

# EXAMPLES OF PLANAR TIGHT CONTACT STRUCTURES WITH SUPPORT NORM ONE

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ABSTRACT. We exhibit an infinite family of tight contact structures with the property that none of the supporting open books minimizes the genus and maximizes the Euler characteristic of the page simultaneously, answering a question of Baldwin and Etnyre in [2].

Let  $Y$  be a closed oriented 3-manifold and  $\xi$  be a contact structure on  $Y$ . Recall that an open book is a fibration  $\pi : Y - B \rightarrow S^1$  where  $B$  is an oriented link in  $Y$  such that the fibres of  $\pi$  are Seifert surfaces for  $B$ . The contact structure  $\xi$  is said to be supported by an open book  $\pi$  if  $\xi$  is the kernel of a one-form  $\alpha$  such that  $\alpha$  evaluates positively on the positively oriented tangent vectors of  $B$  and  $d\alpha$  restricts to a positive area form on each fibre of  $\pi$ . It is well known that every contact structure  $\xi$  is supported by an open book on  $Y$  and all open book decompositions of  $Y$  supporting  $\xi$  are equivalent up to positive stabilizations and destabilizations [5], but given a contact 3-manifold  $(Y, \xi)$  it is not always easy to find a “simple” supporting open book. One natural measure of simplicity comes from the Euler characteristic of a page which is decreased by stabilization. The genus of a page is another useful indicator of simplicity.

In [4], Etnyre and Ozbagci define three numerical invariants of  $\xi$ , called the *support norm*, *support genus* and *binding number*, respectively, in terms of its supporting open books:

$$\text{sn}(\xi) = \min\{-\chi(\pi^{-1}(\theta)) \mid \pi : Y - B \rightarrow S^1 \text{ supports } \xi\}$$

$$\text{sg}(\xi) = \min\{g(\pi^{-1}(\theta)) \mid \pi : Y - B \rightarrow S^1 \text{ supports } \xi\}$$

$$\text{bn}(\xi) = \min\{|B| \mid \pi : Y - B \rightarrow S^1 \text{ supports } \xi \text{ and } g(\pi^{-1}(\theta)) = \text{sg}(\xi)\},$$

where  $\theta$  is any point in  $S^1$ ,  $g(\cdot)$  is the genus, and  $|\cdot|$  is the number of components. In general, these invariants are hard to compute. It is known that  $\text{sg}(\xi) = 0$  if  $\xi$  is overtwisted [3], and in general there are obstructions for a contact structure to have support genus zero ([3, 9]). However, there is no known example of a contact structure with support genus greater than one. Even if  $\xi$  is overtwisted, it is not easy to determine  $\text{bn}(\xi)$ . Furthermore, it is known that no two of these invariants determine the third [2].

It is obvious from the above definitions that

$$\text{sn}(\xi) \leq 2\text{sg}(\xi) + \text{bn}(\xi) - 2,$$

and that equality holds when  $\text{bn}(\xi) \leq 3$ .

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In [2], Baldwin and Etnyre exhibit examples of *overtwisted* contact structures which make the above inequality strict and ask whether the inequality can be strict for tight contact structures. Here we give an infinite family of tight contact structures (exactly one of which is Stein fillable) for which this inequality is strict.

Let  $T_0$  be genus one surface with one boundary component and consider the family of diffeomorphisms  $\phi_m = (\tau_a \tau_b)^3 \tau_a^{-m-4}$ , for  $m \geq 0$ , where  $a$  and  $b$  are simple closed curves given in Figure 1 and  $\tau$  denotes the right-handed Dehn twist along the corresponding curve.

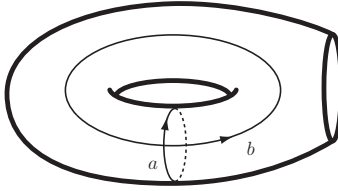


FIGURE 1.

For later use, we orient  $a$  and  $b$  so that  $a \cdot b = -1$ . Let  $(Y_m, \xi_m)$  denote the contact manifold supported by the open book decomposition  $(T_0, \phi_m)$ .

**Theorem.** *The contact structure  $\xi_0$  is Stein fillable and  $\xi_m$  is tight but not Stein fillable for  $m > 0$ . Furthermore,*

$$sn(\xi_m) = 1, \quad sg(\xi_m) = 0, \quad 3 < bn(\xi_m) \leq m + 5$$

*In particular,  $sn(\xi_m) < 2sg(\xi_m) + bn(\xi_m) - 2$ .*

*Proof.* The fact that every  $\xi_m$  is tight with nontrivial Heegaard Floer invariant  $c(Y_m, \xi_m) \in \widehat{HF}(-Y_m, \mathfrak{s}_{\xi_m})$  follows from Theorem 4.3 in [6]. Now, using the relations  $\tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b$  and  $\tau_{f(\gamma)} = f \tau_\gamma f^{-1}$  for any automorphism  $f$  and simple closed curve  $\gamma$ ,

$$\phi_0 = (\tau_a \tau_b)^3 \tau_a^{-4} = \tau_a^{-2} \tau_a \tau_b \tau_a \tau_b \tau_a \tau_b \tau_a^{-2} = \tau_a^{-1} \tau_b \tau_a \tau_a \tau_b \tau_a^{-1} = \tau_{a+b} \tau_{a-b}$$

Since it is supported by an open book whose monodromy is a product of right-handed Dehn twists,  $\xi_0$  is Stein fillable. In general, we have  $\phi_m = \tau_{a+b} \tau_{a-b} \tau_a^{-m}$ . Using this factorization, we draw a handlebody diagram of a 4-manifold  $X_m$  with boundary  $Y_m$  in Figure 2.

Figure 3 describes a way to see that  $Y_m$  is diffeomorphic to the Seifert fibered 3-manifold  $M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{m+2})$ . A complete classification of tight contact structures on  $M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{m+2})$  is given in Section 4 of [8]. From their classification, it follows that all the tight contact structures on these manifolds are supported by planar open books, i.e.  $sg(\xi_m) = 0$ . In fact, we can pinpoint precisely the contact isotopy class of  $\xi_m$  from this classification by calculating a Hopf invariant,  $d_3(\xi_m)$ . Indeed, in Theorem 4.1 and Proposition 5.1 of [1] Baldwin shows that  $Y_m$  is an L-space and calculates the correction term  $d(Y_m, \mathfrak{s}_{\xi_m}) = -\frac{m}{4}$ . Since we also know that  $c(Y_m, \xi_m)$  is non-zero, and it has grading equal to  $-d_3(\xi_m)$ <sup>1</sup> by

<sup>1</sup>Here we follow the convention in [8] where the Hopf invariant is shifted by 1/2 so that it is 0 for the standard contact structure on  $S^3$ .

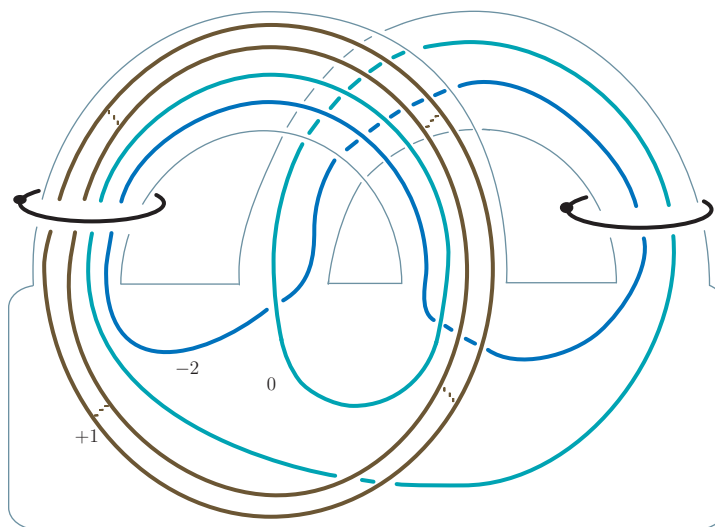


FIGURE 2. Handlebody diagram of  $X_m$  with two 1-handles,  $m + 1$ -framed 2-handles, a  $-2$ -framed 2-handle, and a  $0$ -framed 2-handle

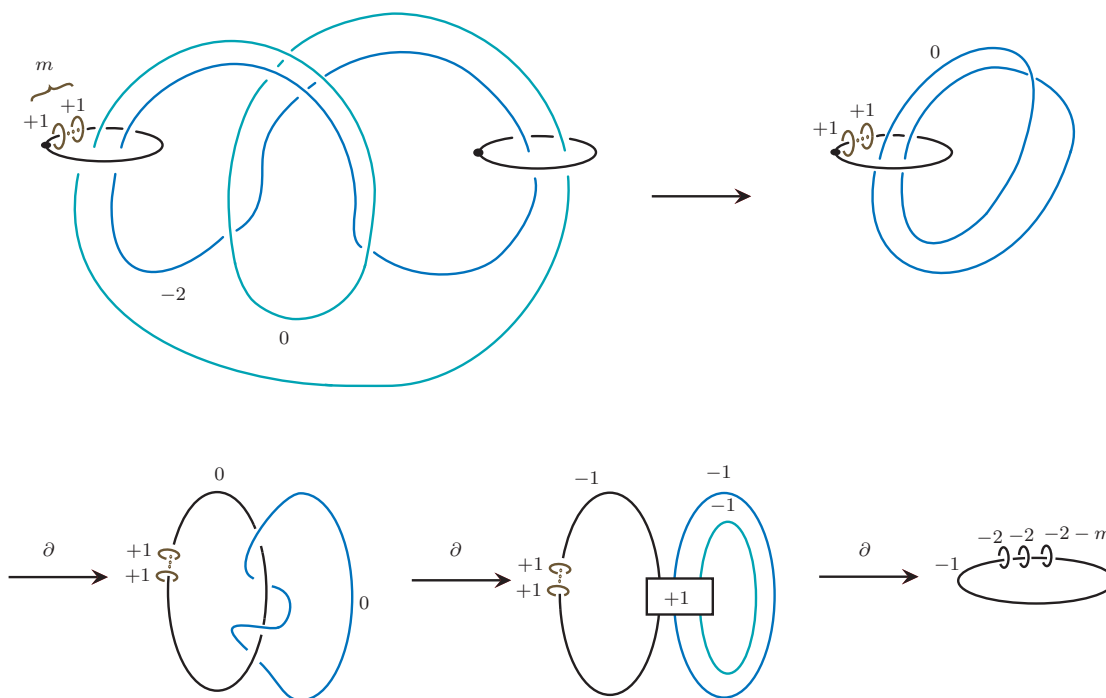


FIGURE 3. Seifert fibered 3-manifold description of  $Y_m$

Proposition 4.6 of [10], we conclude that  $d_3(\xi_m) = d(Y_m, \mathfrak{s}_{\xi_m}) = -\frac{m}{4}$ . (Note that this calculation can also be done by drawing a contact surgery diagram associated with  $\phi_m$ ).

For each  $m > 0$  there are three tight contact structures on  $Y_m$  [8] exactly one of which has  $d_3$  invariant equal to  $-\frac{m}{4}$ , and it is given by the contact surgery diagram in Figure 4. Note that the fact that  $\xi_m$  is Stein fillable if and only if  $m = 0$  follows from Theorem 4.13 in [8].

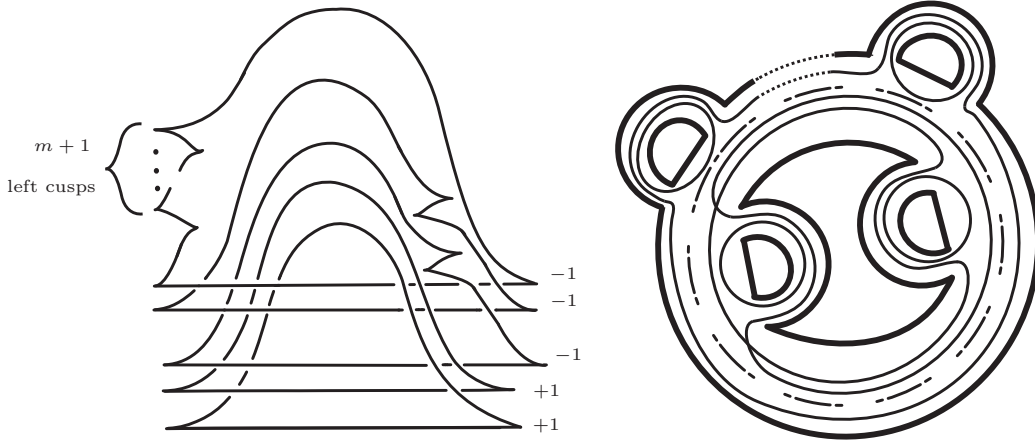


FIGURE 4. On the left: Contact surgery diagram of  $\xi_m$  as given by Figure 7 in [8]. On the right: A planar open book supporting  $\xi_m$  with  $m + 5$  boundary components, the monodromy is negative Dehn twist around the middle dashed curve and positive Dehn twist around all the other curves.

So far, we have shown that  $\text{sg}(\xi_m) = 0$ , and  $\text{sn}(\xi_m) \leq 1$ , where the latter follows because we started with an open book supporting  $\xi_m$  with pages a genus one surface with one boundary component. Furthermore, Figure 4 gives a planar open book supporting  $\xi_m$  with  $m + 5$  boundary components, hence  $\text{bn}(\xi_m) \leq m + 5$ . Next, observe that  $\text{sn}(\xi) < 1$  implies that  $\xi$  is a contact structure on a lens space. Hence  $\text{sn}(\xi_m) = 1$ . To finish, we need to show that  $\text{bn}(\xi_m) \neq 3$ . Any 3-manifold with a planar open book with three binding components is given by a surgery diagram as in Figure 5. These are connected sums of lens spaces if  $\{0, \pm 1\} \cap \{p, q, r\} \neq \emptyset$ , and small Seifert fibered spaces with  $e_0 = \lfloor -\frac{1}{p} \rfloor + \lfloor -\frac{1}{q} \rfloor + \lfloor -\frac{1}{r} \rfloor$  otherwise. Since  $e_0(Y_m) = -1$  any open book decomposition of  $Y_m$  with planar pages and three binding components must have exactly two of  $p, q$  and  $r$  negative, and in that case the monodromy is not right-veering. Therefore, these open books cannot support the tight contact structures  $\xi_m$  by [7].  $\square$

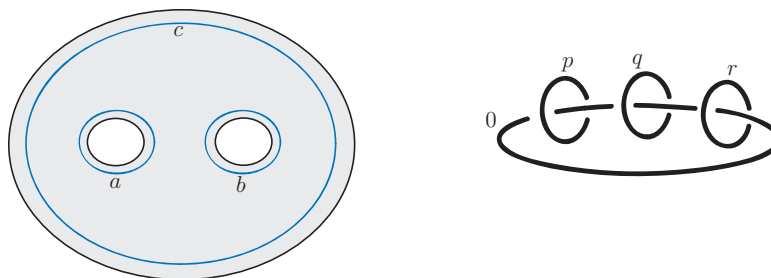


FIGURE 5. A surgery picture of the 3–manifold given by the planar open book with three binding components and monodromy  $\phi = \tau_a^p \tau_b^q \tau_c^r$ , where  $\tau$  denotes the right-handed Dehn twist along the corresponding curve

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