

$$\mathcal{O}_{R_{i+1,j}}(2(k-i)) \xrightarrow{c_{1,i+1}^{-1}} \mathcal{O}_{R_{i,j} \cup R_{i+1,j}} \rightarrow \mathcal{O}_{R_{i,j}} \xrightarrow{\delta'_{ij}} \mathcal{O}_{R_{i+1,j}}[1](2(k-i)),$$

well defined near $R_{i+1,j} \cap R_{i+1,j}$. For $i = k$, we have

$$\gamma_{kj} = \gamma'_{kj} \cdot \frac{1}{f_1^* p_{k-1}^* z_0}, \quad \delta_{kj} = \delta'_{0j} \cdot f_1^* p_{k-1}^* z_0,$$

where γ'_{kj} and δ'_{kj} come from the above triangles with $i = k$.

(iii) *For $i - j > 1$, the space $\text{Hom}(P_{i+1,j+1}, P_{ij})$ (resp., $\text{Hom}(P_{ij}, P_{i+1,j+1})$) is a free $\text{End}(P_{ij}, P_{i+1,j+1})$ -module generated by $\gamma_{i,j+1}\alpha_{ij} = \alpha_{i+1,j}\gamma_{ij}$ (resp., $\delta_{ij}\beta_{i+1,j} = \beta_{ij}\delta_{i,j+1}$).*

The similar result holds for $\text{Hom}(P_{i+1,j-1}, P_{ij})$ (resp., $\text{Hom}(P_{ij}, P_{i+1,j-1})$). The space $\text{Hom}(P_{i+1,i+2}, P_{i,i+1})$

(resp., $\text{Hom}(P_{i,i+1}, P_{i+1,i+2})$) is a free $\text{End}(P_{i,i+1}, P_{i+1,i+2})$ -module generated by $\gamma_{i,i+2}\alpha_{i,i+1}$ (resp., $\beta_{i,i+1}\delta_{i,i+2}$).

Proof. As before, we first compute the sheaves of homomorphisms using the complete intersection equations for \overline{R}_{ij} and Lemma 2.3.6. Furthermore, when calculating $\text{Hom}(P_{ij}, P_{i'j'})$ we only need to study the situation in a neighborhood of $(\overline{R}_{ij} \cap \overline{R}_{i'j'}) \times \mathbb{A}^k$.

For example, let us compute $\underline{\text{Hom}}(P_{i,j+1}, P_{i,j})$, where $i < k$. We can work near $(R_{i,j} \cap R_{i,j+1}) \times \mathbb{A}^k$, so we can replace $f_2^*s_1(j+1)$ with V_{j+2} and $f_2^*s_2(j+1)$ with $C_{2,j}$ in the equations of $R_{i,j+1}$. Similarly, we can replace $f_2^*s_1(j)$ with $C_{2,j+1}^{-1}$ and $f_2^*s_2(j)$ with V_j in the equations of $R_{i,j}$. Thus, we split W as

$$W = \sum_{r \neq i, i+1, j+1} V_r U_r + f_1^*s_1(i) \cdot f_1^*p_i^*z_1 \cdot U_{i+1} + f_1^*s_2(i) \cdot f_1^*p_i^*z_0 \cdot U_i + C_{2,j+1}^{-1} C_{2,j} \cdot U_{j+1},$$

and then take the tensor product of the following sheaves of homomorphisms between Koszul matrix factorizations:

- for $r \neq i, i+1, j+1$, $\underline{\text{End}}(\{U_r, V_r\}) \simeq \mathcal{O}/(U_r, V_r)$;
- $\underline{\text{End}}(\{f_1^*p_i^*z_1 \cdot U_{i+1}, f_1^*s_1(i)\}) \simeq \mathcal{O}/(f_1^*p_i^*z_1 \cdot U_{i+1}, f_1^*s_1(i))$;
- $\underline{\text{End}}(\{f_1^*p_i^*z_0 \cdot U_i, f_1^*s_2(i)\}) \simeq \mathcal{O}/(f_1^*p_i^*z_0 \cdot U_i, f_1^*s_2(i))$;
- $\underline{\text{Hom}}(\{C_{2,j+1}^{-1} \cdot U_{j+1}, C_{2,j}\}, \{C_{2,j} \cdot U_{j+1}, C_{2,j+1}^{-1}\}) \simeq \mathcal{O}/(C_{2,j+1}^{-1}, C_{2,j})[-1]$.

Thus, we get an isomorphism on $\overline{R}_{ij} \cap \overline{R}_{i,j+1} \times \mathbb{A}^k$,

$$\underline{\text{Hom}}(P_{i,j+1}, P_{i,j}) \simeq \mathcal{O}/((U_r \mid r \neq i, i+1, j+1), f_1^*p_i^*z_1 \cdot U_{i+1}, f_1^*p_i^*z_0 \cdot U_i)[-1] \simeq \mathcal{O}_{C(U_i, U_{i+1}) \times \mathbb{A}_{U_{j+1}}^1}[-1],$$

where $C(U_i, U_{i+1})$ is the nodal chain with the components $\mathbb{A}_{U_i}^1$, \mathbb{P}^1 , and $\mathbb{A}_{U_{i+1}}^1$. Thus, we get

$$\text{Hom}(P_{i,j+1}, P_{i,j}) \simeq \mathbf{k}[U_i, U_{i+1}, U_{j+1}]/(U_i U_{i+1})[-1] = \text{End}(P_{i,j+1}, P_{i,j})[-1].$$

Chasing the generators from Lemma 2.3.6 we get the statement in this case.

In the case of $\underline{\text{Hom}}(P_{i+1,j}, P_{i,j})$ we take the tensor product of the sheaves of homomorphisms,

- for $r \neq i+1, j, j+1$, $\underline{\text{End}}(\{U_r, V_r\}) \simeq \mathcal{O}/(U_r, V_r)$;
- $\underline{\text{End}}(\{f_2^*p_j^*z_1 \cdot U_{j+1}, f_2^*s_1(j)\}) \simeq \mathcal{O}/(f_2^*p_j^*z_1 \cdot U_{j+1}, f_2^*s_1(j))$;
- $\underline{\text{End}}(\{f_2^*p_j^*z_0 \cdot U_j, f_2^*s_2(j)\}) \simeq \mathcal{O}/(f_2^*p_j^*z_0 \cdot U_j, f_2^*s_2(j))$;
- $\underline{\text{Hom}}(\{C_{1,i+1}^{-1} \cdot U_{i+1}, C_{1,i}\}, \{C_{1,i} \cdot U_{i+1}, C_{1,i+1}^{-1}\}) \simeq \mathcal{O}/(C_{1,i+1}^{-1}, C_{1,i})[-1]$;

(we can omit the twists by line bundles in the definition of P_{ij} and $P_{i+1,j}$ since we are working near $\overline{R}_{ij} \cap \overline{R}_{i+1,j}$). Using trivializations of the relevant line bundles over $\overline{R}_{ij} \cap \overline{R}_{i+1,j} \times \mathbb{A}^k$, we deduce the result in this case.

Similarly we consider other Hom-spaces in (i) and (ii). Let us show how a similar computation goes in (iii). For example, for $i-j > 1$ we compute $\underline{\text{Hom}}(P_{i+1,j+1}, P_{i,j})$ in the neighborhood of $(R_{i,j} \cap R_{i+1,j+1}) \times \mathbb{A}^k$ as the tensor product of

- for $r \neq i+1, j+1$, $\underline{\text{End}}(\{U_r, V_r\}) \simeq \mathcal{O}/(U_r, V_r)$;
- $\underline{\text{Hom}}(\{C_{1,i+1}^{-1} \cdot U_{i+1}, C_{1,i}\}, \{C_{1,i} \cdot U_{i+1}, C_{1,i+1}^{-1}\}) \simeq \mathcal{O}/(C_{1,i+1}^{-1}, C_{1,i})[-1]$;
- $\underline{\text{Hom}}(\{C_{2,j+1}^{-1} \cdot U_{j+1}, C_{2,j}\}, \{C_{2,j} \cdot U_{j+1}, C_{2,j+1}^{-1}\}) \simeq \mathcal{O}/(C_{2,j+1}^{-1}, C_{2,j})[-1]$.

Recall that $R_{ij} \cap R_{i+1,j+1}$ is a single point $p_{i+1,j+1}$. Thus, we get an isomorphism on $p_{i+1,j+1} \times \mathbb{A}^k$,

$$\underline{\text{Hom}}(P_{i+1,j+1}, P_{i,j}) \simeq \mathcal{O}_{p_{i+1,j+1} \times \mathbb{A}_{U_{i+1}, U_{j+1}}^2}[-2],$$

so

$$\text{Hom}(P_{i+1,j+1}, P_{i,j}) \simeq \mathbf{k}[U_{i+1}, U_{j+1}][-2] \simeq \text{End}(P_{i,j}, P_{i+1,j+1})[-2]$$

and we can chase the generator in the above calculation. \square

5.3.3. *Morphisms between Q_i 's.* We have the following analog of Lemma 5.3.5. The proof is analogous (but simpler), so we omit it.

Lemma 5.3.6. (i) For $j \geq 0$, we have generators

$$\alpha_j \in \text{Hom}^{-1-2(k-j)}(Q_{j+1}, Q_j), \quad \beta_j \in \text{Hom}^{1+2(k-j)}(Q_j, Q_{j+1})$$

such that $\alpha_j \beta_j = \beta_j \alpha_j = U_{j+1}$, and such that

$$\text{Hom}^*(Q_{j+1}, Q_j) = \text{End}(Q_j, Q_{j+1}) \cdot \alpha_j, \quad \text{Hom}^*(Q_j, Q_{j+1}) = \text{End}(Q_j, Q_{j+1}) \cdot \beta_j$$

in a way compatible with the $\text{End}(Q_j) - \text{End}(Q_{j+1})$ -bimodule structure. The generator α_j (resp., β_j) comes from the exact triangle

$$\mathcal{O}_{R_{*j}}(-2 - 2(k-j)) \xrightarrow{c_{2,j}} \mathcal{O}_{R_{*j} \cup R_{*j+1}} \rightarrow \mathcal{O}_{R_{*j+1}} \xrightarrow{\alpha_j} \mathcal{O}_{R_{*j+1}}[1](-2 - 2(k-j))$$

(resp., from the triangle

$$\mathcal{O}_{R_{*j+1}}(2(k-j)) \xrightarrow{c_{2,j+1}^{-1}} \mathcal{O}_{R_{*j} \cup R_{*j+1}} \rightarrow \mathcal{O}_{R_{*j}} \xrightarrow{\beta_j} \mathcal{O}_{R_{*j+1}}[1](2(k-j)),$$

well defined near $R_{*j} \cap R_{*j+1}$.

5.3.4. *Morphisms between Q_j and P_{ij} .*

Lemma 5.3.7. For $k > i > j$ and any r , one has $\text{Hom}(P_{ij}, Q_r) = \text{Hom}(Q_r, P_{ij}) = 0$.

Proof. We only have to consider the cases $r = j - 1$, j or $j + 1$. In addition, we can use the vanishing criterion for the proper morphism $\overline{\mathcal{X}}_{2,k} \rightarrow \mathbb{A}_{U_1, \dots, U_k, V_1, \dots, V_k, X_1, X_2}^{2k+2}$ as in Lemma 2.3.4(ii). Namely, by this Lemma, assuming that either $j > 0$ or $r > 0$, we obtain that the vanishing of $\text{Hom}(Q_r, P_{ij})$ implies the vanishing of $\text{Hom}(P_{ij}, Q_r)$. Indeed, the local computation shows that $\underline{\text{Hom}}(P_{ij}, Q_r)$ is a push-forward of a coherent sheaf from $R_{ij} \cap R_{*r}$, so we can use the fact that for $j > 0$ (resp., $r > 0$), the relative canonical bundle has trivial restriction to R_{ij} (resp., R_{*j}) (which follows from Corollary 2.4.3).

However, in the case $r = j = 0$, we have to deal with both Hom-spaces.

Case $r = j$, $\text{Hom}(Q_j, P_{ij})$. As before, we use the equations of \overline{R}_{*j} and Lemma 2.3.6. For $i - 1 > j > 0$, we use the splitting (5.2) of W , and consider the tensor product of sheaf homomorphisms between the following Koszul matrix factorizations, twisted by $f_1^* p_i^* \mathcal{O}(-1)$ (due to the definition of P_{ij}),

- for $r \neq i, i + 1, j, j + 1$, $\underline{\text{End}}(\{U_r, V_r\}) \simeq \mathcal{O}/(U_r, V_r)$;
- $\underline{\text{Hom}}(\{U_{i+1}, V_{i+1}\}, \{f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* s_1(i)\}) \simeq \mathcal{O}/(U_{i+1}, f_1^* s_1(i))$;
- $\underline{\text{Hom}}(\{U_i, V_i\}, \{f_1^* p_i^* z_0 \cdot U_i, f_1^* s_2(i)\}) \simeq \mathcal{O}/(U_i, f_1^* s_2(i))$;
- $\underline{\text{End}}(\{f_2^* p_j^* z_1 \cdot U_{j+1}, f_2^* s_1(j)\}) \simeq \mathcal{O}/(f_2^* p_j^* z_1 \cdot U_{j+1}, f_2^* s_1(j))$;

- $\underline{\text{End}}(\{f_2^* p_j^* z_0 \cdot U_j, f_2^* s_2(j)\}) \simeq \mathcal{O}/(f_2^* p_j^* z_0 \cdot U_j, f_2^* s_2(j))$.

Thus, we get an isomorphism on $\overline{R}_{ij} \times \mathbb{A}^k$,

$$\underline{\text{Hom}}(Q_j, P_{ij}) \simeq \mathcal{O}/((U_r \mid r \neq j, j+1), U_{j+1} \cdot f_2^* p_j^* z_1, f_2^* p_j^* z_0 \cdot U_j) \otimes f_1^* p_i^* \mathcal{O}(-1) = \mathcal{O}_Z(-1),$$

where $Z \simeq \mathbb{P}^1 \times C$, with $\mathcal{O}(-1)$ coming from the factor \mathbb{P}^1 . Here, for $j > 0$, C is the nodal curve $C_2(U_j, U_{j+1})$, as in the proof of Lemma 5.3.3. For $j = 0$, we have $C = C_2(U_0)$. Calculating the cohomology of $\mathcal{O}(-1)$ on this product we get zero.

Case $r = j = 0$, $\text{Hom}(P_{i0}, Q_0)$. The sheaf $\underline{\text{Hom}}(P_{i0}, Q_0) \otimes f_1^* p_i^* \mathcal{O}(-1)$ is given by the tensor product of

- $\underline{\text{End}}(\{U_r, V_r\}) \simeq \mathcal{O}/(U_r, V_r)$, for $r \neq 0, i, i+1$;
- $\underline{\text{Hom}}(\{f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* s_1(i)\}, \{U_{i+1}, V_{i+1}\}) \simeq \mathcal{O}/(f_1^* s_1(i), U_{i+1}) \otimes f_1^* p_i^* \mathcal{O}(-1)$;
- $\underline{\text{Hom}}(\{f_1^* p_i^* z_0 \cdot U_i, f_1^* s_2(i)\}, \{(U_i, V_i)\}) \simeq \mathcal{O}/(f_1^* s_2(i), U_i) \otimes f_1^* p_i^* \mathcal{O}(-1)$;
- $\underline{\text{End}}(\{f_2^* p_0^* z_1 \cdot U_1, f_2^* s_1(0)\}) \simeq \mathcal{O}/(f_2^* p_0^* z_1 \cdot U_1, f_2^* s_1(0))$,

where for $i = 1$ we replace the last two factors with

$$\underline{\text{Hom}}(\{f_1^* p_1^* z_0 \cdot f_2^* p_0^* z_1 \cdot U_1, s_1\}, \{f_2^* p_0^* z_1 \cdot U_1, f_2^* s_1(0)\}) \simeq \mathcal{O}/(f_2^* p_0^* z_1 \cdot U_1, s_1) \otimes f_1^* p_1^* \mathcal{O}(-1).$$

This leads to an isomorphism over $\overline{R}_{i0} \times \mathbb{A}^k$,

$$\underline{\text{Hom}}(P_{i0}, Q_0) \simeq \mathcal{O}/((U_r \mid r \neq 0), f_2^* p_0^* z_1 \cdot U_1) \otimes f_1^* p_i^* \mathcal{O}(-1) \simeq \mathcal{O}_Z(-1),$$

where $Z = \mathbb{P}^1 \times C_2(U_0)$, with $\mathcal{O}(-1)$ coming from the factor \mathbb{P}^1 . Hence, the cohomology of this sheaf vanish.

Case $r = j + 1$, $\text{Hom}(Q_{j+1}, P_{ij})$. Assume first that $i > j + 1$. We can work locally near $(R_{ij} \cap R_{*,j+1}) \times \mathbb{A}^k$, so we can replace $f_2^* s_1(j)$ with $C_{2,j+1}^{-1}$ and $f_2^* s_2(j)$ with V_j (resp., (resp., $f_2^* s_1(j+1)$ with V_{j+2} and $f_2^* s_2(j+1)$ with $C_{2,j}$) in the equations of R_{ij} (resp., $R_{*,j+1}$). Thus, $\underline{\text{Hom}}(Q_{j+1}, P_{ij}) \otimes f_1^* p_i^* \mathcal{O}(1)$ is the tensor product of

- $\underline{\text{End}}(\{U_r, V_r\}) \simeq \mathcal{O}/(U_r, V_r)$, for $r \neq i, i+1, j+1$;
- $\underline{\text{Hom}}(\{U_{i+1}, V_{i+1}\}, \{f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* s_1(i)\}) \simeq \mathcal{O}/(U_{i+1}, f_1^* s_1(i))$;
- $\underline{\text{Hom}}(\{U_i, V_i\}, \{f_1^* p_i^* z_0 \cdot U_i, f_1^* s_2(i)\}) \simeq \mathcal{O}/(U_i, f_1^* s_2(i))$;
- $\underline{\text{Hom}}(\{C_{2,j+1}^{-1} \cdot U_{j+1}, C_{2,j}\}, \{C_{2,j} \cdot U_{j+1}, C_{2,j+1}^{-1}\}) \simeq \mathcal{O}/(C_{2,j+1}^{-1}, C_{2,j})[-1]$.

Hence, we get an isomorphism on $(R_{ij} \cap R_{*,j+1}) \times \mathbb{A}^k$,

$$\underline{\text{Hom}}(Q_{j+1}, P_{ij}) \simeq \mathcal{O}/(U_r \mid r \neq j+1) \otimes f_1^* p_i^* \mathcal{O}(-1)[-1] = \mathcal{O}_{(R_{ij} \cap R_{*,j+1}) \times \mathbb{A}_{U_{j+1}}^1} \otimes f_1^* p_i^* \mathcal{O}(-1)[1], \quad (5.3)$$

which has vanishing cohomology.

In the case $i = j + 1$ we can use

$$\dots, V_{j+3}, f_1^* s_1(j+1), s_{j+1}, V_j, \dots$$

as equations of $R_{j+1,j}$ and

$$\dots, V_{j+3}, V_{j+2}, f_2^* p_j^* z_1, V_j, \dots$$

as equations of $Q_{*,j+1}$, so $\underline{\text{Hom}}(Q_{j+1}, P_{j+1,j}) \otimes f_1^* p_i^* \mathcal{O}(1)$ is the tensor product of

- $\underline{\text{End}}(\{U_r, V_r\}) \simeq \mathcal{O}/(U_r, V_r)$, for $r \neq j+1, j+2$;
- $\underline{\text{Hom}}(\{U_{j+2}, V_{j+2}\}, \{f_1^* p_{j+1}^* z_1 \cdot U_{j+2}, f_1^* s_1(j+1)\}) \simeq \mathcal{O}/(U_{j+2}, f_1^* s_1(j+1))$;
- $\underline{\text{Hom}}(\{s_{j+1} f_1^* p_{j+1}^* z_0 \cdot U_{j+1}, f_2^* p_j^* z_1\}, \{f_1^* p_{j+1}^* z_0 f_2^* p_j^* z_1 \cdot U_{j+1}, s_{j+1}\}) \simeq \mathcal{O}/(f_2^* p_j^* z_1, s_{j+1}) \otimes f_2^* p_j^* \mathcal{O}(1)[-1]$.

Taking into account the trivialization of $f_2^* p_j^* \mathcal{O}(1)$, we see that isomorphism (5.3) still holds, and so the Hom-space vanishes.

Case $r = j - 1$, $\text{Hom}(Q_{j-1}, P_{ij})$. We work locally near $(R_{ij} \cap R_{*,j-1}) \times \mathbb{A}^k$, so we can use

$$\dots, V_{i+2}, f_1^* s_1(i), f_1^* s_2(i), V_{i-1}, \dots, V_{j+1}, C_{2,j-1}, V_{j-1}, \dots$$

as equations of R_{ij} (even in the case $i = j + 1$), and

$$\dots, V_{j+1}, C_{2,j}^{-1}, V_{j-1}, \dots$$

as equations of $R_{*,j-1}$. Thus, we can compute $\underline{\text{Hom}}(Q_{j-1}, P_{ij}) \otimes f_1^* p_i \mathcal{O}(1)$ as the tensor product of

- $\underline{\text{End}}(\{U_r, V_r\}) \simeq \mathcal{O}/(U_r, V_r)$, for $r \neq i, i+1, j$;
- $\underline{\text{Hom}}(\{U_{i+1}, V_{i+1}\}, \{f_1^* p_i^* z_1 \cdot U_{i+1}, f_1^* s_1(i)\}) \simeq \mathcal{O}/(U_{i+1}, f_1^* s_1(i))$;
- $\underline{\text{Hom}}(\{U_i, V_i\}, \{f_1^* p_i^* z_0 \cdot U_i, f_1^* s_2(i)\}) \simeq \mathcal{O}/(U_i, f_1^* s_2(i))$;
- $\underline{\text{Hom}}(\{C_{2,j-1} \cdot U_j, C_{2,j}^{-1}\}, \{C_{2,j}^{-1} \cdot U_j, C_{2,j-1}\}) \simeq \mathcal{O}/(C_{2,j}^{-1}, C_{2,j-1})[-1]$.

This leads to

$$\underline{\text{Hom}}(Q_{j-1}, P_{ij}) \simeq \mathcal{O}_{(R_{ij} \cap R_{*,j-1}) \times \mathbb{A}_{U_j}^1} \otimes f_1^* p_i^* \mathcal{O}(-1)[1],$$

which has vanishing cohomology. \square

Lemma 5.3.8. (i) For $j < k$, the space $\text{Hom}(Q_j, P_{kj})$ is freely generated over $\text{End}(Q_j, P_{kj})$ by the generator δ_j of degree 0 that corresponds to the morphism

$$\mathcal{O}_{R_{*j}} \rightarrow \mathcal{O}_{R_{kj}}.$$

On the other hand, the space $\text{Hom}(P_{kj}, Q_j)$ is freely generated over $\text{End}(Q_j, P_{kj})$ by the generator γ_j of degree 0 that corresponds to the morphism

$$\mathcal{O}_{R_{kj}} \xrightarrow{X_1?} \mathcal{O}_{R_{*j}}.$$

We have $\gamma_j \delta_j = \delta_j \gamma_j = X_1$.

(ii) For $j \leq k$, the space $\text{Hom}(Q_j, P_{k,j-1})$ (resp., $\text{Hom}(P_{k,j-1}, Q_j)$) is freely generated over $\text{End}(Q_j, P_{k,j-1})$ by $\delta_{j-1} \alpha_{j-1}$ (resp., $\beta_{j-1} \gamma_{j-1}$), where we use generators from Lemma 5.3.6 and from part (i). For $j < k - 1$ the space $\text{Hom}(Q_j, P_{k,j+1})$ (resp., $\text{Hom}(P_{k,j+1}, Q_j)$) is freely generated over $\text{End}(Q_j, P_{0,j+1})$ by $\delta_{j+1} \beta_j$ (resp., $\alpha_j \gamma_{j+1}$).

(iii) For $j < k$, we have $\delta_{j-1} \alpha_{j-1} = \alpha_{k,j-1} \delta_j$ in $\text{Hom}(Q_j, P_{k,j-1})$ (resp., $\beta_{j-1} \gamma_{j-1} = \gamma_j \beta_{k,j-1}$ in $\text{Hom}(P_{k,j-1}, Q_j)$). We also have $\delta_j \beta_{j-1} = \beta_{k,j-1} \delta_{j-1}$ in $\text{Hom}(Q_{j-1}, P_{kj})$ (resp., $\alpha_{j-1} \gamma_j = \gamma_{j-1} \alpha_{k,j-1}$ in $\text{Hom}(P_{kj}, Q_{j-1})$).

Proof. (i) As in Lemma 5.3.7, we get an isomorphism on $\overline{R}_{kj} \times \mathbb{A}^k$,

$$\underline{\text{Hom}}(Q_j, P_{kj}) \simeq \mathcal{O}_{p^{-1} \overline{R}_{kj}} / (U_k).$$

Now the locus $U_k = 0$ in $p^{-1}\overline{R}_{kj}$ is the product $\mathbb{A}_{X_1}^1 \times C_2(U_j, U_{j+1})$ for $j > 0$, and $\mathbb{A}_{X_1}^1 \times C_2(U_1)$ for $j = 0$. The space of global functions on this locus coincides with $\text{End}(Q_j, P_{kj})$.

Assume now that $j < k - 1$. Then as before, we get an isomorphism on $\overline{R}_{*j} \times \mathbb{A}^k$,

$$\begin{aligned} \underline{\text{Hom}}(P_{kj}, Q_j) &\simeq \mathcal{O}/(U_r \mid r \neq k, j, j+1) \otimes \mathcal{O}/(U_k, C_{1,k-1}) \otimes \\ &\mathcal{O}/(f_2^* p_j^* z_1 \cdot U_{j+1}, f_2^* p_j^* z_0 \cdot U_j) \simeq \mathcal{O}_{p^{-1}\overline{R}_{kj}}/(U_k), \end{aligned}$$

which is the same sheaf as before.

In the case $j = k - 1$, the sheaf $\underline{\text{Hom}}(P_{k,k-1}, Q_{k-1})$ is the tensor product of

- $\underline{\text{End}}(\{U_r, V_r\}) \simeq \mathcal{O}/(U_r, V_r)$, for $r < k - 1$;
- $\underline{\text{Hom}}(\{X_1 f_2^* p_{k-1}^* z_1 \cdot U_k, s_k\}, \{f_2^* p_{k-1}^* z_1 \cdot U_k, f_2^* s_{k-1}(1)\}) \simeq \mathcal{O}/(f_2^* p_{k-1}^* z_1 \cdot U_k, s_1)$;
- $\underline{\text{End}}(\{f_2^* p_{k-1}^* z_0 \cdot U_{k-1}, f_2^* s_2(k-1)\}) \simeq \mathcal{O}/(f_2^* p_{k-1}^* z_0 \cdot U_{k-1}, f_2^* s_2(k-1))$,

so we get an isomorphism on $R_{k,k-1} \times \mathbb{A}^k$,

$$\underline{\text{Hom}}(P_{k,k-1}, Q_{k-1}) \simeq \mathcal{O}/((U_r \mid r < k - 1), f_2^* p_{k-1}^* z_1 \cdot U_k, f_2^* p_{k-1}^* z_0 \cdot U_{k-1}) = \mathcal{O}_Z,$$

where $Z \simeq \mathbb{A}_{X_1}^1 \times C_2(U_{k-1}, U_k)$. Hence,

$$\text{Hom}(P_{k,k-1}, Q_{k-1}) \simeq \mathbf{k}[X_1, U_{k-1}, U_k]/(U_{k-1}U_k) = \text{End}(P_{k,k-1}, Q_{k-1}).$$

(ii) Similarly to Lemma 5.3.7 we get the following local isomorphisms

$$\underline{\text{Hom}}(Q_j, P_{k,j-1}) \simeq \underline{\text{Hom}}(P_{k,j-1}, Q_j) \simeq \mathcal{O}_{(R_{k,j-1} \cap R_{*j}) \times \mathbb{A}_{U_j}^1}[-1],$$

$$\underline{\text{Hom}}(Q_j, P_{k,j+1}) \simeq \underline{\text{Hom}}(P_{k,j+1}, Q_j) \simeq \mathcal{O}_{(R_{k,j+1} \cap R_{*j}) \times \mathbb{A}_{U_{j+1}}^1}[-1].$$

Chasing the generators from Lemma 2.3.6 we get the assertion.

(iii) The relation $\delta_{j-1}\alpha_{j-1} = \alpha_{k,j-1}\delta_j$ comes from the morphism of exact triangles

$$\begin{array}{ccccccc} \mathcal{O}_{R_{*j-1}} & \xrightarrow{c_{2,j-1}} & \mathcal{O}_{R_{*j-1} \cup R_{*j}} & \longrightarrow & \mathcal{O}_{R_{*j}} & \xrightarrow{\alpha_{j-1}} & \mathcal{O}_{R_{*j-1}}[1] \\ \delta_{j-1} \downarrow & & \downarrow & & \downarrow \delta_j & & \downarrow \delta_{j-1} \\ \mathcal{O}_{R_{0,j-1}} & \xrightarrow{c_{2,j-1}} & \mathcal{O}_{R_{k,j-1} \cup R_{k,j}} & \longrightarrow & \mathcal{O}_{R_{k,j}} & \xrightarrow{\alpha_{k,j-1}} & \mathcal{O}_{R_{k,j-1}}[1] \end{array}$$

The other relations are proved similarly. \square

5.3.5. *Recovering the multigrading; shifted generators.* Recall that we are interested in the \mathbb{L}_B -graded category of \tilde{T} -equivariant matrix factorizations

$$\mathcal{B}_{2,k} := \text{MF}_{\tilde{T}}(\overline{\mathcal{X}}_{2,k}, \mathbf{w}_{2,k}),$$

and our generating matrix factorizations P_{ij} and Q_j have natural lifts to this category (recall that we equip $f_1^* p_i^* \mathcal{O}(1)$ with the T -equivariant structure, so that $f_1^* p_i^* z_0$ has weight 0).

The group \tilde{H} of characters of the torus \tilde{T} is a free abelian group with generators $x_1, x_2, u_1, \dots, u_k, l_0$. Hence, the grading group $\mathbb{L}_B = 2\tilde{H} + \mathbb{Z}l_0$ is free on generators

$2x_1, 2x_2, 2u_1, \dots, 2u_k, l_0$. So far, we computed morphisms in the \mathbb{Z} -graded category obtained from $\mathcal{B}_{2,k}$ by collapsing the \mathbb{L}_B -grading via the homomorphism $f : \mathbb{L}_B \rightarrow \mathbb{Z}$ given by

$$f(2x_1) = f(2x_2) = f(2u_i) = 0, f(l_0) = 1.$$

for which we have $f(2v_i) = 2$, $f(2c_{1i}) = f(2c_{2i}) = 2(k - i)$, and $f(2x_3) = 2k$.

Now for all morphisms between the generators (P_{ij}, Q_j) in this \mathbb{Z} -graded category we can identify morphisms in $\mathcal{B}_{2,k}$ that they come from, and in this way we will determine completely morphisms between our generators in $\mathcal{B}_{2,k}$.

We start with endomorphisms of P_{ij} and Q_j . Recall that all of them are generated by the multiplication with functions X_1, X_2 and U_i . Hence, we lift them to the same endomorphisms in $\mathcal{B}_{2,k}$, and they acquire the natural (doubled) grading, so X_i will have the grading $2x_i$ and U_i will have the grading $2u_i$.

The exact triangles in Lemmas 5.3.5, 5.3.8 and 5.3.6 show that the generators of morphisms between (P_{ij}) have the following lifts:

$$\begin{aligned} \alpha_{ij} &\in \text{Hom}^{l_0 - 2c_{2,j}}(P_{i,j+1}, P_{i,j}), \quad \alpha_j \in \text{Hom}^{l_0 - 2c_{2,j}}(Q_{j+1}, Q_j), \\ \gamma_{ij} &\in \text{Hom}^{l_0 - 2c_{1,i} + 2c_{1,i+1}}(P_{i+1,j}, P_{i,j}) \quad \text{for } i < k, \quad \gamma_{kj} \in \text{Hom}^{l_0 - 2c_{1,k-1}}(P_{k,j}, P_{k-1,j}) \\ \delta_j &\in \text{Hom}^0(Q_j, P_{kj}). \end{aligned}$$

The gradings of the remaining generators are determined by the relations and by the gradings of the endomorphisms.

Now let us define the following shifted generators in $\mathcal{B}_{2,k}$:

$$\begin{aligned} \tilde{Q}_j &:= Q_j[k - j] \otimes (-2c_{2,j} - 2c_{2,j+1} - \dots - 2c_{2,k-1}), \\ \tilde{P}_{ij} &:= P_{ij}[2k - i - j] \otimes (-2c_{1,i} - 2c_{2,j} - 2c_{2,j+1} - \dots - 2c_{2,k-1}). \end{aligned}$$

Then the morphisms α_{ij} , α_j , γ_{ij} and δ_j induce morphisms of degree 0 between these shifted generators.

Theorem 5.3.9. *The \mathbb{L}_B -graded A_∞ -algebra of endomorphisms of $\bigoplus \tilde{Q}_j \oplus \bigoplus \tilde{P}_{ij}$ is formal. The \mathbb{L}_B -grading on its cohomology is uniquely determined by the condition that the morphisms α_{ij} , α_j , γ_{ij} and δ_j have degree 0, while X_r (resp., U_r), viewed as an endomorphism of \tilde{Q}_j or \tilde{P}_{ij} , has degree $2x_r \in \mathbb{L}_B$ (resp., $2u_r \in \mathbb{L}_B$).*

Proof. Let us denote by S a multiplicatively closed subset of arrows in $\text{End}(\bigoplus \tilde{Q}_j \oplus \bigoplus \tilde{P}_{ij})$ (viewed as a quiver with relations), which includes all arrows X_r and U_r (forming a loop at some vertex), as well as the arrows α_{ij} , α_j , γ_{ij} and δ_j . The computations in Sec. 5.3.1, 5.3.2, 5.3.4, 5.3.3 show that for every other generator t of the End-algebra, there exists an element $s \in S$ such that st is a nonzero element of S . Hence, the grading of the entire End-algebra is uniquely determined by the grading of the elements of S .

The formality statement follows from Lemma 2.1.3, due to the fact that with the respect to the \mathbb{Z} -grading induced by $f : \mathbb{L}_B \rightarrow \mathbb{Z}$, the subset S has degree 0. \square

5.4. **Localization for $n = 2$.** As in Section 4.2, we denote by $\mathcal{B}_{2,k}^{\mathbb{Z}}$ (resp., $\overline{\mathcal{B}}_{2,k}^{\mathbb{Z}}$) the category of \mathbb{Z} -graded matrix factorizations on $\mathbf{w}_{2,k}$ on $\mathcal{X}_{2,k}$ (resp., $\overline{\mathcal{X}}_{2,k}$), corresponding to the grading

$$|U_i| = 2, \quad |X_i| = |C_{ij}| = |V_i| = 0.$$

We have equivalences

$$\mathcal{B}_{2,k}^{\mathbb{Z}} \simeq D^b \text{Coh}(\mathcal{Z}_{2,k}), \quad \overline{\mathcal{B}}_{2,k}^{\mathbb{Z}} \simeq D^b \text{Coh}(\overline{\mathcal{Z}}_{2,k}),$$

and we can identify $D^b \text{Coh}(\mathcal{Z}_{2,k})$ with the quotient of $\overline{\mathcal{B}}_{2,k}^{\mathbb{Z}}$ by the subcategory of matrix factorizations supported on the boundary of $\overline{\mathcal{Z}}_{2,k}$,

$$D := f_2^{-1}(p) = \text{crit}(w) \cap (\overline{\mathcal{Y}}_{2,k} \setminus \mathcal{Y}_{2,k}),$$

where p is the infinite point on $\overline{R}_0 = \mathbb{P}^1 \subset \overline{\mathcal{Y}}_{1,k}$.

Recall that D is a nodal chain,

$$D = D'_0 \cup D_0 \cup D_1 \cup \dots \cup D_k,$$

where $D_i = \overline{R}_{i0} \setminus R_{i0}$ for $i > 0$ and $D'_0 \cup D_0 = \overline{R}_{00} \setminus R_{00}$. Thus, by Lemma 4.1.4, our subcategory is generated by the matrix factorizations corresponding to the structure sheaves of the components D_0, \dots, D_k , as well as the structure sheaf of D . Actually, the same proof shows that the objects

$$\mathcal{O}_D, \mathcal{O}_{D_0}(-1), \dots, \mathcal{O}_{D_k}(-1)$$

generate $D^b \text{Coh}(D)$.

Similarly to Lemma 4.2.1 we can find the resolutions for these objects. Let us set for brevity $\mathcal{O}(H_i^v) := f_1^* p_i^* \mathcal{O}(1)$, $\mathcal{O}(H_i^h) := f_2^* p_i^* \mathcal{O}(1)$ (here v stands for ‘‘vertical’’, h stands for ‘‘horizontal’’). Let us set $\overline{R}_{i,\leq j} := \cup_{j' \leq j} \overline{R}_{ij'}$, $\overline{R}_{i*} := \cup_j \overline{R}_{ij}$, $\overline{R}_{\leq i,j} := \cup_{i' \leq i} \overline{R}_{i'j}$, and consider the objects, for $i < j$,

$$\begin{aligned} P_{i,\leq j} &:= \mathcal{O}_{\overline{R}_{i,\leq j}}(H_{j-1}^h + \dots + H_0^h - H_i^v), \\ P_{i,\leq i} &:= \mathcal{O}_{\overline{R}_{i*}}(H_i^h + \dots + H_0^h - H_i^v), \\ P'_{i,\leq j} &:= \mathcal{O}_{\overline{R}_{i,\leq j}}(H_j^h + \dots + H_1^h - H_i^v), \\ P'_{i,\leq i} &:= \mathcal{O}_{\overline{R}_{i*}}(H_i^h + \dots + H_1^h - H_i^v), \\ P_{\leq i,j} &:= \mathcal{O}_{\overline{R}_{\leq i,j}}(-H_i^v), \quad P_{ii} = P_{\leq i,i} := \mathcal{O}_{\overline{R}_{ii}}(H_i^h - H_i^v), \\ P'_{\leq i,j} &:= \mathcal{O}_{\overline{R}_{\leq i,j}}, \quad P'_{ii} = P'_{\leq i,i} := \mathcal{O}_{\overline{R}_{ii}}. \end{aligned}$$

Note that for $i > 0$,

$$P_{i,\leq 0} = P'_{i,\leq 0} = P_{i0}, \quad P_{\leq k,j} = P'_{\leq k,j} = Q_j.$$

It is also convenient to set $P'_{ij} = P_{ij}$ for $i > j$.

Lemma 5.4.1. (i) We have the following exact triangles, for $i > j$,

$$\begin{aligned} P_{ij}[-1] &\rightarrow P_{\leq i-1, j} \rightarrow P_{\leq i, j} \rightarrow \dots, \\ P'_{\leq i, j} &\rightarrow P'_{\leq i-1, j} \rightarrow P_{ij}[1] \rightarrow \dots, \end{aligned}$$

and for $i \geq j$,

$$\begin{aligned} P_{i, \leq j-1} &\rightarrow P_{i, \leq j} \rightarrow P_{ij} \rightarrow \dots, \\ P'_{ij} &\rightarrow P'_{i, \leq j} \rightarrow P'_{i, \leq j-1} \rightarrow \dots, \end{aligned}$$

which recursively express the objects $(P_{\leq i, j})$, $(P_{i, \leq j})$ in terms of the generators. Finally, we have the exact triangles, for $i > 0$,

$$P'_{i, \leq i} \rightarrow P_{i, \leq i} \rightarrow \mathcal{O}_{D_i}(-1),$$

and

$$P'_{00} \rightarrow P_{00} \rightarrow \mathcal{O}_{D_0}(-1).$$

(ii) One has $\mathcal{O}_D \simeq f_2^* \mathcal{O}_p$, $Q_j = f_2^* \mathcal{O}_{R_j}$, so the pull-back of the exact triangles of Lemma 4.2.1 under f_2 give a resolution for \mathcal{O}_D .

Proof. These are pretty straightforward. The only nontrivial point is to use the following relations between divisors on R_{ii} (and \bar{R}_{00}), which is isomorphic to the blow up of a point on $\mathbb{A}^1 \times \mathbb{P}^1$. Let us set $E_i^v := R_{ii} \cap R_{i+1, i}$, $E_i^h := R_{ii} \cap R_{i, i-1}$. Then one has

$$\mathcal{O}_{R_{ii}}(E_i^v) \simeq \mathcal{O}_{R_{ii}}(-E_i^h) \simeq \mathcal{O}_{R_{ii}}(H_i^v - H_i^h).$$

□

6. MATCHING THE A-SIDE WITH THE B-SIDE

6.1. Equivalence of the Fukaya category with one stop with the compactified LG-model. Here we prove Theorem 1.4.2.

First, we observe that we can match the grading groups. Namely, for this we use canonical decompositions

$$\begin{aligned} \mathbb{L} &= \mathbb{Z} \oplus H, \\ \mathbb{L}_B &= \mathbb{Z} \cdot l_0 \oplus 2H, \end{aligned}$$

where $H = H_1(\Sigma)$ (see Sec. 3.1). The isomorphism $\mathbb{L} \rightarrow \mathbb{L}_B$ sends (n, h) to $nl_0 + 2h$. Recall that by definition the homomorphism $\sigma : \mathbb{L} \rightarrow \mathbb{Z}/2$ sends H to 0, hence our isomorphism is compatible with the homomorphisms to $\mathbb{Z}/2$.

In the case $n = 1$, we consider the correspondence between the generating objects

$$P_i \leftrightarrow L_i, \quad i = 0, \dots, k.$$

It is easy to see that it extends to the isomorphism between the corresponding (cohomology) endomorphism algebras, so that endomorphisms U_i and X_1 on the B-side correspond to u_i and x_1 on the A-side, and $a_i : P_i \rightarrow P_{i-1}$ (resp., $b_i : P_{i-1} \rightarrow P_i$) corresponds to $r_i : L_i \rightarrow L_{i-1}$ (resp. $\ell_i : L_{i-1} \rightarrow L_i$). It is easy to see that there is a unique (up to common shift by an element of \mathbb{L}) choice of gradings of L_i , so that the gradings match. Since both endomorphism algebras are formal, the equivalence follows.

In the case $n = 2$, we consider the following correspondence between the generating objects

$$\begin{aligned} P_{ij} &\leftrightarrow L_i \times L_j, \quad \text{for } 0 \leq j < i \leq k, \\ Q_j &\leftrightarrow L_j \times L_{k+1}, \quad \text{for } 0 \leq j \leq k. \end{aligned}$$

A straightforward check shows that the morphisms on both sides match, where the generators u_i (resp., x_1, x_2) on the A-side correspond to U_i (resp., X_1, X_2) on the B-side.

By [14, Thm. 3.2.5(iii)], up to shifts of the generators, the grading on the A-side is uniquely determined by the degrees of u_r and x_r . Hence, by choosing appropriate shifts, we can make it match with the grading on the B-side. Again both endomorphism algebras are formal, so the equivalence follows.

6.2. Matching the localizations. Now we can deduce Theorem 1.2.2 by localization.

6.2.1. Case $n = 1$. We have to match the resolution (3.2) for the object T in the Fukaya category with the resolution for \mathcal{O}_p obtained in Lemma 4.2.1. The exact triangles of that Lemma show recursively the following correspondence between the A-side and the B-side:

$$\begin{aligned} \mathcal{O}_{R_{[0,i]}}(1, \dots, 1, 0) &\leftrightarrow [L_i \xrightarrow{r_i} \dots \xrightarrow{r_1} L_0], \quad i < k, \\ \mathcal{O}_{R_{[0,k]}}(1, 1, \dots, 1) &\leftrightarrow [L_k \xrightarrow{r_k} \dots \xrightarrow{r_1} L_0], \\ \mathcal{O}_{R_{[0,i]}}(0, 1, \dots, 1) &\leftrightarrow [L_0 \xrightarrow{\ell_1} \dots \xrightarrow{\ell_i} L_i], \quad i \leq k. \end{aligned}$$

Finally, we can identify the morphism

$$\mathcal{O}_{R_{0,k}}(0, 1, \dots, 1) \rightarrow \mathcal{O}_{R_{[0,k]}}(1, \dots, 1)$$

in the resolution for \mathcal{O}_p with the degree 0 morphism of twisted complexes

$$[L_0 \rightarrow \dots \rightarrow L_k] \rightarrow [L_k \rightarrow \dots \rightarrow L_0]$$

given by $x_1 : L_k \rightarrow L_k$. The cone of this morphism is exactly the resolution of the arc T , which therefore corresponds to \mathcal{O}_p .

6.2.2. Case $n = 2$. We will match the resolutions of $L_i \times T$ obtained from (3.3) with the resolutions in Lemma 5.4.1. The exact triangles of that Lemma show recursively the following correspondences for $i \leq k$:

- for $j = 0, \dots, i-1$, $P_{i,\leq j} \leftrightarrow [L_{i,j} \rightarrow \dots \rightarrow L_{i,0}]$,
- for $r = k-1, \dots, i$, $P_{\leq r,i} \leftrightarrow [L_{k+1,i} \rightarrow L_{k,i}[-1] \rightarrow \dots \rightarrow L_{r+1,i}[-1]]$,
- for $j = 0, \dots, i-1$, $P'_{i,\leq j} \leftrightarrow [L_{i,0} \rightarrow \dots \rightarrow L_{i,j}]$,
- for $r = k-1, \dots, i$, $P'_{\leq r,i} \leftrightarrow [L_{r+1,i}[1] \rightarrow \dots \rightarrow L_{k,i}[1] \rightarrow L_{k+1,i}]$.

Next, using the exact triangles for $P_{i,\leq i}$ and $P'_{i,\leq i}$ we sew together the above complexes and get the correspondences

$$\begin{aligned} P_{i,\leq i} &\leftrightarrow [L_{k+1,i} \rightarrow L_{k,i}[-1] \rightarrow \dots \rightarrow L_{i+1,i}[-1] \rightarrow L_{i,i-1} \rightarrow \dots \rightarrow L_0], \\ P'_{i,\leq i} &\leftrightarrow [L_{i,0} \rightarrow \dots \rightarrow L_{i,i-1} \rightarrow L_{i+1,i}[1] \rightarrow \dots \rightarrow L_{k,i}[1] \rightarrow L_{k+1,i}]. \end{aligned}$$

Finally, using the exact triangle for $\mathcal{O}_{D_i}(-1)$, we see that it corresponds to the complex (3.3) representing $T \times L_i$.

For $i = k + 1$, similarly to the case $n = 1$, we check that the complex (3.4) corresponds to the resolution for \mathcal{O}_D in Lemma 5.4.1(ii).

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