A SYMPLECTIC LOOK AT THE FARGUES-FONTAINE CURVE

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1. Introduction

This paper discusses homological mirror symmetry for the Fargues-Fontaine curve.

1.1. **The Fargues-Fontaine curve.** Let E be a local field and let C be a perfectoid field of characteristic p. For each such pair (E, C), Fargues and Fontaine have defined an E-scheme that we will denote by $FF_E(C)$ — it is denoted by $X_{C,E}$ in [FF, Def. 6.5.1]. It is in no sense a "curve over E" or even a variety: it is not of finite type over E or over any other field. A scheme that is not of finite type over some base cannot be smooth or proper in the usual sense ([EGA, Vol 4, §17; Vol 2, §5]) and yet $FF_E(C)$ resembles a closed Riemann surface is some peculiar ways:

- It is noetherian of Krull dimension one. Moreover it is regular, so that the local ring at each closed point of $FF_E(C)$ has a discrete valuation.
- A nonzero rational function f (that is, a section of $\mathcal{O}_{FF}$ over the generic point) has $v(f) \neq 0$ for at most finitely many of these valuations v, and $\sum_v v(f) = 0$

In fact $FF_E(C)$ even resembles the Riemann sphere: one has

$$\mathrm{Pic}(FF_E(C)) = \mathbf{Z} \text{ and } H^1(\mathcal{O}_{FF}) = 0$$

There are some contrasts with the Riemann sphere: $FF_E(C)$ has indecomposable vector bundles of higher rank, and its étale fundamental group is naturally isomorphic to the absolute Galois group of E. Fargues has a program to apply these properties of $FF_E(C)$ to the local Langlands correspondence [Fa].

When $E = \mathbf{Q}_p$, $FF_E(C)$ is an important object in p-adic cohomology — it was introduced to organize some of the structures of p-adic Hodge theory. When $E = \mathbf{F}_p(\!(z)\!)$, the analogous structures are those of Hartl [H]. We have nothing to say about $FF_{\mathbf{Q}_p}(C)$ but we are able to touch $FF_{\mathbf{F}_p(\!(z)\!)}(C)$ with mirror symmetry.

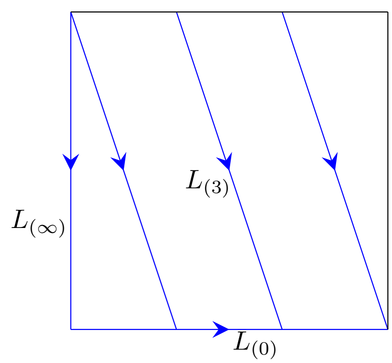
1.2. **Homological mirror symmetry.** Homological mirror symmetry (HMS) is a framework for relating Lagrangian Floer theory on a symplectic manifold to the homological algebra of coherent sheaves on a scheme — often, a scheme that is seemingly unrelated to the symplectic manifold.

What symplectic structure could be mirror to $FF_{\mathbf{F}_p(\!(z)\!)}(C)$? We suggest the answer is a two-dimensional torus. There is already a very well-studied mirror relationship between the symplectic torus and the Tate elliptic curve (over $\mathbf{Z}(\!(t)\!)$), which we review in §2. To get the Fargues-Fontaine curve in place of the Tate curve, we introduce two changes:

(1) We couple Lagrangian Floer theory to a locally constant sheaf of rings on the torus — the fiber of this sheaf of rings has characteristic p, and going around one of the circles is the pth power map. (Going around the other circle is the identity map).
(2) We set the Novikov parameter (this is the element $t \in \mathbf{Z}(\!(t)\!)$ in the ground ring of the Tate curve) to $t = 1$ — symplectically this is sort of like studying the limit as the symplectic form goes to 0.

Both of these maneuvers are unusual in symplectic geometry. The first turns out to be straightforward, so that one obtains a Fukaya A_∞-category with the usual properties. The second is much more delicate and touches some folklore questions about "convergent power series Floer homology."

1.3. **Lagrangian Floer theory on the torus.** Let T be a 2-dimensional torus, which we present as a quotient of $\mathbf{R}^2$ by $\mathbf{Z}^2$ and endow with the standard symplectic form $dx\,dy$. For each integer m, let $L_{(m)} \subset T$ denote the image of the line in $\mathbf{R}^2$ through the origin of slope $-m$. Let $L_{(\infty)}$ denote the image of the vertical line through the origin. We orient $L_{(m)}$ from left to right and $L_{(\infty)}$ from top to bottom. The figure shows $L_{(0)}$, $L_{(\infty)}$, and $L_{(3)}$ in a fundamental domain of T:



If $m_1 > m_0$, then $L_{(m_1)}$ and $L_{(m_0)}$ meet transversely in $(m_1 - m_0)$ points. Lagrangian Floer theory gives algebraic structures to the free modules

$$\mathrm{CF}(L_{(m_0)}, L_{(m_1)}) := \bigoplus_{x \in L_{(m_0)} \cap L_{(m_1)}} \Lambda, \tag{1.3.1}$$

where Λ is a suitable ring, about which more in §1.6. "CF" stands for "Floer cochains." The orientations of $L_{(m_0)}, L_{(m_1)}$ endow (1.3.1) with a $\mathbf{Z}/2$-grading (which can be lifted to a $\mathbf{Z}$-grading by making some additional topological choices), and $\mathrm{CF}^*(L_{(m_0)}, L_{(m_1)})$ supports a differential of degree $+1$ §2.5. In the case at hand these gradings are concentrated in a single degree 0, and the differential is zero.

We will return to the differential in §1.15 (and a little in §1.5), but to start we will be very interested in the "triangle product":

$$\mathrm{CF}(L_{(m_1)}, L_{(m_2)}) \times \mathrm{CF}(L_{(m_0)}, L_{(m_1)}) \to \mathrm{CF}(L_{(m_0)}, L_{(m_2)}) \tag{1.3.2}$$

whose value on $(x_1, x_2) \in (L_{(m_1)} \cap L_{(m_2)}) \times (L_{(m_0)} \cap L_{(m_1)})$ is the summation

$$\sum_{y \in L_{(m_0)} \cap L_{(m_2)}} y \left(\sum_{u \in \mathcal{M}(y, x_2, x_1)} \pm t^{\mathrm{area}(u)} \right) \tag{1.3.3}$$

The inner sum is infinite: it is indexed by the set of rigid pseudoholomorphic triangles

$$u : \Delta^2 \to T \tag{1.3.4}$$

with vertices at x_1, x_2, y and one edge each along $L_{(m_2)}, L_{(m_1)}, L_{(m_0)}$. The sign in $\pm t^{\mathrm{area}(u)}$ depends on u, and on the choice of a spin structure on each of the oriented 1-manifolds $L_{(m_i)}$, see §2.4. In the present case it is possible to make those choices so that the signs are identically $+1$.

1.4. **Dehn twist.** There is a canonical identification of $\mathrm{CF}(L_{(m)}, L_{(n)})$ with $\mathrm{CF}(L_{(0)}, L_{(n-m)})$, induced by

$$(x, y) \mapsto (x, y - mx)$$

the m-fold Dehn twist around $L_{(\infty)}$. An old suggestion of Seidel's [Z] is to use this identification to package the triangle products as a graded ring structure on the sum

$$\Lambda \oplus \bigoplus_{m=1}^{\infty} \mathrm{CF}(L_{(0)}, L_{(m)}) \tag{1.4.1}$$

The multiplication on (1.4.1) is associative and commutative for nontrivial reasons. The associativity is a consequence of a very general Floer-theoretic argument that studies 1-dimensional moduli spaces of pseudoholomorphic quadrilaterals §2.11, and the commutativity is a consequence of a more particular observation about the Dehn twist [Z, §3].

1.5. **The Floer cohomology of $(L_{(0)}, L_{(0)})$.** In (1.4.1), we have inserted the unit of the ring by hand (the summand Λ, which we place in degree zero), but this can also be motivated Floer-theoretically. The definition of CF in (1.3.1) is not the right one when $L_{(m_0)} = L_{(m_1)}$, or for any other pair that do not meet transversely. But if $\phi = \{\phi^s\}_{s \in \mathbf{R}}$ is a general Hamiltonian isotopy, then

$$\mathrm{CF}(\phi^s L_{(0)}, L_{(0)}) := \bigoplus_{x \in \phi^s L_{(0)} \cap L_{(0)}} x \cdot \Lambda$$

together with its differential, gives a cochain complex whose cohomology groups do not depend on ϕ. These cohomology groups are $\mathbf{Z}/2$-graded, the Λ piece of (1.4.1) is naturally identified with $\mathrm{HF}^0(L_{(0)}, L_{(0)})$, cf. §2.12.

1.6. **Novikov ring Λ and Floer theory relative to a divisor.** There is some flexibility in choosing the ground ring Λ, but it should contain a ring of constants (let us use C for this ring — later on it will be the same as the C of §1.1) a parameter t and all necessary powers of it, and it should carry a topology in which all the sums (1.4.1) converge. The conventional choice is the Novikov ring (2.7.3), which we will denote by Λ_C:

$$\Lambda_C = \left\{ \sum_{i=0}^{\infty} a_i t^{\lambda_i} \mid a_i \in C, \lambda_i \in \mathbf{R} \text{ and } \lim_{i\to\infty} \lambda_i = \infty \right\} \tag{1.6.1}$$

We can shrink those coefficients to $C[\![t]\!]$ by the following device of Seidel's, called Floer theory "relative to a divisor." Rather than computing the area of the triangles u, we fix a basepoint $D \in T^2$ (in general, a symplectic divisor $D \subset T^2$) and use the cardinality of $u^{-1}(D)$ in place of symplectic area. If D is in the first quadrant and extremely close to $(0,0)$, then this cardinality is given by a simple formula which is independent of D unless the triangle u is extremely acute — let us write $\mathrm{area}_{\mathbf{Z}}(u)$ for this discretized notion of area. With some additional care, by letting $D \to (0,0)$ (see [LPe2, §7.2.3] and [LPe2, Prop. 9.1]), we get a graded ring

$$C[\![t]\!] \cdot 1 \oplus \bigoplus_{m=1}^{\infty} \mathrm{CF}(L_{(0)}, L_{(m)}) \tag{1.6.2}$$

1.7. **Theorem** [LPe2]**.** The $C[\![t]\!]$-scheme

$$\mathrm{Proj}\left(C[\![t]\!] \cdot 1 \oplus \bigoplus_{m=1}^{\infty} \mathrm{CF}(L_{(0)}, L_{(m)}) \right) \tag{1.7.1}$$

is isomorphic to $E_{\mathrm{Tate}} \times_{\mathbf{Z}[\![t]\!]} C[\![t]\!]$; the Tate elliptic curve over $C[\![t]\!]$ whose Weierstrass equation is

$$y^2 + xy = x^3 - b_2 x - b_3$$

where $b_2, b_3 \in \mathbf{Z}[\![t]\!]$ are the series

$$b_2 = \sum_{n=1}^{\infty} 5n^3 \frac{t^n}{1-t^n} \qquad b_3 = \sum_{n=1}^{\infty} \left(\frac{7n^5 + 5n^3}{12} \right) \frac{t^n}{1-t^n} \tag{1.7.2}$$

1.8. **Theta series.** The relationship between (1.6.2) and functions on the Tate elliptic curve is more transparent when those functions are described in terms of θ-series. Set

$$\theta_{m,k} := \sum_{i=-\infty}^{\infty} t^{m\frac{i(i-1)}{2} + ki} z^{mi+k}; \qquad \theta_{m,k}^{\mathrm{abs}} := \sum_{i=-\infty}^{\infty} t^{(mi+k)^2/(2m)} z^{mi+k} \tag{1.8.1}$$

The simplest of these series is the Jacobi function

$$\theta_{1,0} = \sum_{i=-\infty}^{\infty} t^{\frac{i(i-1)}{2}} z^i = (1+z) \prod_{i=1}^{\infty} \left[(1-t^i)(1+t^i z)(1+t^i z^{-1}) \right]$$

The others $\theta_{m,k}$ are obtained by a change of variables from $\theta_{1,0}$. These series are doubly infinite in z, but in formally expanding the product of two of them, the coefficient of $z^i t^j$ has only finitely many nonzero contributions. The $C[\![t]\!]$-linear span of the $\theta_{m,k}$ (resp. the Λ_C-linear span the $\theta_{m,k}^{\mathrm{abs}}$) is closed under multiplication and graded by m, and is isomorphic as a graded ring to (1.6.2) (or (1.4.1) in the absolute case). The isomorphisms send $(k/m, 0) \in L_{(0)} \cap L_{(m)}$ to $\theta_{m,k}$ or to $\theta^{\mathrm{abs}}_{m,k}$.

1.9. **Fukaya category and homological mirror symmetry.** The triangle product (1.3.2) resembles a composition law in a category. It is part of a sequence of structures on the $\mathrm{CF}(L, L')$,

$$\mu_n : \mathrm{CF}(L_{n-1}, L_n) \times \cdots \times \mathrm{CF}(L_0, L_1) \to \mathrm{CF}(L_0, L_n) \tag{1.9.1}$$

that are obtained by summing over the $(n+1)$-gons with sides along $L_0, L_1, \ldots, L_n$ §2.7. When one takes extra care to treat sets of Lagrangians that are not transverse §2.12, these structures define an A_∞-category. After passing to a triangulated envelope and splitting idempotents, we will call any of these A_∞-structures a "Fukaya category" and denote it by $\mathrm{Fuk}(T)$ (in the absolute case) or $\mathrm{Fuk}(T, D)$ (in the relative case).

Kontsevich's homological mirror symmetry conjecture, specialized to T, asks for a quasi-equivalence between $\mathrm{Fuk}(T)$ and the derived category of coherent sheaves on an elliptic curve. A version of this for complex elliptic curves was obtained in [PoZa]. When $C = \mathbf{Z}$, Theorem 1.7, together with a generation result for $\mathrm{Fuk}(T, D)$ [LPe2, §6.3], constitute "homological mirror symmetry over $\mathbf{Z}$":

$$\mathrm{Fuk}(T, D) \cong D^b(\mathrm{Coh}(E_{\mathrm{Tate}}))$$

The structure sheaf of E_{Tate} is the image of $L_{(0)}$ under this equivalence.

1.10. F**-fields.** Let $\underline{\Lambda}$ be a local system of rings on T, so that at each point $x \in T$ we are given a ring $\underline{\Lambda}_x$, and along each path γ from x to y we are given a ring isomorphism

$$\nabla\gamma : \underline{\Lambda}_x \xrightarrow{\sim} \underline{\Lambda}_y \tag{1.10.1}$$

Suppose that each ring $\underline{\Lambda}_x$ has the structures that we asked for in §1.6: it contains a ring of constants (we can denote it by $\underline{C}_x$), distinguished elements of the form t^a, and carries a topology. The maps (1.10.1) should be continuous, carry each $\underline{C}_x$ to $\underline{C}_y$, but leave the elements of the form t^a alone ($\nabla\gamma(t^a) = t^a$).

We will develop a version of Floer theory "with coefficients in $\underline{\Lambda}$." As in §1.6, we could work either relative to a divisor, or absolutely. In the relative case we would take $\underline{\Lambda} := \underline{C}[\![t]\!]$, where $\underline{C}$ is a locally constant sheaf of rings. In the absolute case, we would take $\underline{\Lambda} := \Lambda_{\underline{C}}$ (1.6.1).

In the example of interest to us, the sheaf of rings is pulled back from S^1, along the projection map

$$\mathfrak{f} : T \to S^1 \tag{1.10.2}$$

Then $\underline{\Lambda}$ is determined by a ring C and an automorphism (the monodromy around the base S^1) σ of C. It induces an automorphism of $C[\![t]\!]$ and of Λ_C that fixes each t^a. We are interested in the case when C is perfect of characteristic p and σ is the pth root map.

The map (1.10.2) (and the monodromy map σ) is just for book-keeping, but it also has an "occult" interpretation, in a way fitting in to the old analogy between number fields and three-manifolds, and between primes and knots. The generator in the fundamental group of the base circle S^1 and the Frobenius in the absolute Galois group of $\mathbf{F}_p$ act on C

in the same way: by pth powers. See [T] for a little bit more about this. There is also a natural map from the set of closed points of $FF_E(C)$ (for $E = \mathbf{F}_p(\!(z)\!)$ or any other local field) to S^1, and in some sense this paper explores the idea that the SYZ mechanism for mirror symmetry could apply §4.5.

1.11. **Lagrangian Floer theory — coupled to $\underline{\Lambda}$.** Let us put

$$\mathrm{CF}(L_{(m_0)}, L_{(m_1)}; \underline{\Lambda}) := \bigoplus_{x \in L_{(m_0)} \cap L_{(m_1)}} \underline{\Lambda}_x \tag{1.11.1}$$

Lagrangian Floer theory coupled to $\underline{\Lambda}$ concerns algebraic structures on (1.11.1), for instance a triangle product

$$\mathrm{CF}(L_{(m_1)}, L_{(m_2)}; \underline{\Lambda}) \times \mathrm{CF}(L_{(m_0)}, L_{(m_1)}; \underline{\Lambda}) \to \mathrm{CF}(L_{(m_0)}, L_{(m_2)}; \underline{\Lambda}) \tag{1.11.2}$$

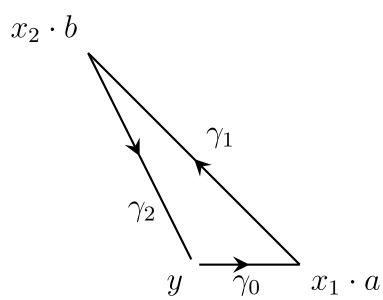
In some sense (1.11.1) is another free Λ-module on the intersection points $L_{(m_0)} \cap L_{(m_1)}$, but with many different Λ-module structures. The product (1.11.2) is not Λ-bilinear with respect to any of them. To define it we give its value on a pair

$$(x_2 \cdot b, x_1 \cdot a) \in \mathrm{CF}(L_{(m_1)}, L_{(m_2)}; \underline{\Lambda}) \times \mathrm{CF}(L_{(m_0)}, L_{(m_1)}; \underline{\Lambda}) \tag{1.11.3}$$

for any $a \in \underline{\Lambda}_{x_1}$ and $b \in \underline{\Lambda}_{x_2}$, and extend bi-additively (or more precisely, $\Lambda_{\mathbf{Z}}$-bilinearly). The value on $(x_2 \cdot b, x_1 \cdot a)$ is

$$\sum_{y \in L_{(m_0)} \cap L_{(m_2)}} y \sum_{u \in \mathcal{M}(y, x_2, x_1)} \pm t^{\mathrm{area}(u) \text{ or } \mathrm{area}_{\mathbf{Z}}(u)} \nabla\gamma_2(b \nabla\gamma_1(a \nabla\gamma_0(1))) \tag{1.11.4}$$

where $\gamma_0 : y \to x_1$, $\gamma_1 : x_1 \to x_2$, and $\gamma_2 : x_2 \to y$ are the three sides of the triangle u, appearing in counterclockwise order.



The $\pm$ signs in the formula (1.11.4) are the same as they are in (1.3.3); in particular one can arrange that they are identically $+1$.

When the monodromy of $\underline{\Lambda}$ around $L_{(\infty)}$ is trivial — equivalently, when $\underline{\Lambda}$ is pulled back along (1.10.2) — it is possible to package these triangle products into a graded ring structure

$$\bigoplus_{m=1}^{\infty} \mathrm{CF}(L_{(0)}, L_{(m)}; \underline{\Lambda}) \tag{1.11.5}$$

1.12. **Theorem.** For each $a \in C$, and each pair of integers m, k with $m > k \geq 0$, let $\theta_{m,k}[a]$ denote the formal series

$$\theta_{m,k}[a] := \sum_{i=-\infty}^{\infty} t^{m\frac{i(i-1)}{2}+ki} z^{mi+k} \sigma^i(a)$$

Let $\theta^{\mathrm{abs}}_{m,k}[a]$ denote the formal series

$$\theta^{\mathrm{abs}}_{m,k}[a] := \sum_{i=-\infty}^{\infty} t^{\frac{1}{2m}(mi+k)^2} z^{mi+k} \sigma^i(a)$$

Then the relative (resp. absolute) version of (1.11.5) is isomorphic, as a ring-without-unit, to the $\mathbf{Z}[\![t]\!]$-linear span of the $\theta_{m,k}[a]$ (resp. to the $\Lambda_{\mathbf{Z}}$-linear span of the $\theta^{\mathrm{abs}}_{m,k}[a]$).

1.13. **Specializations of t.** Fix a commutative ring C and an automorphism σ, cf. (1.10.2). The groups (1.11.1) are linear over Λ^{σ}_C in the absolute case, and over $C^{\sigma}[\![t]\!]$ in the relative case. We will discuss the specializations $t = 0$ and $t = 1$. The case $t = 0$ we treat only briefly in §4.1 — the absolute case is not of interest, while the relative case is parallel to the "large volume limit" of T, and its mirror relationship with the nodal cubic curve at the "large complex structure limit."

The case $t = 1$ is more delicate. There is a class of symplectic manifolds and Lagrangian submanifolds (for instance, monotone Lagrangians in a Fano manifold, or in a genus two surface) for which setting $t = 1$ is unproblematic, but the torus does not belong to this class. And indeed the series (1.3.3), (1.7.2) do not converge, in any archimedean or nonarchimedean ring, when $t = 1$.

An F-field can repair some (but only some) of the convergence. Define

$$\mathrm{CF}(L_{(m_0)}, L_{(m_1)}; \underline{C}) = \bigoplus_{x \in L_{(m_0)} \cap L_{(m_1)}} \underline{C}_x \tag{1.13.1}$$

Setting $t = 1$ in (1.11.4) suggests, in a formal way, a map

$$\mathrm{CF}(L_{(m_1)}, L_{(m_2)}; \underline{C}) \times \mathrm{CF}(L_{(m_0)}, L_{(m_1)}; \underline{C}) \dashrightarrow \mathrm{CF}(L_{(m_0)}, L_{(m_2)}; \underline{C})$$

When C is complete with respect to a norm $|\cdot|$, σ is the pth root map, and $m_0 < m_1 < m_2$, this map has a nontrivial domain of convergence. In particular it defines a multiplication on

$$\bigoplus_{m>0} \mathrm{CF}(L_{(0)}, L_{(m)}; \underline{\mathfrak{m}})$$

where $\mathfrak{m} = \{x \in C : |x| < 1\}$.

1.14. **The Fargues-Fontaine graded ring.** A perfect field of characteristic p, complete with respect to a norm $|\cdot|$, is since [Sc] known as a "perfectoid field of characteristic p." Suppose $(C, |\cdot|)$ is such a field, and suppose furthermore that C is algebraically closed. Let $E = \mathbf{F}_p(\!(z)\!)$, and let $B \supset E$ be the set of bi-infinite formal series $\sum_{i \in \mathbf{Z}} b_i z^i \in \prod_{i \in \mathbf{Z}} C z^i$ with coefficients $b_i \in C$, and which obey

$$\forall r \in (0,1) \qquad |b_i| r^i \to 0 \text{ as } |i| \to \infty \tag{1.14.1}$$

This ring B coincides with what is called $B_{(0,1)}$ in [FF, Ex. 1.6.5], and what is called $\mathcal{O}_{\mathbf{R}^1}((0,1))$ in [KS, Def. 21].

The automorphism $\varphi : B \to B$ given by

$$\varphi : \left(\sum c_i z^i\right) \mapsto \sum c_i^p z^i \qquad \text{(i.e. } \varphi(f(z)) = f(z^{1/p})^p) \tag{1.14.2}$$

cuts B into "eigenspaces" $B^{\varphi=z^n} := \{f \in B \mid \varphi(f) = z^n f\}$. The Fargues-Fontaine curve attached to (E, C) is

$$FF_E(C) := \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} B^{\varphi=z^n}\right) \tag{1.14.3}$$

Theorem. (4.3.2) is isomorphic as a graded-ring-without-unit to the irrelevant ideal of (1.14.3), i.e.

$$\bigoplus_{n=1}^{\infty} \mathrm{CF}(L_{(0)}, L_{(n)}; \underline{\mathfrak{m}}) \cong \bigoplus_{n=1}^{\infty} B^{\varphi=z^n} \tag{1.14.4}$$

1.15. **Annuli.** The degree zero piece $B^{\varphi=1}$ of the Fargues-Fontaine graded ring (1.14.3) is isomorphic to E ($= \mathbf{F}_p(\!(z)\!)$ in our case). The theorem (1.14.4) does not explain how this part arises Floer-theoretically. As in §1.5, it should come from the Floer cohomology of $L_{(0)}$ against itself — a version of Floer cohomology with the F-field turned on — but the usual rules for making sense of the nontransverse intersection $L_{(0)} \cap L_{(0)}$ have to be revisited when $t = 1$.

As we mentioned in §1.5, and review in §2.13, the usual rules involve choosing a Hamiltonian isotopy $\{\phi^s\}_{s\in\mathbf{R}}$ so that $\phi^s L_{(0)}$ and $L_{(0)}$ do meet transversely. The problem that we encounter is that the quasi-isomorphism type of $\mathrm{CF}(\phi^s L_{(0)}, L_{(0)}; \underline{C})$, with its bigon differential, is no longer independent of ϕ. One still has natural maps between cochain complexes for different ϕ, but they are not quasi-isomorphisms: the usual formula for the necessary cochain homotopies does not converge.

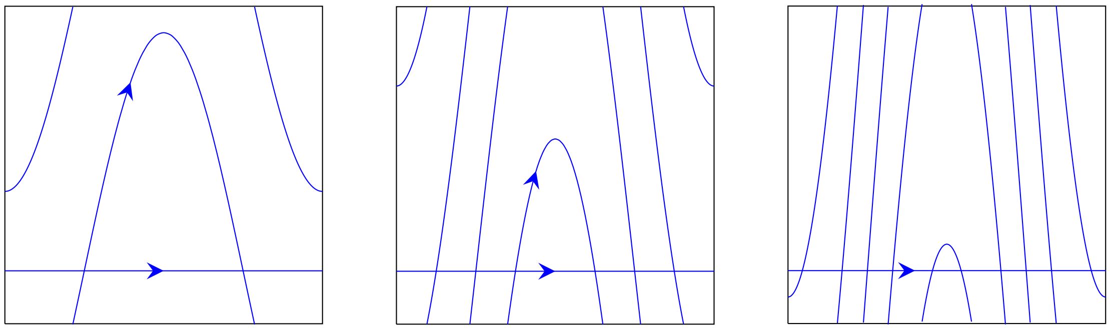
This is a well-known problem with "convergent power series Floer cohomology." It is discussed in print in [ChOh, p.3] and [Aur2, §4.2], and perhaps elsewhere, but there is not much theory available for addressing it. Still, we take the following point of view (which is only heuristic):

> "Continuation", i.e. the independence of the Hamiltonian displacement ϕ, fails because there are pseudoholomorphic annuli in T that have one side on $\phi^s L_{(0)}$ and the other side on $L_{(0)}$.

For instance, Oszváth and Szabó stick to "admissible" Heegaard diagrams to avoid problems with annuli like these [OsSz, §4.2.2]. The problem they pose in Lagrangian Floer theory is closely related to the problem that closed gradient orbits pose in circle-valued Morse theory [HuLe]. There is some speculation about incorporating them directly into Floer-theoretic invariants in [Aur2].

1.16. **Loud Floer cochains.** If L and L' are in different homology classes there are no annuli between them. But there are infinitely many annuli between $\phi^s L_{(0)}$ and $L_{(0)}$ for any

ϕ. If we fix an "autonomous" ϕ, then we can get these under control by considering larger and larger s: for s large, all of the annuli between $\phi^s L_{(0)}$ and $L_{(0)}$ have large area. For instance:



Some of the constructions of [Lee] have inspired us, here. The complexes $\mathrm{CF}(\phi^s L_{(0)}, L_{(0)}; \underline{C})$ for different s do not all have the same cohomology but there are natural cochain maps

$$\mathrm{CF}(\phi^s L_{(0)}, L_{(0)}; \underline{C}) \to \mathrm{CF}(\phi^{s'} L_{(0)}, L_{(0)}; \underline{C})$$

whenever $s' > s$. We will study the colimit of this filtered diagram. For large s the picture of $\phi^s L_{(0)}$ is a sine wave with large amplitude (wrapped up around the torus), we call

$$\mathrm{CF}_{\mathrm{loud}}(L_{(0)}, L_{(0)}; \underline{C}) := \varinjlim_{s} \mathrm{CF}(\phi^s L_{(0)}, L_{(0)}; \underline{C}) \tag{1.16.1}$$

the loud Floer cochains on $(L_{(0)}, L_{(0)})$. The name was suggested to us by Johnson-Freyd. Now our point of view is the following:

> By shouting infinitely loud, all of the annuli break, along with whatever problems they posed for noninvariance.

We will not try to make this precise, but for a somewhat analogous precedent in the setting of periodic orbits, see what is called the "Latour trick" in [Hutc]. The Latour trick breaks up the periodic orbits of a closed 1-form by adding a large multiple of an exact 1-form. One could equivalently think of pushing the graph of the closed 1-form, for a long time, by the Hamiltonian flow of a primitive for the exact form. A large *finite* multiple of the exact form suffices to break up all the periodic orbits, while in (1.16.1) one has to pass to the limit, but maybe "shouting loud" is not a worse metaphor for one process than for the other.

We will show that the triangles with sides on $L_{(0)}, \phi^s L_{(0)}, \phi^{s+s'} L_{(0)}$ induce a multiplication on $\mathrm{CF}_{\mathrm{loud}}(L_{(0)}, L_{(0)}; \underline{C})$ and on $\mathrm{HF}_{\mathrm{loud}}(L_{(0)}, L_{(0)}; \underline{C})$. Our construction of this multiplication is quite crude: a better analysis would follow the construction of an A_∞-structure on wrapped Floer cochains [AS], which we expect to apply here and give a richer structure. But our computations give an isomorphism of rings

$$\mathrm{HF}^0_{\mathrm{loud}}(L_{(0)}, L_{(0)}; \underline{C}) \cong C^{\sigma}[z, z^{-1}] \tag{1.16.2}$$

When σ is the pth root map, this is the Laurent polynomial ring $\mathbf{F}_p[z, z^{-1}]$, a dense subring of $\mathbf{F}_p(\!(z)\!)$.

One can similarly define $\mathrm{HF}_{\mathrm{loud}}(L_{(m)}, L_{(m)}; \underline{C})$, and a ring structure on it, for any $m \in \mathbf{Q}$. One gets the same answer (1.16.2) when m is an integer. For $m = d/r$, we expect but will not prove that $\mathrm{HF}^0_{\mathrm{loud}}$ is a dense subring of an r^2-dimensional division algebra over $\mathbf{F}_p(\!(z)\!)$ whose invariant is $m + \mathbf{Z} \in \mathbf{Q}/\mathbf{Z}$. The indecomposable vector bundles on the Fargues-Fontaine curve are classified by $\mathbf{Q}$ and those division algebras arise as their endomorphism rings [FF, Thm. 8.2.10].

1.17. **Acknowledgments.** We have benefitted from conversations, suggestions and correspondence from D. Auroux, B. Bhatt, A. Caraiani, K. Fukaya, M. Kim, K. Madapusi-Pera, B. Poonen and P. Seidel. YL is partially supported by the Royal Society URF\R\180024. DT was supported by NSF-DMS 1811971.

2. Some Floer-theoretic background

In this section we review some of Floer theory, making what simplifications are possible when the target manifold is a 2d torus T.

2.1. ***J*-holomorphic polygons.** Let $D \subset \mathbf{C}$ be the closed unit disk in the complex plane. We denote by D° the open unit disk and $\partial D = D - D^\circ$ the boundary of D. Let $\mathbf{z} = (z_0, \ldots, z_n)$ denote an ordered $(n+1)$-tuple of points in ∂D. We require that the points of $\mathbf{z}$ are pairwise distinct and that the counterclockwise arc subtending z_{i-1} and z_i (or z_n and z_0) does not contain any other point of $\mathbf{z}$.

Let $L_0, L_1, \ldots, L_n$ be an $(n+1)$-tuple of one-dimensional submanifolds of T, and let $(x_0, \ldots, x_n)$ be a tuple of points in T with

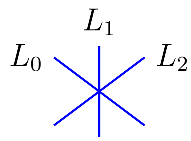
$$x_0 \in L_0 \cap L_n, \qquad x_1 \in L_1 \cap L_0, \qquad \ldots, \qquad x_n \in L_n \cap L_{n-1}$$

We write $\mathcal{W}(x_0, \ldots, x_n)$ for the set of pairs $(\mathbf{z}, u)$ where $u : D \to T$ is a continuous map, smooth away from $\mathbf{z}$, that carries z_i to x_i and that maps the counterclockwise arc between z_i and z_{i+1} into L_i. It carries a topology in which a sequence $(\mathbf{z}_i, u_i)$ converges $(\mathbf{z}, u)$ if $\mathbf{z}_i \to \mathbf{z}$ in $(\partial D)^{\times(n+1)}$ and $u_i \to u$ in a suitable Sobolev space.

Fix an almost complex structure J on T. A polygon $(\mathbf{z}, u) \in \mathcal{W}(x_0, \ldots, x_n)$ is called J-holomorphic if the differential of u is $\mathbf{C}$-linear on each tangent space of the interior D°. Write $\widetilde{\mathcal{M}}(x_0, \ldots, x_n) \subset \mathcal{W}(x_0, \ldots, x_n)$ for the subspace of J-holomorphic polygons. The group $\mathrm{PSL}_2(\mathbf{R})$ of biholomorphisms of D° acts on $\widetilde{\mathcal{M}}$ by reparametrization, and we denote the quotient by $\mathcal{M}(x_0, \ldots, x_n)$.

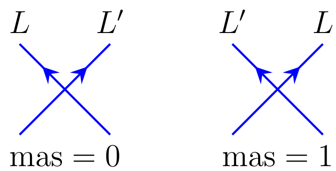
2.2. **Transversely cut criteria.** Each connected component of $\mathcal{M}(x_0, \ldots, x_n)$ is labeled by a nonnegative integer called the analytic index of the component (or of any map in the component). In case the conditions that cut $\widetilde{\mathcal{M}}$ out of $\mathcal{W}$ are transverse in a sense that we will not review here, then each component is a topological manifold and the analytic index coincides with the dimension of this component. A formula for this dimension is given below (2.3.1). When the "transversely cut" condition is satisfied, we call the components of dimension zero rigid polygons.

The "transversely cut" condition is satisfied whenever the $(L_0, \ldots, L_n)$ are in general position — that is, whenever the L_i are pairwise transverse and $L_i \cap L_j \cap L_k$ is empty. On T or another surface, this triple intersection condition can be relaxed [Se2, Lem. 13.2], for instance $\mathcal{M}$ is transversely cut in a neighborhood of u as soon as u is not constant. Even some constant maps u are transversely cut, for instance: If (L_0, L_1, L_2) are pairwise transverse, then at any triple intersection point $x \in L_0 \cap L_1 \cap L_2$, the tangent lines $(T_x L_0, T_x L_1, T_x L_2)$ come in either clockwise or counterclockwise order. A constant map $D \to L_0 \cap L_1 \cap L_2$ contributes to $\mathcal{M}(x, x, x)$ if and only if they come in clockwise order



An example of a non-transversely cut quadrilateral in T (necessarily constant) is analyzed in [LPe1, Thm. 8]. We will encounter some non-transversely cut triangles in §4.9.

2.3. **Maslov index of an intersection point.** Let (L, L') be an ordered pair of connected one-dimensional submanifolds of T, and fix an orientation of both L and L'. Suppose that L and L' meet transversely at the point x, then we define $\mathrm{mas}(x) \in \mathbf{Z}/2$ by the rule indicated in the diagram



If L and L' are not homologous to zero, one may lift the Maslov index to a $\mathbf{Z}$-valued invariant by equipping L and L' (and T) with gradings, see [LPe2, §6] — let us denote this $\mathbf{Z}$-valued Maslov index by $\mathrm{mas}_{\mathbf{Z}}(x)$. A formula for the dimension near $u \in \mathcal{M}(x_0, \ldots, x_n)$ is

$$\mathrm{mas}_{\mathbf{Z}}(x_0) - \mathrm{mas}_{\mathbf{Z}}(x_1) - \cdots - \mathrm{mas}_{\mathbf{Z}}(x_n) \tag{2.3.1}$$

2.4. **Sign of a rigid polygon.** By making some additional choices one may attach a sign to each rigid polygon with boundary on $(L_0, \ldots, L_n)$ — in other words one may define a map

$$\mathcal{M}(x_0, \ldots, x_n) \to \{1, -1\} \tag{2.4.1}$$

We recall the recipe for (2.4.1) given in [Se1, §7] — it depends on the choice of orientation for each L_i, and on the additional data of a basepoint $\star_i \in L_i$ in each L_i. One requires that $\star_i \notin L_j$ for any $j \neq i$. The point $\star_i$ endows L_i with a nontrivial spin structure (also known as bounding or Neveu-Schwarz spin structure) which is trivialized away from $\star_i$.

If $u|_{\partial D} : \partial D \to \cup_{i=0}^n L_i$ preserves the counter-clockwise orientation of D, the sign is $+1$ or -1 according to whether one encounters an even or odd number of stars going around ∂D — i.e. it is $(-1)^{\# u^{-1}\{\star_0, \ldots, \star_n\}}$. Changing the orientation of L_0 does not change this sign, changing the orientation of L_n changes the signs by $(-1)^{\mathrm{mas}(x_0) + \mathrm{mas}(x_n)}$, while changing the orientation of any of the other L_i changes the sign by $(-1)^{\mathrm{mas}(x_i)}$.

For short, we will sometimes write

$$(-1)^{|x|} := (-1)^{\mathrm{mas}(x)} \tag{2.4.2}$$

2.5. **Floer cochain complexes.** Let t be a formal variable, and let $\mathbf{Z}[t^{\mathbf{R}_{\geq 0}}]$ denote the semigroup algebra of $\mathbf{R}_{\geq 0}$, i.e. the group of finite $\mathbf{Z}$-linear combinations of symbols of the form t^a, where $a \in \mathbf{R}_{\geq 0}$, with the multiplication $t^a t^b = t^{a+b}$. Let Λ be a $\mathbf{Z}[t^{\mathbf{R}_{\geq 0}}]$-algebra. In a moment we will take Λ to be the Novikov completion of $\mathbf{Z}[t^{\mathbf{R}_{\geq 0}}]$ but the differential in $\mathrm{CF}(L, L')$ is given by a finite sum, so that in this section it might as well be $\mathbf{Z}[t^{\mathbf{R}_{\geq 0}}]$ itself.

When L and L' intersect transversely, then we let

$$\mathrm{CF}(L, L') := \bigoplus_{x \in L \cap L'} \Lambda \tag{2.5.1}$$

The choice of orientation for L and L' endows this group with a $\mathbf{Z}/2$-grading

$$\mathrm{CF} = \mathrm{CF}^0 \oplus \mathrm{CF}^1, \qquad \text{where } \mathrm{CF}^i(L, L') := \bigoplus_{x | \mathrm{mas}(x) = i} \Lambda$$

Further equipping L and L' with stars §2.4 gives us the bigon differential:

$$\mu_1 : \mathrm{CF}^i(L, L') \to \mathrm{CF}^{i+1}(L, L') : x \mapsto \sum_{y | \mathrm{mas}(y) = i+1} y \left(\sum_{u \in \mathcal{M}(x,y)} \pm t^{\mathrm{area}(u)} \right) \tag{2.5.2}$$

where the sign is given in §2.4, and $\mathrm{area}(u) := \int_D u^*(dx\, dy)$. The inner sum is finite. A general argument using non-rigid bigons shows that $\mu_1 \mu_1 = 0$ — this is a case of the A_∞-relations §2.11. In T, this can be proved more simply by lifting the grading from $\mathbf{Z}/2$ to $\mathbf{Z}$: the $\mathbf{Z}$-grading is always concentrated in only two degrees.

We have just described the "absolute" Floer cochain complex. After fixing a point $D \in T$, not on L or L', we also have a "relative to D" complex in which Λ in (2.5.1) can be shrunk to $\mathbf{Z}[t]$ (or another $\mathbf{Z}[t]$-algebra), and the expression $t^{\mathrm{area}(u)}$ in is replaced by $t^{\# u^{-1}(D)}$.

2.6. **Example — special Lagrangians.** We will call a circle $L \subset T$ a "special Lagrangian" if it is the image under $\mathbf{R}^2 \to T$ of a straight line. If that straight line has the form $y + mx = b$ then we will call m the slope of the special Lagrangian, otherwise we say L has slope ∞; thus the possible slopes are $m \in \mathbf{Q} \cup \{\infty\}$. If L is special with finite slope, let us call the orientation under the parametrization $x \mapsto (x, b - mx)$ the "default orientation."

If $L \neq L'$ are two special Lagrangians, of finite slopes m and m', then they meet transversely in a set of cardinality $|nd' - n'd|$, if $m = n/d$ and $m' = n'/d'$. All the intersection points are in a single Maslov degree; with the default orientations, these degrees are

$$\mathrm{CF}(L, L') = \begin{cases} \mathrm{CF}^0(L, L') & \text{if } m' > m \\ \mathrm{CF}^1(L, L') & \text{if } m' < m \end{cases}$$

There are no bigons and the differential (2.5.2) is zero.

2.7. **Polygon maps.** Suppose that $(L_0, \ldots, L_n)$ are in sufficiently general position that all the $\mathcal{M}(x_0, \ldots, x_n)$ are transversely cut. The $(n+1)$-gon map is a multilinear map

$$\mu_n : \mathrm{CF}^{i_n}(L_{n-1}, L_n) \times \cdots \times \mathrm{CF}^{i_1}(L_0, L_1) \to \mathrm{CF}^{i_1+\cdots+i_n+2-n}(L_0, L_n) \tag{2.7.1}$$

which carries $(x_n, x_{n-1}, \ldots, x_1)$ to

$$\sum_y y \left(\sum_{u \in \mathcal{M}(y, x_1, x_2, \ldots, x_n)} \pm t^{\mathrm{area}(u)} \right) \tag{2.7.2}$$

It is not defined until the L_i are equipped with orientations and stars. Furthermore, the inner sum in (2.7.2) is usually infinite, so Λ should carry a topology in which it converges. The standard choice for Λ is one of Λ^0 or $\Lambda^0[t^{-1}]$, where Λ^0 is the Novikov ring

$$\Lambda^0 = \Lambda^0_{\mathbf{Z}} = \left\{ \sum_{i=0}^{\infty} a_i t^{\lambda_i} \mid a_i \in \mathbf{Z}, \lambda_i \in \mathbf{R}_{\geq 0} \text{ and } \lim_{i \to \infty} \lambda_i = \infty \right\} \tag{2.7.3}$$

The fact that Λ^0 can be taken to have $\mathbf{Z}$-coefficients is a reflection of the fact that the moduli spaces $\mathcal{M}$ are not orbifolds — this holds for T and more generally for semipositive symplectic manifolds.

In the relative setting, as long as D does not lie on any L_i, we replace $t^{\mathrm{area}}(u)$ with $t^{\# u^{-1}(D)}$, and (2.7.1) is multilinear over $\mathbf{Z}[\![t]\!]$.

2.8. **Example — some triangle maps.** Suppose $L_0, \ldots, L_k$ are special Lagrangians of slopes

$$m_0 < m_1 < \cdots < m_k < \infty \tag{2.8.1}$$

If $k \neq 2$, then the set of rigid $(k+1)$-gons with boundary on $L_0, \ldots, L_k$ is empty, and the maps

$$\mu_k : \mathrm{CF}(L_{k-1}, L_k) \times \cdots \times \mathrm{CF}(L_0, L_1) \to \mathrm{CF}(L_0, L_k)$$

are zero — this is a consequence of §2.3.1. In other words when (2.8.1) is satisfied μ_2 is the only interesting polygon map. In this section we explain how to compute μ_2 in detail for Lagrangians of the form $L_{(m_0)}, L_{(m_1)}, L_{(m_2)}$ §1.3. These are the maps that can be packed into a product structure on $\bigoplus \mathrm{CF}(L_{(0)}, L_{(m)})$ §1.4, and our notation is adapted to describing this product structure.

One consequence of the vanishing of the μ_k for $k \neq 2$ is that this product is strictly associative §2.11, and we can describe the product in terms of the theta functions. If (2.8.1) is not satisfied then the maps μ_k do not all vanish for $k \geq 3$ — they can written in terms of Hecke's indefinite theta series [Poli].

If $m_1 < m_2$, there are $m_2 - m_1$ intersection points between $L_{(m_1)}$ and $L_{(m_2)}$, each contributing a basis element to $\mathrm{CF}(L_{(m_1)}, L_{(m_2)})$. Let us index those intersection points in the following way.

$$\tau^{m_1}(x_{m_2 - m_1, \kappa}) := (\kappa, -m_2 \kappa) \tag{2.8.2}$$

where $\kappa \in \{0, \frac{1}{m_2-m_1}, \ldots, \frac{m_2-m_1-1}{m_2-m_1}\}$ and τ denotes the Dehn twist map

$$\tau : (x, y) \mapsto (x, y - x)$$

Fix an irrational number ε and equip each $L_{(m)}$ with a star §2.4

$$\star_{(m)} = \star_{(m),\varepsilon} := (\varepsilon, -m\varepsilon) \tag{2.8.3}$$

The Dehn twist carries $L_{(m)}$ to $L_{(m+1)}$ and preserves the stars. As ε is irrational, the stars avoid the intersection points $L_{(m_1)} \cap L_{(m_2)}$. We will compute the triangle maps

$$\mathrm{CF}(L_{(m_1)}, L_{(m_1+m_2)}) \times \mathrm{CF}(L_{(0)}, L_{((m_1))}) \to \mathrm{CF}(L_{(0)}, L_{(m_1+m_2)})$$

by computing $\mu_2(\tau^{m_1} x_{m_2,\kappa_2}, x_{m_1,\kappa_1})$. For example

$$\mu_2(\tau x_{1,0}, x_{1,0}) = \begin{cases} x_{2,0}\Big(\sum_{i\in\mathbf{Z}} t^{i^2}\Big) + x_{2,\frac{1}{2}}\Big(\sum_{i\in\mathbf{Z}} t^{(i+\frac{1}{2})^2}\Big) & \text{(absolute setting)} \\ x_{2,0}\Big(\sum_{i\in\mathbf{Z}} t^{i^2}\Big) + x_{2,\frac{1}{2}}\Big(\sum_{i\in\mathbf{Z}} t^{i(i+1)}\Big) & \text{(relative setting)} \end{cases} \tag{2.8.4}$$

Theorem. Let

$$E(\kappa_1, \kappa_2) = E_{m_1,m_2}(\kappa_1, \kappa_2) := \frac{m_1\kappa_1 + m_2\kappa_2}{m_1 + m_2} \tag{2.8.5}$$

(which carries $\frac{1}{m_1}\mathbf{Z} \times \frac{1}{m_2}\mathbf{Z}$ into $\frac{1}{m_1+m_2}\mathbf{Z}$). Then in the absolute setting $\mu_2(\tau^{m_1} x_{m_2,\kappa_2}, x_{m_1,\kappa_1})$ is given by

$$\mu_2(\tau^{m_1} x_{m_2,\kappa_2}, x_{m_1,\kappa_1}) = \sum_{\ell\in\mathbf{Z}} x_{m_1+m_2,E(\kappa_1,\kappa_2+\ell)} t^{(\ell+\kappa_2-\kappa_1)^2/(2(\frac{1}{m_1}+\frac{1}{m_2}))} \tag{2.8.6}$$

Let the functions $\phi(s)$ and $\lambda(u, v) = \lambda_{m_1,m_2}(u, v)$ be given by (cf. [LPe2, p. 83])

$$\phi(s) := \lfloor s \rfloor s - \frac{1}{2}\lfloor s \rfloor \lfloor s+1 \rfloor; \qquad \lambda(u, v) := m_1\phi(u) + m_2\phi(v) - (m_1+m_2)\phi(E(u, v)) \tag{2.8.7}$$

Then in the relative setting $\mu_2(\tau^{m_1} x_{m_2,\kappa_2}, x_{m_1,\kappa_1})$ is given by

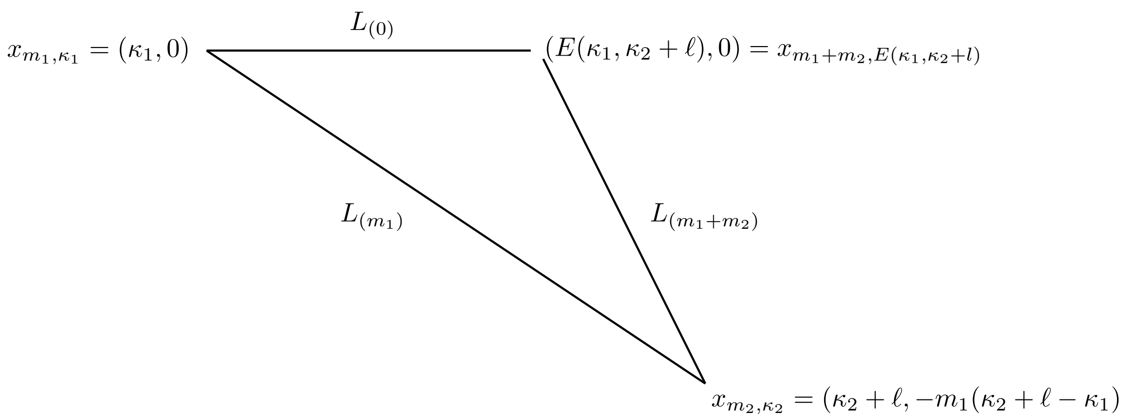
$$\mu_2(\tau^{m_1} x_{m_2,.\kappa_2}, x_{m_1,\kappa_1}) = \sum_{\ell\in\mathbf{Z}} x_{m_1+m_2,E(\kappa_1,\kappa_2+\ell)} t^{\lambda(\kappa_1,\kappa_2+\ell)} \tag{2.8.8}$$

where we understand $x_{m_1+m_2,E(\kappa_1,\kappa_2+\ell)} := x_{m_1+m_2,\kappa}$ if $E(\kappa_1, \kappa_2 + \ell) = \kappa$ modulo 1.

Proof. The index ℓ in either sum (2.8.6), (2.8.8) determines a triangle, two of whose vertices are at x_{m_1,κ_1} and $\tau^{m_1} x_{m_2,\kappa_2}$. In the universal cover, the coordinates of all three vertices are

$$(\kappa_1, 0), \qquad (E(\kappa_1, \kappa_2 + \ell), 0), \qquad (\kappa_2 + \ell, -m_1(\kappa_2 + \ell - \kappa_1))$$

as in the diagram



On any of these triangles there are an even number of stars (2.8.3): for each $i \in \mathbf{Z}$ with $\kappa_1 < \varepsilon + i < \kappa_2 + \ell$ there are exactly two stars whose x-coordinate (in the universal cover) is $\varepsilon + i$, one along $L_{(m_1)}$ and the other along either $L_{(0)}$ or $L_{(m_1+m_2)}$. Thus every summand in the triangle has sign $+1$.

The exponent of t in (2.8.6) is the area of the ℓth triangle, i.e.

$$\frac{1}{2}m_1(\kappa_2 + \ell - \kappa_1)(E(\kappa_1, \kappa_2 + \ell) - \kappa_1) = (\ell + \kappa_2 - \kappa_1)^2/(2(\frac{1}{m_1} + \frac{1}{m_2})).$$

The more complicated exponent of t in (2.8.8) is the lattice area §1.6 of the same triangle, which coincides with the cardinality of $u^{-1}(D)$ when D is in the first quadrant very close to $(0, 0)$ — see [LPe2]. □

2.9. **Theta functions.** Let $\theta_{m,k}$, $\theta^{\mathrm{abs}}_{m,k}$ be as in (1.8.1)

$$\theta_{m,k} := \sum_{i=-\infty}^{\infty} t^{m\frac{i(i-1)}{2}+ki} z^{mi+k}, \qquad \theta^{\mathrm{abs}}_{m,k} := \sum_{i=-\infty}^{\infty} t^{(mi+k)^2/(2m)} z^{mi+k}$$

The Jacobi theta function is $\theta_{1,0}$, and the others are obtained by a simple change of variables

$$\theta_{m,k}(t, z) = z^k \theta_{1,0}(t^m, t^k z^m), \qquad \theta_{m,k}(t, t^{\frac{1}{2}} z) = t^{k(m-k)/(2m)} \theta^{\mathrm{abs}}_{m,k}(t, z)$$

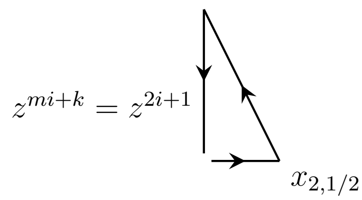
Although these series are doubly infinite in z, when formally expanding the product of $\theta_{m,k}$ and $\theta_{m',k'}$ only finitely many terms contribute to the coefficient of any monomial $z^e t^f$ — the same goes for $\theta^{\mathrm{abs}}_{m,k}$ and $\theta^{\mathrm{abs}}_{m',k'}$. This is a consequence of the convexity of the functions $i \mapsto m\binom{i}{2} + ki$ and $i \mapsto (mi+k)^2/(2m)$ in the exponent of t. That the resulting series $\theta_{m,k} \cdot \theta_{m',k'}$ or $\theta^{\mathrm{abs}}_{m,k} \cdot \theta^{\mathrm{abs}}_{m',k'}$ can be written as a linear combination of $\theta_{m+m',0}, \cdots, \theta_{m+m',m+m'-1}$ is a standard but nontrivial fact about the theta functions, the formulas for these coefficients is the same as (2.8.8) and (2.8.6): putting $\kappa_1 = k_1/m_1$ and $\kappa_2 = k_2/m_2$,

$$\theta_{m_2,k_2} \cdot \theta_{m_1,k_1} = \sum_{\ell \in \mathbf{Z}} \theta_{m_1+m_2,E(\kappa_1,\kappa_2+\ell)} t^{\lambda(\kappa_1,\kappa_2+\ell)}$$

and

$$\theta^{\text{abs}}_{m_2,k_2} \cdot \theta^{\text{abs}}_{m_1,k_1} = \sum_{\ell \in \mathbf{Z}} \theta^{\text{abs}}_{m_1+m_2,E(\kappa_1,\kappa_2+\ell)} t^{(\ell+\kappa_2-\kappa_1)^2/(2(\frac{1}{m_1}+\frac{1}{m_2}))}$$

In other words the map $x_{m,k} \mapsto \theta_{m,k}$ or $\theta^{\text{abs}}_{m,k}$ is a ring homomorphism. One may verify this directly (and we will do so in the next section when we turn on an F-field), but it is natural to ask what is the Floer-theoretic origin of these series. Each summand of (1.8.1) is indexed by a *right* triangle, with one vertex at $x_{m,k/m}$ and sides along $L_{(0)}, L_{(m)}, L_{(\infty)}$. The exponent of t carries the area (or lattice area) of this right triangle and the exponent of z carries the number of times the vertical edge of the triangle wraps around $L_{(\infty)}$ — this z can be interpreted as the monodromy of a rank one local system §3.3. For instance the right triangles contributing to $\theta_{2,1}$ have the form (for i positive)

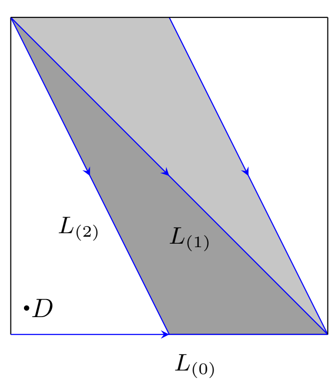


2.10. **The punctured torus, the large complex structure limit.** Part of the motivation for relative Floer theory is to make sense of the specialization $t = 0$ in the polygon sums. Setting $t = 0$ has the effect of discarding the polygons that contribute a positive power of t, which in the relative case is the same as discarding the polygons that touch D. One gets the same effect by doing Floer theory for Lagrangians in the punctured torus $T - D$. When $T - D$ is equipped with the right symplectic structure, one with infinite area in a neighborhood of D, this is called the "large volume limit" of Floer theory.

We can treat the Lagrangians $L_{(m)}$ as boundary conditions for triangles $u : \Delta^2 \to T - D$, and sum over them to obtain a map μ_2 as before, this time defined on the free $\mathbf{Z}$-modules spanned by $L_{(m_1)} \cap L_{(m_2)}$. For instance,

$$\mu_2(\tau x_{1,0}, x_{1,0}) = x_{2,0} + x_{2,1/2} \cdot 2$$

where $x_{2,0}$ comes from a constant map and the two copies of $x_{2,1/2}$ come from the two shaded triangles in the figure



We also record

$$\begin{aligned}
\mu_2(\tau^2 x_{1,0}, x_{2,0}) &= x_{3,0} + x_{3,1/3} + x_{3,2/3} \\
\mu_2(\tau^2 x_{1,0}, x_{2,1/2}) &= x_{3,1/3} + x_{3,2/3} \\
\mu_2(\tau^3 x_{3,1/3}, x_{3,2/3}) &= x_{6,3/6}
\end{aligned}$$

and

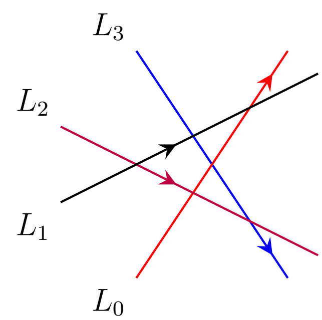
$$\mu_2(\tau^4 x_{2,1/2}, \mu_2(\tau^2 x_{2,1/2}, x_{2,1/2})) = x_{6,3/6}.$$

If one sets $z = x_{1,0}$, $x = x_{2,1/2}$, $y = x_{3,2/3}$, these equation imply in particular that

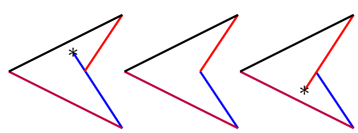
$$y^2 z + x^3 = xyz$$

This is the equation for a nodal plane cubic curve — the "large complex structure limit" that matches the large volume limit under mirror symmetry.

2.11. A_∞**-relations.** The moduli spaces $\mathcal{M}$ that parametrize non-rigid polygons are not usually compact. For example, suppose L_0, L_1, L_2, L_3 are as in the diagram



Denote the red-black, black-purple, and purple-blue intersection points by f, g, and h respectively, and the blue-red intersection point by hgf. Then $\mathcal{M}(hgf, f, g, h)$ includes the following one-parameter family of quadrilaterals:



(2.11.1)

They all have the same image closure, but along the boundary may back-track along either the blue or the red line. Near the $*$s, the map u is biholomorphic to the map from the upper half-plane to $\mathbf{C}$ that sends z to z^2. At the extreme parameters, where the $*$ reaches all the way to the black or to the purple line, there is no such J-holomorphic quadrilateral — so $\mathcal{M}$ is not compact — but each of those extremes is occupied by a pair of J-holomorphic triangles.

Since the quadrilaterals are not rigid, they do not contribute to $\mu_3(h, g, f)$. But the triangles at the extremes are rigid, at one end they contribute to $\mu_2(g, f)$ and $\mu_2(h, gf)$, and at the other end to $\mu_2(h, g)$ and $\mu_2(hg, f)$. The interpolating family of quadrilaterals exhibits a relation between them.

More generally there is a compactification (the Deligne-Mumford-Stasheff compactification) of $\mathcal{M}$. When everything is transversely cut §2.2, the compactification is a topological manifold-with-corners, whose corners are indexed by tuples of rigid polygons. Equipping the L_i with orientations and stars induces an orientation on $\mathcal{M}$ and its compactification — (2.4.1) is a special case of this orientation. The oriented compactification of $\mathcal{M}$ is used in the proof (it essentially is the proof) of the following equations among the polygon maps μ_n:

$$\sum_{i+j=n+1} \sum_{\ell<i} (-1)^{|f_1|+\cdots+|f_\ell|-\ell} \mu_i(f_n,,\ldots,f_{\ell+j+1},\mu_j(f_{\ell+j},\ldots,f_{\ell+1}),f_\ell,\ldots,f_1) = 0 \quad (2.11.2)$$

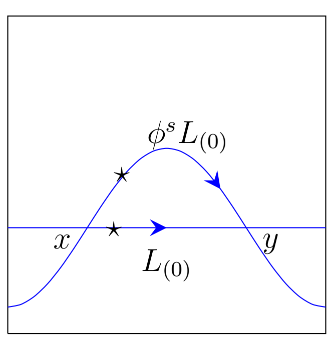
The algebraic structure formed by the $\mathrm{CF}(L,L')$ and the maps μ_n, subject to the relations (2.11.2), is called an "A_∞-precategory" in [KS, §4.3]. Each L is like an object, and each $f \in \mathrm{CF}(L,L')$ is like a morphism between objects. (2.11.2) expresses the fact that these morphism spaces are cochain complexes and that the composition law is associative up to chain homotopy in a strong sense. It falls short of being an A_∞-category, for instance because $\mathrm{CF}(L,L')$ is defined only when L and L' meet transversely, so $\mathrm{CF}(L,L)$ is undefined and there is no "morphism" that could play the role of the identity map. A standard way to address this problem is by analyzing the sense in which $\mathrm{CF}(L,L')$ are invariant under Hamiltonian isotopies — Floer's theory of continuation.

2.12. **Example — identity maps.** There is a Floer cochain complex $\mathrm{CF}(L,L')$, well-defined up to quasi-isomorphism, even if L and L' do not intersect transversely. In case L' meets both L_0 and L_1 transversely, then in the absolute setting (resp. relative setting) any Hamiltonian isotopy (resp. any Hamiltonian isotopy supported on the complement of D) induces a quasi-isomorphism between $\mathrm{CF}(L_0,L')$ and $\mathrm{CF}(L_1,L')$ §2.13. One accesses $\mathrm{CF}(L,L')$ by perturbing L.

We illustrate this in an example that fills in the zeroth graded piece of (1.4.1). Let ϕ^s denote the flow of $H(x,y) = \sin(2\pi x)$, i.e.

$$\phi^s(x,y) = (x, y - s\cos(2\pi x)) \quad (2.12.1)$$

Then for $0 < s < 1$, $\phi^s L_{(0)}$ meets $L_{(0)}$ transversely at the points $x = (.25, 0)$ and $y = (.75, 0)$, so that $\mathrm{CF}(\phi^s L_{(0)}, L_{(0)}) = \Lambda x \oplus \Lambda y$. In a suitable fundamental domain the picture is this:



Equipping $L_{(0)}$ with its default orientation and $\phi^s L_{(0)}$ with the orientation induced by $\phi^s : L_{(0)} \cong \phi^s L_{(0)}$, the Maslov degrees are

$$\mathrm{CF}^0(\phi^s L_{(0)}, L_{(0)}) = x\Lambda \qquad \mathrm{CF}^1(\phi^s L_{(0)}, L_{(0)}) = y\Lambda \tag{2.12.2}$$

There are two bigons contributing to the differential μ_1, of equal area A. Both bigons have input x and output y, and their signs §2.4 are opposite to each other (no matter where the stars are placed), so that the differential is

$$\mu_1(x) = y(t^A - t^A) = 0$$

Some variations of this computation are made in §2.14 and §3.5.

2.13. **Continuation.** Let X_H denote the Hamiltonian vector field of a function $H : [0,1] \times T \to \mathbf{R}$, and write ϕ^s for its time s flow, $\phi^s : T \to T$. Let us review how the quasi-isomorphism

$$\mathrm{CF}(L, L') \to \mathrm{CF}(\phi^s L, L') \tag{2.13.1}$$

works in the absolute setting. The map (2.13.1) goes back to [Fl, Thm. 4], our notation is closer to the appendix of [Aur1]. It is defined in terms of a set $\mathcal{M}(x, y, \phi, \beta)$ of maps

$$u : [-\infty, \infty] \times [0,1] \to T$$

that obey the boundary conditions

$$\begin{array}{rcl} u([-\infty,\infty] \times \{0\}) & \subset & L \\ u([-\infty,\infty] \times \{1\}) & \subset & L' \end{array} \qquad u(\{\infty\} \times [0,1]) = \{x\} \qquad u(-\infty, \tau) = \phi^{s \cdot \tau}(y)$$

and (with analytic index zero §2.2) Floer's X_H-perturbed J-holomorphic curve equation:

$$\partial u/\partial \sigma + J(\partial u/\partial \tau - \beta(\sigma) s X_H) = 0 \tag{2.13.2}$$

Here β (the "profile function") is a monotone decreasing $\mathbf{R}$-valued function on $[-\infty, \infty]$ with $\beta(\sigma) = 1$ for $\sigma \ll 0$ and $\beta(\sigma) = 0$ for $\sigma \gg 0$. The formula for (2.13.1) is

$$x \mapsto \sum_{y \in \phi^s L \cap L'} y \sum_{u \in \mathcal{M}(x,y,\phi)} \pm t^{\text{topological energy of } u} \tag{2.13.3}$$

where the "topological energy" is (see Lemma 14.4.5 [Oh]).

$$\int u^* \omega + \int_0^1 H(\tau, u(\infty, \tau)) d\tau - \int_{-\infty}^{\infty} \beta'(\sigma) s \int_0^1 (H_\tau \circ u) d\tau d\sigma$$

This is sometimes a negative quantity, so we must allow $t^{-1} \in \Lambda$.

The homotopy inverse to (2.13.1) is just the continuation map for the reversed flow ϕ^{-s}. To describe the chain homotopy between the composite

$$\mathrm{CF}(\phi^s L, L') \to \mathrm{CF}(\phi^{-s}\phi^s L, L') = \mathrm{CF}(L, L') \to \mathrm{CF}(\phi^s L, L') \tag{2.13.4}$$

and the identity, map, let $B_r(\sigma)$ (for each $r > 0$) be a function that vanishes on an interval of length r and that agrees up to translation of σ with $\beta(\sigma)$ when σ is to the left of that interval and with $\beta(-\sigma)$ when σ is to the right of that interval. Then

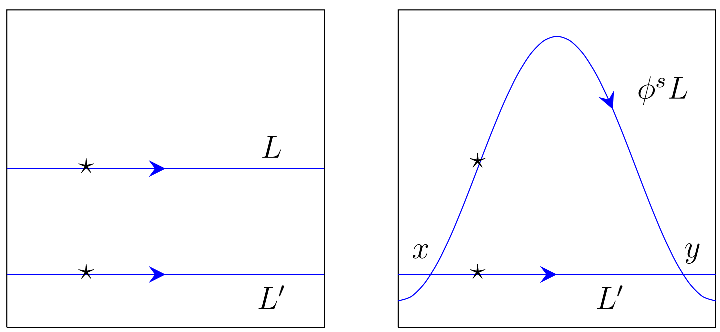
$$\Delta(y) = \sum_x x \sum_u \pm t^{\text{topological energy of } u} \tag{2.13.5}$$

where the inner sum is over strips $u : [-\infty, \infty] \times [0, 1] \to T$ that solve (for some r, with index -1)

$$\partial u/\partial\sigma + J(\partial u/\partial\tau - B_r(\sigma)sX_H) = 0 \qquad r \in \mathbf{R}_{\geq 0}$$

and that have $u(-, 0) \subset L$, $u(-, 1) \subset L'$, and $u(-\infty, \tau) = \phi^{s\cdot\tau}(x)$ and $u(\infty, \tau) = \phi^{s\cdot\tau}(y)$.

2.14. **Example.** Suppose that L and L' are a pair of parallel, horizontal circles, at distance c apart. With ϕ^s as in (2.12.1), $\phi^s(L) \cap L'$ is empty unless $|s| > c$. If $|s|$ only slightly exceeds c, then $\phi^s(L) \cap L'$ has two intersection points, say x and y as in the diagram:



Thus, $\mathrm{CF}(L, L') = 0$, while (noting that there are two bigons in the right picture, a small one of area A and a large one of area $A + c$) and $\mathrm{CF}(\phi^s L, L') = x\Lambda \oplus y\Lambda$, with differential

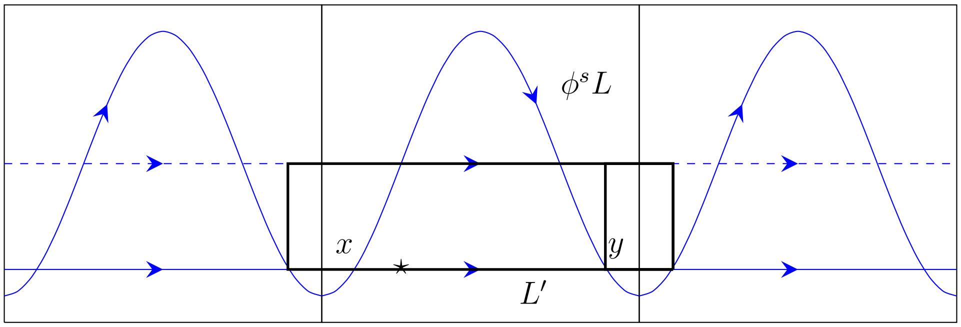
$$\mu_1(x) = y(t^A - t^{A+c})$$

(see §2.4 for the signs).

As $\mathrm{CF}(L, L') = 0$, the composite (2.13.4) is zero. By Floer's theorem, the identity map on $\mathrm{CF}(\phi^s L, L')$ is chain homotopic to zero, with (2.13.5) supplying the contracting homotopy. Indeed the contracting homotopy is the geometric series

$$\Delta(y) = x \cdot (t^{-A} + t^{-A+c} + \cdots + t^{-A+nc} + \cdots)$$

The strip u_n contributing the nth term in the series (of topological energy $nc - A$) stretches horizontally across $n + 2$ fundamental domains, crossing the boundary of the fundamental domain exactly $n + 1$ times. Here is a picture of $u_0|_{\partial([-\infty,\infty]\times[0,1])}$ and $u_1|_{\partial([-\infty,\infty]\times[0,1])}$



3. Floer theory coupled to an F-field

In this section we go over the constructions and calculations of §2, coupling all the sums over polygons to a sheaf of rings $\underline{\Lambda}$ of the kind discussed in §1.10. We are interested in the case when $\underline{\Lambda}$ is pulled back along the projection map $\mathfrak{f} : T \to S^1$ (1.10.2), and set up some notation for dealing with that case in §3.4.

3.1. **Cochain complex.** If L and L' are two embedded circles in T, meeting transversely, let us write (just as in (1.11.1))

$$\mathrm{CF}(L, L'; \underline{\Lambda}) := \bigoplus_{x \in L \cap L'} \underline{\Lambda}_x \tag{3.1.1}$$

If $a \in \underline{\Lambda}_x$ and we wish to regard it as an element of (3.1.1), we will write it as $x \cdot a$. We equip (3.1.1) with a $\mathbf{Z}/2$-grading by equipping L and L' with orientations, just as in §2.5. After further equipping L and L' with stars, we define a differential

$$\mu_1 : \mathrm{CF}^i(L, L'; \underline{\Lambda}) \to \mathrm{CF}^{i+1}(L, L'; \underline{\Lambda})$$

by the following analog of (2.5.2):

$$x \cdot a \mapsto \sum_{y | \mathrm{mas}(y) = i+1} y \left(\sum_{u \in \mathcal{M}(y,x)} \pm t^{\mathrm{area}(u)} \nabla\gamma' \left(a \nabla\gamma(1_{\underline{\Lambda}_y}) \right) \right) \tag{3.1.2}$$

where

- $\gamma : I \to L$ is the path along the L-side of the bigon u starting at y and ending at x,
- $\gamma' : I \to L'$ is the path along the L'-side of u starting at x and ending at y.

This differential does not obey anything like $\mu_1(xa) = \mu_1(x)a$ — in fact $\mu_1(x)a$ is typically undefined and in general $\mu_1(xa)$ and $\mu_1(x)$ do not have any useful relationship with each other.

We obtain a $\underline{\Lambda}$-version of the continuation map (2.13.1)

$$\mathrm{CF}(L, L'; \underline{\Lambda}) \to \mathrm{CF}(\phi^s L, L'; \underline{\Lambda}) \tag{3.1.3}$$

by multiplying each summand of (2.13.3) by $\nabla\gamma'(a\nabla\gamma(\nabla(\phi^{-\tau s}(y))(1_{\underline{\Lambda}_y})))$, where $\gamma = u(\tau, 0)$ and $\gamma' = u(-\tau, 1)$ are the paths along L and L' respectively, and $\phi^{-\tau s}(y)$ is the reverse of the trajectory from y to $\phi^s y$, i.e.

$$\sum_{y\in\phi^s L\cap L'} y \sum_{u\in\mathcal{M}(x,y,\phi)} \pm t^{\text{topological energy of } u}\nabla\gamma'(a\nabla\gamma(\nabla(\phi^{-\tau s}(y))(1_{\underline{\Lambda}_y}))) \tag{3.1.4}$$

The same recipe as §2.13 gives the homotopy inverse to (3.1.3), only replacing (2.13.5) by

$$a\cdot y\mapsto \sum_x x\sum_u \pm t^{\text{topological energy of } u}\nabla\gamma'\left(a\cdot\nabla(\phi^{\tau s}(y))\nabla\gamma\nabla(\phi^{-\tau s}(x))(1_{\underline{\Lambda}_x})\right) \tag{3.1.5}$$

3.2. **Polygon maps.** Let $L_0, L_1, \ldots, L_n$ be oriented, starred submanifolds of T as in §2.7. We define a variant of (2.7.1)

$$\mu_n : \mathrm{CF}^{i_n}(L_{n-1}, L_n; \underline{\Lambda})\times\cdots\times \mathrm{CF}^{i_1}(L_0, L_1;\underline{\Lambda})\to \mathrm{CF}^{i_1+\cdots+i_n+2-n}(L_0, L_n;\underline{\Lambda}) \tag{3.2.1}$$

carrying $(x_n\cdot a_n, x_{n-1}\cdot a_{n-1},\ldots,x_1\cdot a_1)$ (with each $a_i\in\underline{\Lambda}_{x_i}$) to

$$\sum_y y\left(\sum_u \pm t^{\mathrm{area}(u)}\nabla\gamma_n(a_n\nabla\gamma_{n-1}(a_{n-1}\cdots\nabla\gamma_2(a_2\nabla\gamma_1(a_1\nabla(\gamma_0(1))\cdots))))\right) \tag{3.2.2}$$

where u runs over the same set of rigid polygons as (2.7.1), the signs are just the same, and γ_i is the path along the boundary of u going from x_i to x_{i+1}, or from x_{n-1} to x_0.

Each connected component of the Deligne-Mumford-Stasheff compactification of the space of non-rigid polygons has the same vertices $x_0,\ldots,x_n$ — that is, every u in that component has those same vertices. Moreover, the path from x_i to x_{i+1} or from x_n to x_0 along u belongs to the same homotopy class, so that $\nabla(\gamma_i):\Lambda_{x_i}\to\Lambda_{x_{i+1}}$ is locally constant in u. Thus the A_∞-relations among the (3.2.1) hold for the usual reasons:

$$\sum_{i+j=n+1}\sum_{\ell<i}(-1)^{|x_1|+\cdots+|x_\ell|-\ell}\mu_i(x_n a_n,\ldots,x_{\ell+j+1}a_{\ell+j+1},\mu_j(\cdots),x_\ell a_\ell,\ldots x_1 a_1)=0 \tag{3.2.3}$$

3.3. **Local systems.** We can put the formulas in §3.2 in context by considering local systems on the L_i. For each i let $\mathcal{E}_i$ be a local system of $\underline{\Lambda}|_{L_i}$-modules on L_i. Then we define

$$\mathrm{CF}((L_i,\mathcal{E}_i),(L_{i+1},\mathcal{E}_{i+1});\underline{\Lambda}) = \bigoplus_{x\in L_i\cap L_{i+1}}\mathrm{Hom}(\mathcal{E}_{i,x},\mathcal{E}_{i+1,x}) \tag{3.3.1}$$

The differential is modified by

$$\mu_1(xf:\mathcal{E}_{i,x}\to\mathcal{E}_{i+1,x}) = \sum y\sum_u \pm t^{\mathrm{area}(u)}\nabla\gamma'\circ f\circ\nabla\gamma \tag{3.3.2}$$

(where xf denotes f placed in the xth summand of (3.3.1)). In case $\mathcal{E}_i=\underline{\Lambda}|_{L_i}$ is the trivial sheaf of modules, then $\mathrm{Hom}(\mathcal{E}_{i,x},\mathcal{E}_{i+1,x})=\mathrm{Hom}(\underline{\Lambda}_x,\underline{\Lambda}_x)$ is naturally identified with $\underline{\Lambda}_x$ and (3.3.2) coincides with (3.1.2).

The polygon maps

$$\mu_n:\mathrm{CF}((L_{n-1},\mathcal{E}_{n-1}),(L_n,\mathcal{E}_n);\underline{\Lambda})\times\cdots\times\mathrm{CF}((L_0,\mathcal{E}_0),(L_1,\mathcal{E}_1);\underline{\Lambda}) \to \mathrm{CF}((L_0,\mathcal{E}_0),(L_n,\mathcal{E}_n);\underline{\Lambda}) \tag{3.3.3}$$

are defined by sending $(f_n, f_{n-1}, \dots, f_1)$ to the formal expression

$$\sum_y y \left(\sum_{u \in \mathcal{M}(y, x_1, x_2, \dots, x_n)} \pm t^{\mathrm{area}(u)} \nabla\gamma_n \circ f_n \circ \nabla\gamma_{n-1} \circ f_{n-1} \circ \cdots \circ \nabla\gamma_1 \circ f_1 \circ \nabla\gamma_0 \right) \tag{3.3.4}$$

We've left the degrees in (3.3.3) off for typesetting reasons; they are the same as in (2.7.1). The formula (3.3.4) specializes to (3.2.2) in case $\mathcal{E}_i = \underline{\Lambda}|_{L_i}$. In general (3.3.4) can fail to converge, unless the following "unitarity" condition is imposed on the $\mathcal{E}_i$:

> Each fiber of $\mathcal{E}_i$ is locally free of finite rank over $\underline{\Lambda}|_{L_i}$, and the monodromy preserves an $\underline{\Lambda}^0|_{L_i}$-lattice.

3.4. *F*-**field.** We would like to package the $\underline{\Lambda}$-coupled triangle products among the $L_{(m)}$ into a graded ring, as in §1.4. The necessary natural isomorphism between $\mathrm{CF}(L_{(m)}, L_{(n)}; \underline{\Lambda})$ and $\mathrm{CF}(L_{(0)}, L_{(n-m)}; \underline{\Lambda})$ exists only when $\underline{\Lambda}$ is pulled back along the projection map

$$\mathfrak{f} : T \to S^1 : (x, y) + \mathbf{Z}^2 \mapsto x + \mathbf{Z} \tag{3.4.1}$$

To make this explicit, let $\sigma : C \to C$ be a ring automorphism, where C is commutative. We also let σ denote induced automorphism of $C[\![t]\!]$ or of Λ_C, with $\sigma(t^a) = t^a$. We are mainly interested in the case that

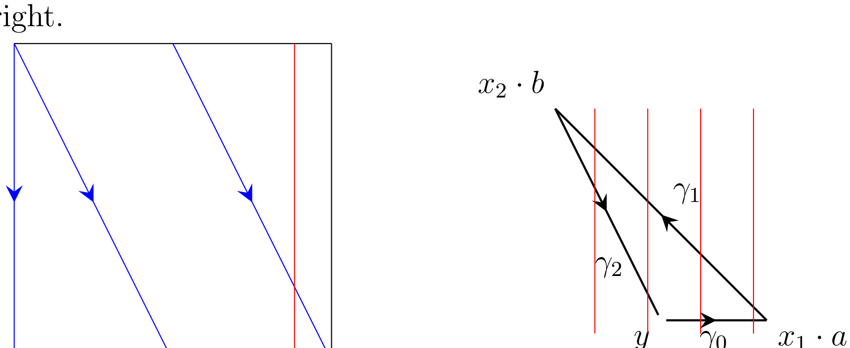
$$C \text{ perfect of characteristic } p, \quad \sigma(c) = c^{1/p} \tag{3.4.2}$$

in which case if $f(t) = \sum c_a t^a$ belongs to $C[\![t]\!]$ or to Λ_C then we can write $\sigma(f)(t) = f(t^p)^{1/p}$. The quotient

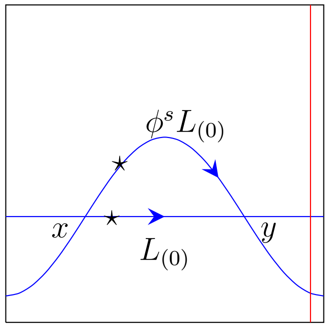
$$(\mathbf{R} \times C)/\sim \tag{3.4.3}$$

of $\mathbf{R} \times C$ by the equivalence relation $(x, c) \sim (x+1, \sigma(c))$ is the étalé space of a locally constant sheaf of rings on S^1 — as are $(\mathbf{R} \times C[\![t]\!])/\sim$ and $(\mathbf{R} \times \Lambda_C)/\sim$. We denote the pullback-to-T of these sheaves of rings by $\underline{C}$, $\underline{C}[\![t]\!]$, and $\Lambda_{\underline{C}}$.

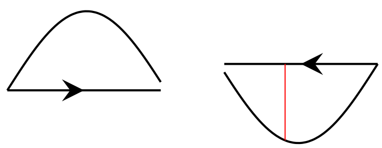
We will call (3.4.1) an "F-field" on T. In diagrams, we keep track of it with a red line — the inverse image of a point close to the right edge of the fundamental domain $[0, 1)$ of S^1, as in the figure on the left below. On the right we have drawn, in a different scale, the preimage of the red line in part of the universal cover of T, along with a triangle that contributes to $\mu_2(x_2 \cdot b, x_1 \cdot a)$. One understands that σ or σ^{-1} is to be applied every time one crosses this "danger line" — σ if one crosses it from right to left, σ^{-1} if one crosses it from left to right.



3.5. **Example.** Let $\underline{\Lambda} = \Lambda_{\underline{C}}$ be as in §3.4 and let $L_{(0)}$ and $\phi^s L_{(0)}$ be as in §2.12.



We will compute the differential on $\mathrm{CF}(\phi^s L_{(0)}, L_{(0)}; \underline{\Lambda})$ — this specializes to the example of §2.12 in case σ is trivial. As in that example we still have $\mathrm{CF}^0(\phi^s L_{(0)}, L_{(0)}; \underline{\Lambda}) = x\Lambda$ and $\mathrm{CF}^1(\phi^s L_{(0)}, L_{(0)}; \underline{\Lambda}) = y\Lambda$, but the map μ_1 is not Λ-linear so we must compute not just $\mu_1(x)$ but $\mu_1(xa)$ for all $a \in \Lambda$. The same two bigons in §2.12, of area A, contribute to $\mu_1(xa)$, but only of them crosses the "danger line"



The left bigon contributes $y \cdot (-t^A a)$ and the right bigon contributes $y \cdot (t^A \sigma(a))$, so that the differential is given by

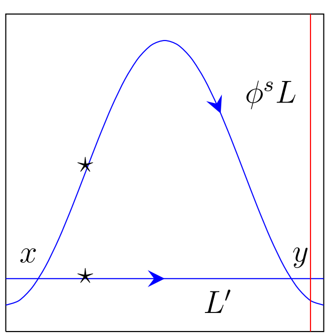
$$\mu_1(xa) = yt^A(\sigma(a) - a)$$

If we make the change of basis $(x, y) \to (x, yt^A)$, then

$$\mathrm{HF}^0 \cong \ker(a \mapsto \sigma(a) - a) \qquad \mathrm{HF}^1 \cong \mathrm{coker}(a \mapsto \sigma(a) - a)$$

More suggestively, $\mathrm{HF}^i(\phi^s L_{(0)}, L_{(0)}; \underline{\Lambda})$ is isomorphic to $H^i(L_{(0)}; \underline{\Lambda}|_{L_{(0)}})$, the cohomology of the circle $L_{(0)}$ with coefficients in $\underline{\Lambda}$.

3.6. **Example.** With $\underline{\Lambda}$ and ϕ as in the previous example, let us compute $\mathrm{CF}(\phi^s L, L'; \underline{\Lambda})$ when L and L' are two special Lagrangians that are parallel to $L_{(0)}$ and to each other, but that do not intersect. Let c be the distance between (and therefore also the area between) L and L'. Suppose that $|s|$ slightly exceed c, so that $\phi^s(L) \cap L'$ has two intersection points that we again denote by x and y.



Then the differential on $\mathrm{CF}(\phi^s L, L'; \underline{\Lambda})$ is $\mu_1(xa) = y(t^a\sigma(a) - t^{A+c}a)$. If c is not zero, the complex is acyclic, but it is interesting to note that the formal series

$$x \cdot \sum_{n \in \mathbf{Z}} \sigma^{-n}(a) t^{nc} \qquad a \in \underline{C}_x \tag{3.6.1}$$

are killed by μ_1. Since (3.6.1) has an infinite "tail", it does not lie in $\underline{\Lambda}_x$ and does not contribute to $\mathrm{CF}(\phi^s L, L'; \underline{\Lambda})$.

Since $L \cap L'$ is empty the continuation maps associated to ϕ^s and its reverse ϕ^{-s}

$$\mathrm{CF}(L, L'; \underline{\Lambda}) \to \mathrm{CF}(\phi^s L, L'; \underline{\Lambda}) \quad \text{and} \quad \mathrm{CF}(\phi^s L, L'; \underline{\Lambda}) \to \mathrm{CF}(L, L'; \underline{\Lambda})$$

both vanish. But the explicit contracting homotopy on $\mathrm{CF}^1(\phi^s L, L'; \underline{\Lambda}) \to \mathrm{CF}^0(\phi^s L, L'; \underline{\Lambda})$ is interesting, it is given by the series

$$\Delta(a \cdot y) = x(t^{-A}\sigma^{-1}(a) + t^{-A+c}\sigma^{-2}(a) + \cdots + t^{-A+nc}\sigma^{-n-1}(a) + \cdots) \tag{3.6.2}$$

The strip u_n contributing the t^{-A+nc} term in this series is the same as in §2.14, but that contribution is now multiplied by $\nabla\gamma'\left(a \cdot \nabla(\phi^{\tau s}(y))\nabla\gamma\nabla(\phi^{-\tau s}(x))(1_{\underline{\Lambda}_x})\right)$, which simplifies to $\sigma^{-n-1}(a)$.

3.7. **Computing the triangle maps.** Let $L_{(m)}$ (equipped with the same orientations and stars), $x_{m,\kappa}$, and τ be as in §2.8. Let $\underline{\Lambda}$ and $\underline{C}$ be pulled back along $\mathfrak{f}$, with σ denoting the nontrivial monodromy, as in §3.4. We will compute

$$\mathrm{CF}(L_{(m_1)}, L_{(m_1+m_2)}; \underline{\Lambda}) \times \mathrm{CF}(L_{(0)}, L_{(m_1)}; \underline{\Lambda}) \to \mathrm{CF}(L_{(0)}, L_{(m_1+m_2)}; \underline{\Lambda})$$

by computing $\mu_2(\tau^{m_1} x_{m_2,\kappa_2} \cdot b, x_{m_1,\kappa_1} \cdot a)$.

Theorem. Let E be as in (2.8.5). Then in the absolute case $\mu_2(\tau^{m_1} x_{m_2,\kappa_2}, x_{m_1,\kappa_1})$ is given by

$$\sum_{\ell \in \mathbf{Z}} x_{m_1+m_2, E(\kappa_1,\kappa_2+\ell)} t^{(\ell+\kappa_2-\kappa_1)^2/(2(\frac{1}{m_1}+\frac{1}{m_2}))} \sigma^{\ell - \lfloor E(\kappa_1,\kappa_2+\ell)\rfloor}(b)\sigma^{-\lfloor E(\kappa_1,\kappa_2+\ell)\rfloor}(a) \tag{3.7.1}$$

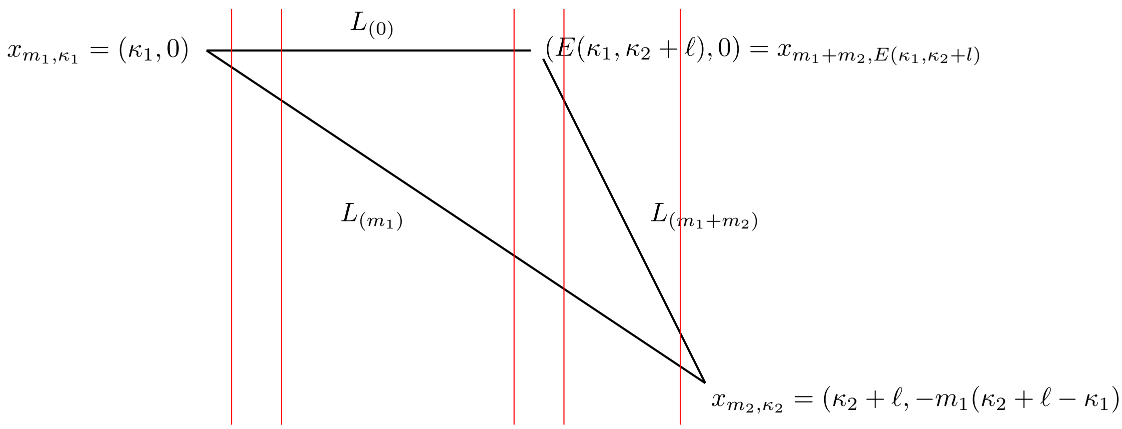
where we understand $x_{m_1+m_2,E(\kappa_1,\kappa_2+\ell)} := x_{m_1+m_2,\kappa}$ if $E(\kappa_1, \kappa_2 + \ell) = \kappa$ modulo 1. In the relative case, $\mu_2(\tau^{m_1} x_{m_2,\kappa_2}, x_{m_1,\kappa_1})$ is given by

$$\sum_{\ell \in \mathbf{Z}} x_{m_1+m_2, E(\kappa_1,\kappa_2+\ell)} t^{\lambda(\kappa_1,\kappa_2+\ell)} \sigma^{\ell - \lfloor E(\kappa_1,\kappa_2+\ell)\rfloor}(b)\sigma^{-\lfloor E(\kappa_1,\kappa_2+\ell)\rfloor}(a) \tag{3.7.2}$$

Proof. The triangles that contribute are exactly as in the proof in §2.8, we will index them again by $\ell \in \mathbf{Z}$. The $\pm$ sign and the exponent of t in (1.11.4) are the same as in §2.8, but it remains to compute $\nabla\gamma_2(b\nabla\gamma_1(a\nabla\gamma_0(1)))$. If I is an interval and $\gamma : I \to T$ is a path in T, write $\mathfrak{f}(\gamma)$ for the number of times γ crosses the "danger line" §3.4, counted with sign. Then

$$\begin{aligned} \nabla\gamma_2(b\nabla\gamma_1(a\nabla\gamma_0(1))) &= \sigma^{\mathfrak{f}(\gamma_2)}(b\sigma^{\mathfrak{f}(\gamma_1)}(a)) & (3.7.3)\\ &= \sigma^{\mathfrak{f}(\gamma_2)}(b)\sigma^{\mathfrak{f}(\gamma_1)+\mathfrak{f}(\gamma_2)}(a) & (3.7.4)\\ &= \sigma^{\mathfrak{f}(\gamma_2)}(b)\sigma^{-\mathfrak{f}(\gamma_0)}(a) & (3.7.5)\end{aligned}$$

The ℓth triangle (pictured below) has $\mathfrak{f}(\gamma_2) = \lfloor E(\kappa_1, \kappa_2+\ell)\rfloor$ and $\mathfrak{f}(\gamma_0) = \ell - \lfloor E(\kappa_1, \kappa_2+\ell)\rfloor$.



□

3.8. **Theta functions with F-field coupling.** Keeping the notation of the previous section, where $\underline{\Lambda}$ and $\underline{C}$ are pulled back along $\mathfrak{f}$, we may give

$$\bigoplus_{m=1}^{\infty} \mathrm{CF}(L_{(0)}, L_{(m)}; \underline{\Lambda}) \tag{3.8.1}$$

the structure of a graded ring without unit. One may equip it with a unit by taking the degree zero piece to be $\Lambda_{C^\sigma} = \mathrm{HF}^0(L_{(0)}, L_{(0)}; \underline{\Lambda})$ §3.5. Here is a description of (3.8.1) analogous to that of §2.9:

Theorem (§1.12)**.** For each $a \in C$ and each pair of integers m, k with $m > k \geq 0$, let $\theta_{m,k}[a]$ denote the formal series

$$\theta_{m,k}[a] := \sum_{i=-\infty}^{\infty} t^{m\frac{i(i-1)}{2}+ki} z^{mi+k} \sigma^i(a) \tag{3.8.2}$$

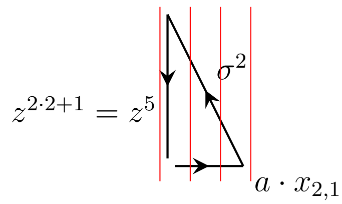
Let $\theta^{\mathrm{abs}}_{m,k}[a]$ denote the formal series

$$\theta^{\mathrm{abs}}_{m,k}[a] := \sum_{i=-\infty}^{\infty} t^{\frac{1}{2m}(mi+k)^2} z^{mi+k} \sigma^i(a) \tag{3.8.3}$$

Then the relative (resp. absolute) version of (3.8.1) is isomorphic (as a graded ring-without-unit) to the $\mathbf{Z}[\![t]\!]$-linear span of the $\theta_{m,k}[a]$ (resp. $\Lambda_{\mathbf{Z}}$-linear span of the $\theta^{\mathrm{abs}}_{m,k}[a]$) via the map

$$x_{m,k/m} \cdot a \mapsto \theta_{m,k}[a] \tag{3.8.4}$$

As in §2.9, the summands of the $\theta[a]$ are indexed by right triangles. The factor of $\sigma^i(a)$ plays the same role as the $\nabla\gamma_2(b\nabla\gamma_1(a\nabla(\gamma_0(1)))$ factor in (1.11.4).



This triangle contributes $t^9 z^5 \sigma^5(a)$ to the relative version of $\theta_{2,1}[a]$ (3.8.2), and $t^{6.25} z^5 \sigma^5(a)$ to the absolute version (3.8.3). Presumably a "family Floer" argument along these lines would prove the Theorem, but we will give a proof in terms of the explicit formulas.

Proof. Let us give the proof first in the relative case. Fix $m_1, m_2 \in \mathbf{Z}_{\geq 0}$, $k_1 \in \{0, \ldots, m_1 - 1\}$, $k_2 \in \{0, \ldots, m_2 - 1\}$, and $a, b \in C[\![t]\!]$. The product of $\theta_{m_2,k_2}[b]$ and $\theta_{m_1,k_1}[a]$ is by definition

$$\sum_{i_1, i_2 \in \mathbf{Z}\times\mathbf{Z}} \sigma^{i_2}(b)\sigma^{i_1}(a) t^{m_2\binom{i_2}{2} + m_1\binom{i_1}{2} + k_2 i_2 + k_1 i_1} z^{m_2 i_2 + m_1 i_1 + k_2 + k_1} \tag{3.8.5}$$

We may also index the sum by triples (r, c, d) where $(c, d) \in \mathbf{Z}\times\mathbf{Z}$ and $r \in \left\{0, 1, \ldots, \frac{m_1+m_2}{\gcd(m_1,m_2)} - 1\right\}$. First, for $\ell \in \mathbb{Z}$, we define d and r via

$$\ell = \frac{m_1 + m_2}{\gcd(m_1, m_2)} d + r$$

It then follows that if we set

$$e(r) = e_{m_1,m_2,k_1,k_2}(r) := \left\lfloor \frac{m_2 r + k_1 + k_2}{m_1 + m_2} \right\rfloor \in \left\{0, 1, \ldots, \frac{m_2}{\gcd(m_1, m_2)} - 1\right\}$$

and $\kappa_1 = k_1/m_1$, $\kappa_2 = k_2/m_2$, we have

$$\lfloor E(\kappa_1, \kappa_2 + \ell) \rfloor = \frac{m_2 d}{\gcd(m_1, m_2)} + e(r) \tag{3.8.6}$$

The triple (r, c, d) (and the integer ℓ) is determined as the unique solution to

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 1 & -m_2/\gcd(m_1, m_2) \\ 1 & m_1/\gcd(m_1, m_2) \end{pmatrix} \begin{pmatrix} c - e(r) \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ r \end{pmatrix}$$

To save space in the exponents, let us write $g := \gcd(m_1, m_2)$. After reindexing the sum (3.8.5) is

$$\sum_r \sum_c \sum_d \sigma^{c - e(r) + m_1 d/g + r}(b) \sigma^{c - e(r) - d m_2/g}(a) t^{\square_1} z^{\square_2} \tag{3.8.7}$$

where

$$\begin{aligned} \square_1 \quad = \quad & m_1\tbinom{c - e(r) - d m_2/g}{2} + k_1(c - e(r) - d m_2/g) \\ & + m_2\tbinom{c - e(r) + d m_1/g + r}{2} + k_2(c - e(r) + d m_1/g + r) \end{aligned} \tag{3.8.8}$$

and

$$\square_2 = m_1(c - e(r) - d m_2/g) + m_2(c - e(r) + d m_1/g + r) + k_1 + k_2 \tag{3.8.9}$$

We note that $\square_2 = (m_1+m_2)c + (m_2 r + k_1 + k_2) - (m_1+m_2)e(r)$ does not depend on d, and furthermore that

$$k(r) = k_{m_1,m_2,k_1,k_2}(r) := m_2 r + k_1 + k_2 - (m_1+m_2)e(r)$$

belongs to $\{0,\ldots,m_1+m_2-1\}$. (Note that $k(r)/(m_1+m_2)$ is the fractional part of $E(\kappa_1,\kappa_2+\ell)$.) Thus the sum (3.8.7) is the same as

$$\sum_r \sum_c \left(\sum_d \sigma^{c-e(r)+dm_1/g+r}(b)\sigma^{c-e(r)-dm_2/g}(a)t^{\square_1}\right) z^{(m_1+m_2)c+k(r)}$$

Since σ acts trivially on t, this is the same as

$$\sum_r \sum_c \sigma^c \left(\sum_d \sigma^{-e(r)+dm_1/g+r}(b)\sigma^{-e(r)-dm_2/g}(a)t^{\square_1}\right) z^{(m_1+m_2)c+k(r)}$$

Now we claim

$$\square_1 = \lambda(\kappa_1,\kappa_2+\ell) + (m_1+m_2)\binom{c}{2} + k(r)c \tag{3.8.10}$$

where $\square_1$ is as in (3.8.8) and λ is as in (2.8.7) and $\ell = \frac{m_1+m_2}{g}d + r$.

Taking (3.8.10) for granted, we obtain that $\theta_{m_2,k_2}[b]\cdot\theta_{m_1,k_1}[a]$ is equal to

$$\sum_{r=0}^{\frac{m_1+m_2}{g}-1} \sum_c \sigma^c \left(\sum_d \sigma^{-e(r)-dm_2/g}(a)\sigma^{-e(r)+dm_1/g+r}(b)t^{\lambda(\kappa_1,\kappa_2+\ell)}\right) t^{(m_1+m_2)\binom{c}{2}+k(r)c} z^{(m_1+m_2)c+k(r)}$$

which is equal to

$$\sum_r \theta_{m_1+m_2,k(r)}\left[\sum_d \sigma^{-e(r)+dm_1/g+r}(b)\sigma^{-e(r)-dm_2/g}(a)t^{\lambda(\kappa_1,\kappa_2+\ell)}\right] \tag{3.8.11}$$

We now compare (3.8.11) to $(b\cdot x_{m_2,k_2/m_2})(a\cdot x_{m_1,k_1/m_1})$ which is given by (3.7.2)

$$\sum_{\ell\in\mathbf{Z}} \left(\sigma^{\ell-\lfloor E(\kappa_1,\kappa_2+\ell)\rfloor}(b)\,\sigma^{-\lfloor E(\kappa_1,\kappa_2+\ell)\rfloor}(a)\,t^{\lambda(\kappa_1,\kappa_2+\ell)}\right) x_{m_1+m_2,E(\kappa_1,\kappa_2+\ell)}$$

Writing $\ell = \frac{m_1+m_2}{\gcd(m_1,m_2)}d + r$ and using the formula (3.8.6), we can rewrite this as

$$\sum_r \left(\sum_d \sigma^{-e(r)+dm_1/g+r}(b)\sigma^{-e(r)-dm_2/g}(a)t^{\lambda(\kappa_1,\kappa_2+\ell)}\right) x_{m_1+m_2,k(r)} \tag{3.8.12}$$

It is now evident that under our map (3.8.4), the two expressions (3.8.11) and (3.8.12) agree.

It remains to verify the claim in (3.8.10). This will be a direct computation. We have

$$\lambda(\kappa_1,\kappa_2+\ell) = m_1\phi(\kappa_1)+m_2\phi(\kappa_2+\ell)-(m_1+m_2)\phi(\frac{m_2 d}{g}+e(r)+\frac{k(r)}{m_1+m_2})$$
$$= m_2(\ell(\ell+\kappa_2)-\frac{\ell(\ell+1)}{2})$$
$$-(m_1+m_2)\left((\frac{dm_2}{g}+e(r))(\frac{dm_2}{g}+e(r)+\frac{k(r)}{m_1+m_2})-\frac{(\frac{dm_2}{g}+e(r))(\frac{dm_2}{g}+e(r)+1)}{2})\right)$$
$$= m_2(\ell\kappa_2+\frac{\ell(\ell-1)}{2})-(\frac{dm_2}{g}+e(r))k(r)-(m_1+m_2)\frac{(\frac{dm_2}{g}+e(r))(\frac{dm_2}{g}+e(r)-1)}{2}$$
$$= m_2\kappa_2(\frac{d(m_1+m_2)}{g}+r)+\frac{m_2(\frac{d(m_1+m_2)}{g}+r)(\frac{d(m_1+m_2)}{g}+r-1)}{2}$$
$$-(\frac{dm_2}{g}+e(r))k(r)-(m_1+m_2)\frac{(\frac{dm_2}{g}+e(r))(\frac{dm_2}{g}+e(r)-1)}{2}$$

Substituting in $k(r)=m_2r+k_1+k_2-(m_1+m_2)e(r)$, we get

$$\begin{aligned}\lambda(\kappa_1,\kappa_2+\ell) &= m_2\kappa_2(\frac{d(m_1+m_2)}{g}+r)+\frac{m_2(\frac{d(m_1+m_2)}{g}+r)(\frac{d(m_1+m_2)}{g}+r-1)}{2}\\ &-(\frac{dm_2}{g}+e(r))(m_1\kappa_1+m_2\kappa_2+m_2r-e(r)(m_1+m_2))\\ &-(m_1+m_2)\frac{(\frac{dm_2}{g}+e(r))(\frac{dm_2}{g}+e(r)-1)}{2}\\ &=\frac{(m_1+m_2)m_1m_2}{2g^2}d^2+\frac{m_1m_2}{g}(r+\kappa_2-\kappa_1)d\\ &+m_2\kappa_2r+\frac{m_2r(r-1)}{2}+\frac{(m_1+m_2)e(r)(e(r)+1)}{2}-(m_1\kappa_1+m_2\kappa_2+m_2r)e(r)\end{aligned}$$

We can rewrite this in a symmetric form as follows:

$$\begin{aligned}\lambda(\kappa_1,\kappa_2+\ell) &= \frac{(m_1+m_2)m_1m_2}{2g^2}d^2+\frac{m_1m_2}{g}(r+\kappa_2-\kappa_1)d\\ &+\frac{m_2(e(r)-r)(e(r)-r+1)}{2}+\frac{m_1e(r)(e(r)+1)}{2}-k_2(e(r)-r)-k_1e(r)\end{aligned}$$

and this in turn can be seen to be equal to

$$\begin{aligned}\lambda(\kappa_1,\kappa_2+\ell) &= m_1\binom{-e(r)-m_2d/g}{2}+k_1(-e(r)-m_2d/g)\\ &+m_2\binom{-e(r)+m_1d/g+r}{2}+k_2(-e(r)+m_1d/g+r)\end{aligned}$$

This completes the proof of the claim (3.8.10) and hence the proof of the theorem in the relative case.

In the absolute case, the product of $\theta^{\text{abs}}_{m_2,k_2}[b]$ and $\theta^{\text{abs}}_{m_1,k_1}[a]$ is given by

$$\sum_{i_1,i_2\in\mathbf{Z}\times\mathbf{Z}} \sigma^{i_2}(b)\sigma^{i_1}(a)t^{\frac{(m_2i_2+k_2)^2}{2m_2}+\frac{(m_1i_1+k_1)^2}{2m_1}} z^{m_2i_2+m_1i_1+k_2+k_1} \tag{3.8.13}$$

Performing the same re-indexing using (3.8), we arrive at

$$\sum_r\sum_c\sum_d \sigma^{c-e(r)+m_1d/g+r}(b)\sigma^{c-e(r)-dm_2/g}(a)t^{\Box_1^{abs}}z^{\Box_2^{abs}} \tag{3.8.14}$$

where

$$\Box_1^{abs} = \frac{(m_1(c-e(r)-dm_2/g)+k_1)^2}{2m_1} + \frac{(m_2(c-e(r)+dm_1/g+r)+k_2)^2}{2m_2} \tag{3.8.15}$$

and $\Box_2^{abs} = \Box_2$ given as before by (3.8.9). Following the same steps, the only difference in the calculation is the verification of the analogue of equation (3.8.10) which now takes the form:

$$\Box_1^{abs} = \frac{(\ell+\kappa_2-\kappa_1)^2 m_1m_2}{2(m_1+m_2)} + \frac{((m_1+m_2)c+k(r))^2}{2(m_1+m_2)} \tag{3.8.16}$$

Recalling that $\ell = (m_1+m_2)d/g+r$, $k(r) = m_2r+k_1+k_2-(m_1+m_2)e(r)$ and $\kappa_i = k_i/m_i$ for $i=1,2$, we can compare the equations (3.8.15) and (3.8.16) directly to verify the claim. This completes the proof in the absolute case. □

4. Specializing the Novikov parameter

4.1. **At** $t=0$**.** The specialization $t=0$ renders uninteresting the absolute version of the maps μ_n, at least if we also set $t^a=0$ for every $a>0$. But it is a standard part of relative Floer theory. In fact it is part of the motivation for relative Floer theory — in any sum over triangles (say), the contribution from triangles which are not disjoint from D vanishes, so that working with $t=0$ is closely related to replacing the closed symplectic manifold T with the open $T-D$. See [LPe2, §6.1] for some more context.

For short, let us write S_n for the nth graded piece of (3.8.1). Let us also put $S_0 := C^\sigma[\![t]\!]$ — here C^σ denotes the σ-fixed subring of C. Then $S_\bullet$ is a graded $C^\sigma[\![t]\!]$-algebra — it is associative and commutative by Theorem 1.12, §3.8. If C is a perfect field and σ is the pth root map, then $C^\sigma = \mathbf{F}_p$. In any case there is an isomorphism in the category of C^σ-schemes

$$\mathrm{Proj}(S\times_{C^\sigma[\![t]\!]} C^\sigma) = \mathrm{colim}\left[\ \mathrm{Spec}(C) \underset{i_\infty}{\overset{i_0\circ\sigma}{\rightrightarrows}} \mathbf{P}^1_{/C}\ \right]$$

where i_0 and i_∞ are the inclusions of C-schemes $\mathrm{Spec}(C)\to\mathbf{P}^1_{/C}$ with coordinates 0 and ∞, respectively.

If C is a field then $\mathrm{Proj}(S\times_{S_0} C^\sigma)$ is a one-dimensional scheme, which can be covered by two affine charts. It fails to be regular at a unique point and the complement of this point is isomorphic to $\mathrm{Spec}(C[x,x^{-1}])$. For the other chart take the complement of any other point — one obtains an affine Zariski neighborhood of the non-regular point that is isomorphic to the spectrum of a subring of $C[y]$, namely

$$\{f \in C[y] : \sigma(f(0)) = f(1)\}$$

This ring is in some sense an order in a Dedekind domain but if C has infinite degree over C^σ then it is not of finite type.

4.2. **Floer cochains at $t = 1$.** Let C and σ be as in §3.4, with $\underline{C}$ pulled back along $\mathfrak{f}$ from the sheaf whose étalé space is (3.4.3). If L and L' are one-dimensional submanifolds that intersect transversely, we will write (similar to (2.5.1))

$$\mathrm{CF}(L, L'; \underline{C}) = \bigoplus_{x \in L \cap L'} \underline{C}_x \tag{4.2.1}$$

This supports a $\mathbf{Z}/2$-grading and a bigon differential

$$\mu_1(x \cdot a) = \sum_{y \mid \mathrm{mas}(y) = \mathrm{mas}(x)+1} y \left(\sum_{u \in \mathcal{M}(y,x)} \pm \nabla\gamma' \left(a \nabla\gamma(1_{\underline{\Lambda_y}}) \right) \right) \tag{4.2.2}$$

with γ and γ' as in (3.1.2). (4.2.2) is a finite sum

If we further endow C with a topology, for which σ is continuous, we can investigate the algebraic structures on (4.2.1) induced by (3.2.1). That is, we study the sums

$$\sum_y y \left(\sum_u \pm \nabla\gamma_n(a_n \nabla\gamma_{n-1}(a_{n-1} \cdots \nabla\gamma_2(a_2 \nabla(\gamma_1(a_1 \nabla\gamma_0(1)) \cdots)))) \right) \tag{4.2.3}$$

(4.2.2) and (4.2.3) are simply the specializations one obtains by setting t and every power t^a to 1 in the formulas from §3. The sums $\sum_u$ in (4.2.3) might diverge or converge in the topological ring C, so that at best the map

$$\mathrm{CF}(L_{n-1}, L_n; \underline{C}) \times \cdots \times \mathrm{CF}(L_0, L_1; \underline{C}) \dashrightarrow \mathrm{CF}(L_0, L_n; \underline{C}) \tag{4.2.4}$$

is only partially defined. In many cases, the domain of convergence is reduced to a point, but we will see that the triangle maps are not trivial.

4.3. **The triangle products at $t = 1$.** Suppose that C is complete with respect to a nonarchimedean norm $|\cdot|$, and that

$$\sigma(c) = |c|^{1/p}$$

for some $p > 1$. With p prime and C, σ as in (3.4.2), the pair $(C, |\cdot|)$ is a perfectoid field of characteristic p [Sc, §3]. The maps $\nabla\gamma$ for the sheaf of rings $\underline{C}$ are continuous but (crucially) they d not preserve the norms.

Write

$$\mathcal{O}_C := \{c \in C : |c| \leq 1\} \qquad \mathfrak{m}_C := \{c \in C : |c| < 1\};$$

then $\mathcal{O}_C$ is the ring of integers in C and $\mathfrak{m}$ is the unique maximal ideal of $\mathcal{O}_C$. They are both stable by the σ-action so that they determine locally constant subsheaves of $\underline{C}$ that we denote by $\underline{\mathcal{O}}_C$ and $\underline{\mathfrak{m}}_C$. The fiber of $\underline{\mathfrak{m}}$ at x is the set of topologically nilpotent elements in $\underline{C}_x$.

Following the notation of (4.2.1) set

$$\mathrm{CF}(L, L'; \underline{\mathfrak{m}}) := \bigoplus_{x \in L \cap L'} \underline{\mathfrak{m}}_x \tag{4.3.1}$$

It is an open subgroup of $\mathrm{CF}(L, L'; \underline{C})$.

Suppose L_0, L_1, and L_2 are special of finite slopes m_0, m_1, and m_2, in the sense of §2.6. The contribution of a triangle with vertices

$$x_1 \in L_0 \cap L_1 \qquad x_2 \in L_1 \cap L_2 \qquad y \in L_2 \cap L_0$$

to $\mu_2(x_2 \cdot b, x_1 \cdot a)$ has the form (3.7.5) $\sigma^{\mathfrak{f}(\gamma_2)}(b)\sigma^{-\mathfrak{f}(\gamma_0)}(a)$, where γ_0 and γ_2 are the two edges of u incident with the output vertex y. If (and only if) $m_0 < m_1 < m_2$, then $\mathfrak{f}(\gamma_2)$ and $\mathfrak{f}(\gamma_0)$ all have the same sign — with perhaps finitely many exceptions where one of $\mathfrak{f}(\gamma_0)$ and $\mathfrak{f}(\gamma_2)$ are zero — so that when $|a| < 1$ and $|b| < 1$

$$|\sigma^{\mathfrak{f}(\gamma_2)}(b)\sigma^{-\mathfrak{f}(\gamma_0)}(a)| = |b|^{p^{-\mathfrak{f}(\gamma_2)}}|a|^{p^{\mathfrak{f}(\gamma_0)}}$$

is very rapidly decreasing as the side lengths of the triangles go to infinity. The triangle product

$$\mu_2 : \mathrm{CF}(L_1, L_2; \underline{\mathfrak{m}}) \times \mathrm{CF}(L_0, L_1; \underline{\mathfrak{m}}) \to \mathrm{CF}(L_0, L_2; \underline{\mathfrak{m}})$$

is therefore convergent when $m_0 < m_1 < m_2$. In particular we have a graded ring (for now, without unit)

$$\bigoplus_{m=1}^{\infty} \mathrm{CF}(L_{(0)}, L_{(m)}; \underline{\mathfrak{m}}) \tag{4.3.2}$$

4.4. **The irrelevant ideal in the Fargues-Fontaine graded ring.** Let C be an algebraically closed field of characteristic p that is complete with respect to a norm $|\cdot|$. Let B and φ be as in §1.14, i.e.

$$B = \left\{ \sum_{i \in \mathbf{Z}} b_i z^i \mid \forall r \in (0,1), |b_i| r^i \to 0 \text{ as } |i| \to \infty \right\} \tag{4.4.1}$$

This appears in [KS, Def. 21] and in [FF, Ex. 1.6.5]. Fargues and Fontaine define a version of B for every local field E, (4.4.1) is the case when $E = \mathbf{F}_p(\!(z)\!)$. Below, we are taking advantage of the fact that when E has equal characteristic each element of B has a unique series expansion, something that is not clear when E has mixed characteristic [FF, Rem. 1.6.7].

Let φ be as in (1.14.2), i.e. the automorphism of B given by $\varphi(\sum c_i z^i) = \sum c_i^p z^i$. The homogeneous coordinate ring of $FF_E(C)$ is

$$\mathbf{F}_p(\!(z)\!) \oplus B^{\varphi = z} \oplus B^{\varphi = z^2} \oplus \cdots \tag{4.4.2}$$

We will prove the theorem of §1.14, i.e. that (4.3.2) is isomorphic to the irrelevant ideal of this ring.

Proof of (1.14.4). Suppose that $a \in C$ has $|a| < 1$. Then the sequence $|\sigma^i(a)| = |a|^{p^{-i}}$ of real numbers is bounded as $i \to \infty$ and very rapidly decreasing as $i \to -\infty$, and $r^{mi+k}|a|^{p^{-i}} \to 0$ as $|i| \to \infty$ for any $r \in (0,1)$. Therefore for any m and k the expression

$$\sum_{i \in \mathbf{Z}} z^{mi+k} \sigma^i(a) \tag{4.4.3}$$

belongs to B (1.14.1). Applying φ to (4.4.3) gives $\sum_{i \in \mathbf{Z}} z^{mi+k} \sigma^{i-1}(a)$ — re-indexing this series gives

$$\sum_{i \in \mathbf{Z}} z^{m(i+1)+k} \sigma^i(a) = z^m \sum_{i \in \mathbf{Z}} z^{mi+k} \sigma^i(a)$$

so that (4.4.3) belongs to $B^{\varphi = z^m}$. But (4.4.3) is $\theta_{m,k}[a]|_{t=1}$, so that by §3.8 the map

$$x_{m,k/m} \cdot a \mapsto \theta_{m,k}[a]|_{t=1} \tag{4.4.4}$$

intertwines μ_2 with the ring structure on B.

If $f = \sum b_i z^i$ belongs to $B^{\varphi = z^m}$, then $b_{mi+k} = \sigma^m(b_k)$, so f is determined by $b_0, \ldots, b_{m-1}$. To obey (1.14.1), the elements $b_0, \ldots, b_{m-1}$ must all belong to $\mathfrak{m}$. The map

$$f \mapsto \sum_{k=0}^{m-1} x_{m,k/m} \cdot b_k$$

gives the inverse isomorphism to $\mathrm{CF}(L_{(0)}, L_{(m)}; \underline{\mathfrak{m}}) \cong B^{\varphi = z^m}$. □

4.5. **SYZ duality.** The degree one part of (1.14.4) is an isomorphism

$$\mathrm{CF}(L_{(0)}, L_{(1)}; \underline{\mathfrak{m}}) \cong \mathrm{Hom}_{FF}(\mathcal{O}, \mathcal{O}(1)) \tag{4.5.1}$$

where $\mathcal{O}(1)$ is the Serre line bundle on (1.14.3). In general it seems that $\mathrm{CF}(L, L'; \underline{\mathfrak{m}})$ captures the set of homomorphisms between two vector bundles on FF whenever L and L' are (or are just isotopic to, if we replace CF by HF) special Lagrangians §2.6 of finite slopes m and m' with m strictly less than m'. But for other kinds of homomorphisms or Ext groups in $\mathrm{Coh}(FF)$, another construction must be necessary — one that we only partially understand. In the next two sections §4.6 and §4.7 we illustrate this in terms of skyscraper sheaves on FF.

The closed points of $FF_E(C)$ are naturally parametrized by the $\mathbf{Z}$-orbits of E-untilts of the perfectoid field C. When $E = \mathbf{F}_p(\!(z)\!)$, an "$E$-untilt" is just a continuous homomorphism $i : E \to C$ — such a homomorphism must carry z to a nonzero element of $\mathfrak{m}$ and conversely every nonzero element of $\mathfrak{m}$ extends to a map from $\mathbf{F}_p(\!(z)\!)$, the $\mathbf{Z}$-action is generated by $i \mapsto \sigma \circ i$. There is a map

$$(\text{closed points of } FF(E, C)) \to \mathbf{R}/\mathbf{Z} \tag{4.5.2}$$

It is defined for any E. When $E = \mathbf{F}_p(\!(z)\!)$ it carries the $\mathbf{Z}$-orbit of $\iota : \mathbf{F}_p(\!(z)\!) \to C$ to the $\mathbf{Z}$-coset of $\log_p(\log(|i(z)|^{-1}))$. We expect that (4.5.2) is the SYZ dual to (3.4.1), and that the skyscraper sheaves have something to do with fibers of (3.4.1).

4.6. **Skyscraper sheaves and $L_{(\infty)}$.** If $\zeta \in C$ is invertible, let us denote by $L^{\zeta}_{(\infty)}$ the special Lagrangian $L_{(\infty)}$ equipped with the rank one local system of $\underline{C}|_{L_{(\infty)}}$-modules (i.e., a local system of C-modules) whose fiber at $(0,0)$ is $\underline{C}_{(0,0)} = C$ and whose monodromy (in the direction of the default orientation, top to bottom) is multiplication by ζ. Let e_m denote $(0,0)$ regarded as the unique intersection point of $L_{(m)}$ and $L_{(\infty)}$, so that

$$\mathrm{CF}(L_{(m)}, L^{\zeta}_{(\infty)}; \underline{C}) = \mathrm{CF}^0(L_{(m)}, L^{\zeta}_{(\infty)}; \underline{C}) = e_m \cdot C. \tag{4.6.1}$$

If C is algebraically closed then one also has $\mathrm{Hom}_{FF}(\mathcal{O}(m), \delta) \cong C$ for any skyscraper sheaf δ. If $|\zeta| < 1$ and δ is the skyscraper sheaf supported at the $\mathbf{Z}$-orbit of the untilt $\mathbf{F}_p(\!(z)\!) \to C$, then we expect that for any surjection $q : \mathcal{O}(1) \to \delta$, there is an isomorphism making the diagram

$$\begin{array}{ccc} \mathrm{CF}(L_{(0)}, L_{(1)}; \underline{\mathfrak{m}}) & \xrightarrow{\mu_2(e_1 \cdot 1, -)} & \mathrm{CF}(L_{(0)}, L^{\zeta}_{(\infty)}; \underline{C}) \\ {\scriptstyle (4.5.1)}\downarrow & & \downarrow \\ \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) & \xrightarrow[q\circ]{} & \mathrm{Hom}(\mathcal{O}, \delta) \end{array} \tag{4.6.2}$$

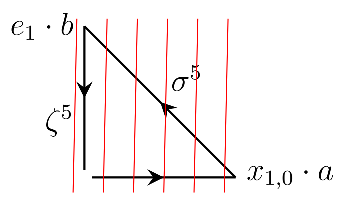
commute; instead of constructing this isomorphism here let us verify that the two rows of (4.6.2) have the same kernel. We may find $i : \mathcal{O} \to \mathcal{O}(1)$ such that

$$0 \to \mathcal{O} \xrightarrow{i} \mathcal{O}(1) \xrightarrow{q} \delta \to 0$$

is exact, so that the kernel of the bottom row in (4.6.2) is isomorphic to $\mathrm{Hom}(\mathcal{O}, \mathcal{O}) = \mathbf{F}_p(\!(z)\!)$, the ground field of (1.14.3). We will show that the kernel of $\mu_2(e_1 \cdot 1, -)$ has the structure of a one-dimensional $\mathbf{F}_p(\!(z)\!)$-module.

In general the triangle map $\mu_2(e_1 \cdot b, x_{1,0} \cdot a)$ (4.2.4) is given by

$$e_0 \cdot \left(\textstyle\sum_{i \in \mathbf{Z}} (-1)^{3i} b \zeta^i \sigma^i(a)\right) \tag{4.6.3}$$



with the figure at the right illustrating the triangle that contributes the $i = 5$ term (for the sign, see §2.4). This is just $b \cdot \theta_{1,0}[a]$ at $t = 1$ and $z = -\zeta$, it converges whenever $|\zeta|$ and $|a|$ are both less than one.

Thus the top row of (4.6.2) is isomorphic to the map $\mathfrak{m} \to C$ sending a to $\vartheta(a) := \sum_{n \in \mathbf{Z}} (-\zeta)^n a^{p^{-n}}$. This map obeys

$$\vartheta(a^p) = (-\zeta)\vartheta(a)$$

i.e. it intertwines the $\mathbf{F}_p(\!(z)\!)$-module structure on C given by the homomorphism $z \mapsto -\zeta$ with the $\mathbf{F}_p(\!(z)\!)$-module structure on $\mathfrak{m}$ given by $(z, a) \mapsto a^p$. The kernel is therefore an $\mathbf{F}_p(\!(z)\!)$-module. The image of this kernel under the isomorphism $\mathfrak{m} \cong B^{\varphi = z}$ (given by

$a \mapsto \sum a^{p^i} z^{-i}$ (1.14.4)) is the set of $b \in B^{\varphi=z}$ whose set of zeroes is exactly $\{(-\zeta)^{p^n}\}_{n\in\mathbf{Z}}$. The function

$$h(z) = \left(\sum_{i\in\mathbf{Z}} a^{p^i} z^{-i}\right)\left(\prod_{n=0}^{\infty}(1+\zeta^{p^n}/z)\right)^{-1}$$

(the meromorphic part of the Weierstrass factorization [FF, Ch. 2]) belongs to $C(\!(z)\!)$ and obeys the functional equation $h(z^{1/p})^p = (\zeta + z)h(z)$, i.e. its coefficients obey the recursion

$$h_n^p - \zeta h_n = h_{n-1} \tag{4.6.4}$$

—for each h_{n-1} there are exactly p solutions in h_n to (4.6.4), so the set of such $h(z)$ is a one-dimensional $\mathbf{F}_p(\!(z)\!)$-submodule of $C(\!(z)\!)$.

4.7. **Ore adjoint.** Let $L^\zeta_{(\infty)}$ be as in §4.6. If we swap the order of $L_{(\infty)}$ and $L_{(m)}$ in (4.6.1), the Maslov index of the intersection point is 1, so that

$$\mathrm{CF}(L^\zeta_{(\infty)}, L_{(m)}; \underline{C}) = \mathrm{CF}^1(L^\zeta_{(\infty)}, L_{(m)}; \underline{C}) = e_m \cdot C$$

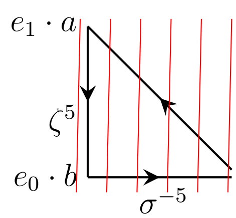
The triangle sum

$$\mathrm{CF}^1(L^\zeta_{(\infty)}, L_{(0)}; \underline{C}) \times \mathrm{CF}^0(L_{(1)}, L^\zeta_{(\infty)}; \underline{C}) \dashrightarrow \mathrm{CF}^1(L_{(1)}, L_{(0)}; \underline{C})$$

is formally given by

$$\sum_{n\in\mathbf{Z}}(-1)^{3n}\sigma^{-n}(b\zeta^n a) = \sum_{n\in\mathbf{Z}}(-\zeta)^{np^n}(ab)^{p^n} \tag{4.7.1}$$

It is the same triangles as (4.6.3) that contribute to (4.7.1), but they are decorated differently. For instance the triangle contributing the $n = 5$ summand is



Even if $|\zeta| < 1$, the $n \to -\infty$ tail of (4.7.1) does not converge unless $ab = 0$. Even so, it is interesting in a formal way. In [Poon], Poonen following [Ore] attaches to each series of the form $f(a) = \sum u_n a^{p^n}$ an "adjoint" series $f^\dagger(a) := \sum u_{-n}^{p^n} a^{p^n}$ — let us call it the Ore adjoint. Evidently (4.7.1) is exactly $\vartheta^\dagger[ba]$.

Under some hypotheses on f Poonen shows that the kernels of f and $f^\dagger$ are Pontrjagin dual to each other in a canonical fashion. These hypotheses are not satisfied by $\vartheta(a)$, but as $\ker(\vartheta)$ (being the additive group of a local field) is Pontrjagin self-dual, and as $\vartheta^\dagger$ does not converge in any case, we are perhaps free to speculate that "$\ker(\vartheta^\dagger)$" is somehow morally isomorphic to $\mathbf{F}_p(\!(z)\!)$. This speculation is consistent with mirror symmetry: on the Fargues-Fontaine curve there indeed is a short exact sequence

$$0 \to \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1)) \to \mathrm{Hom}(\mathcal{O}(1), \delta) \to \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}) \to 0 \tag{4.7.2}$$

coming from the resolution $\mathcal{O} \to \mathcal{O}(1)$ of the skyscraper sheaf δ, and the vanishing of $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(1)) = H^1(FF; \mathcal{O})$. The kernel of (4.7.2) is naturally isomorphic to $H^0(FF; \mathcal{O})$, i.e. to $\mathbf{F}_p(\!(z)\!)$. The middle group is isomorphic to C and to $\mathrm{HF}^0(L_{(1)}, L^{\zeta}_{(\infty)}; \underline{C})$. But we emphasize that $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O})$ is not isomorphic to $\mathrm{HF}(L_{(1)}, L_{(0)}; \underline{C})$, nor to any open subgroup of it.

4.8. **Loud Floer cochains on $L_{(0)}$.** Let $\{\phi^s\}_{s \in \mathbf{R}}$ be as in (2.12.1):

$$\phi^s(x, y) = (x, y - s\cos(2\pi x)) \tag{4.8.1}$$

We can try to compare $\mathrm{CF}(L, L'; \underline{C})$ and $\mathrm{CF}(\phi^s L, L'; \underline{C})$ by specializing to $t = 1$ in (3.1.4). In some cases, for instance if L and L' are parallel to $L_{(0)}$ as in §3.6, the summation (3.1.4) is finite and defines a map

$$\mathrm{CF}(L, L'; \underline{C}) \to \mathrm{CF}(\phi^s L, L'; \underline{C})$$

without any problems. But this map is not always a quasi-isomorphism. The series defining the homotopy (3.1.5) may not converge at $t = 1$ — (3.6.2) is a vivid example of this.

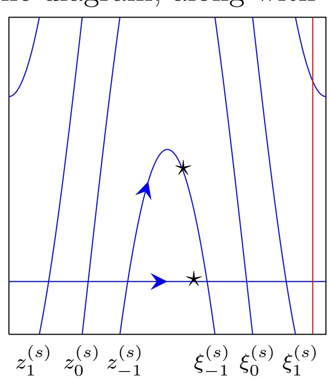
We will analyze the continuation maps between the groups $\mathrm{CF}(\phi^s L_{(0)}, L_{(0)}; \underline{C})$. If n is an integer and $n < s < n + 1$, then $L_{(0)}$ meets $\phi^s L_{(0)}$ in $2n + 2$ points. In the fundamental domain $[0, 1] \times [0, 1]$, half of them have x-coordinate < 0.5 and half of them have x-coordinate > 0.5. They are linearly ordered by the x-coordinate and after listing them in that order we will name them

$$z_n^{(s)}, \ldots, z_{-n}^{(s)}, \xi_{-n}^{(s)}, \ldots, \xi_n^{(s)}$$

More explicitly, $z_i^{(s)}$ and $\xi_i^{(s)}$ are the two solutions to $i - s\cos(2\pi x) = 0$, i.e. for a suitable branch of the inverse cosine function:

$$z_i^{(s)} = \frac{1}{2\pi}\arccos(i/s) \qquad \xi_i^{(s)} = 1 - \frac{1}{2\pi}\arccos(i/s)$$

The case $1 < s < 2$ is shown in the diagram, along with orientations and stars:



The rules of §2.3 give each point $z_i^{(s)}$ the Maslov index 0 and each $\xi_i^{(s)}$ the Maslov index 1, so that

$$\mathrm{CF}^0(\phi^s L_{(0)}, L_{(0)}; \underline{C}) = \bigoplus_{i=-n}^{n} z_i^{(s)} \cdot C \qquad \mathrm{CF}^1(\phi^s L_{(0)}, L_{(0)}; \underline{C}) = \bigoplus_{i=-n}^{n} \xi_i^{(s)} \cdot C$$

Each $\xi_i^{(s)}$ is the output vertex of exactly two bigons, and the other vertex of both bigons in $z_i^{(s)}$. There is an "upward" bigon whose boundary passes through $(0.5, s) + \mathbf{Z}^2$ and a "downward" on whose boundary passes through $(0, -s) + \mathbf{Z}^2$. If one places a star at (or close to, as in the figure above) $(0.5, s) + \mathbf{Z}^2$ and $(0, -s) + \mathbf{Z}^2$, then the sign of every downward bigon is 1 and the sign of every upward bigon is -1. The downward bigons cross the danger line exactly once, and the upward bigons exactly never, so that

$$\mu_1(z_i^{(s)} \cdot a) = \xi_i^{(s)} \cdot (-a + \sigma(a))$$

with the upward bigon contributing $-a$ and the downward bigon contributing $\sigma(a)$.

If $s' > s$ then for a suitable choice of profile function β (on that is very close to the "linear cascades" limit considered in [Aur1]), the continuation map

$$\mathrm{CF}(\phi^s L_{(0)}, L_{(0)}; \underline{C}) \to \mathrm{CF}(\phi^{s'} L_{(0)}, L_{(0)}; \underline{C}) \tag{4.8.2}$$

simply sends $z_i^{(s)} \cdot a$ to $z_i^{(s')} \cdot a$ and $\xi_i^{(s)} \cdot a$ to $\xi_i^{(s')} \cdot a$. In particular it defines a filtered diagram of cochain complexes (indexed by $s > 0$, $s \notin \mathbf{Z}$, with respect to the usual ordering of real numbers s. Let $\mathrm{CF}_{\mathrm{loud}}(L_{(0)}, L_{(0)})$ denote the direct limit of this diagram

$$\mathrm{CF}_{\mathrm{loud}}(L_{(0)}, L_{(0)}; \underline{C}) := \varinjlim_{s>0 | s \notin \mathbf{Z}} \mathrm{CF}(\phi^s L_{(0)}, L_{(0)}; \underline{C})$$

Since each map (4.8.2) is the inclusion of a direct summand of cochain complexes, $\mathrm{CF}_{\mathrm{loud}}$ is a model for the homotopy colimit of cochain complexes as well. Explicitly,

$$\mathrm{CF}^0_{\mathrm{loud}} = \bigoplus_{i \in \mathbf{Z}} z_i \cdot C \qquad \mathrm{CF}^1_{\mathrm{loud}} = \bigoplus_{i \in \mathbf{Z}} \xi_i \cdot C \qquad \mu_1(z_i \cdot a) = \xi_i \cdot (-a + \sigma(a)) \tag{4.8.3}$$

4.9. **Triangles between the $\phi^s L_{(0)}$.** Continuing with the notation of §4.8, let us suppose that none of s, s', and $s + s'$ are in $\mathbf{Z}$, and describe the triangles between $L_{(0)}, \phi^s L_{(0)}$, and $\phi^{s+s'} L_{(0)}$. The "output" corners of these triangles are

$$z_i^{(s+s')} \text{ and } \xi_i^{(s+s')} \text{ on } L_{(0)} \cap \phi^{s+s'} L_{(0)},$$

and the other two corners in counterclockwise order are

$$\phi^s z_i^{(s')}, \phi^s \xi_i^{(s')} \in \phi^s L_{(0)} \cap \phi^{s+s'} L_{(0)}, \qquad z_i^{(s)}, \xi_i^{(s)} \in L_{(0)} \cap \phi^s L_{(0)}.$$

For each such triangle there is a unique pair of integers i and j so that the triangle lifts to $\mathbf{R}^2$ with boundary on the x-axis, the graph of $y = i - s\cos(2\pi x)$, and the graph of $y = i + j - (s + s')\cos(2\pi x)$. The only non-empty moduli spaces of triangles are

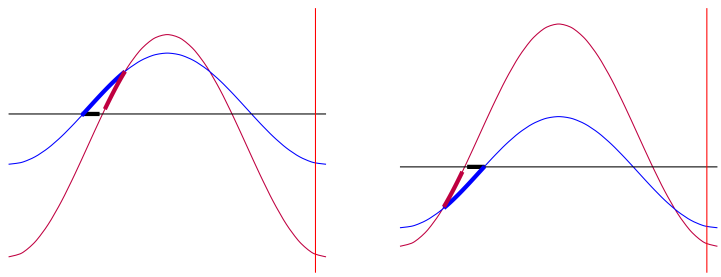
$$\mathcal{M}\left(z_{i+j}^{(s+s')}, \phi^s z_j^{(s')}, z_i^{(s)}\right) \quad \mathcal{M}\left(\xi_{i+j}^{(s+s')}, \phi^s z_j^{(s')}, \xi_i^{(s)}\right) \quad \mathcal{M}\left(\xi_{i+j}^{(s+s')}, \phi^s \xi_j^{(s')}, z_i^{(s)}\right) \tag{4.9.1}$$

with $-s < i < s$ and $-s' < j < s'$. When $i/s = j/s'$, there is something tricky about the latter two moduli spaces (they are not transversely cut §2.2), so let us for a moment assume that $i/s \neq j/s'$. Then each space (4.9.1) contains exactly one triangle. The nature of this triangle depends on which of i/s or j/s' is larger.

The triangle of $\mathcal{M}\left(z_{i+j}^{(s+s')}, \phi^s z_j^{(s')}, z_i^{(s)}\right)$ is the one bounded by

$$\max(0,(i+j)-(s+s')\cos(2\pi x)) \leq y \leq i - s\cos(2\pi x) \quad \text{if } j/s' < i/s \tag{4.9.2}$$
$$\min(0,(i+j)-(s+s')\cos(2\pi x)) \geq y \geq i - s\cos(2\pi x) \quad \text{if } j/s' > i/s \tag{4.9.3}$$

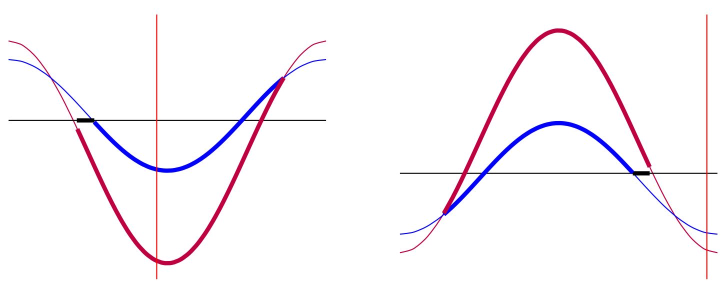
For legibility, in the following illustration of these triangles the curves $\phi^s L_{(0)}$ and $\phi^{s+s'} L_{(0)}$ are drawn in a different aspect ratio than in the diagram of §4.8, vertically compressed. The figure is drawn in a union of $\sim s+s'$ fundamental domains, stacked on top of each other.



In this and the following diagrams, $\phi^{s+s'} L_{(0)}$ is purple, $\phi^s L_{(0)}$ is blue, and $L_{(0)}$ is black. The left side shows the typical case where $i/s > j/s'$, and the right side shows the typical case when $i/s < j/s'$.

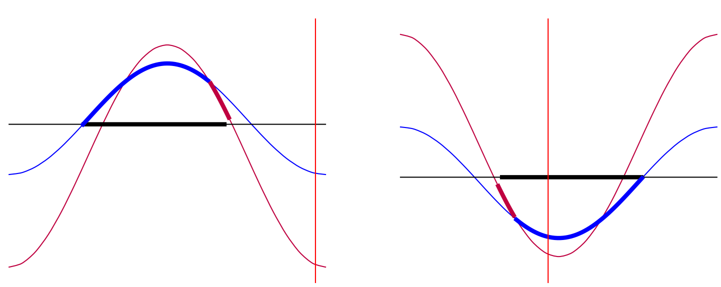
In $\mathcal{M}\left(\xi_{i+j}^{(s+s')}, \phi^s z_j^{(s')}, \xi_i^{(s)}\right)$ we have the triangle

$$\begin{matrix} (4.9.2) \cup \{\min(0, i - s\cos(2\pi x)) \geq y \geq (i+j)-(s+s')\cos(2\pi x)\} & \text{if } \frac{j}{s'} < \frac{i}{s} \\ (4.9.3) \cup \{\max(0, i - s\cos(2\pi x)) \leq y \leq (i+j)-(s+s')\cos(2\pi x)\} & \text{if } \frac{j}{s'} > \frac{i}{s} \end{matrix} \tag{4.9.4}$$

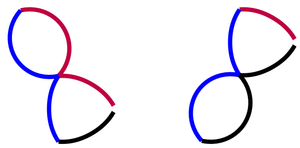


In $\mathcal{M}\left(\xi_{i+j}^{(s+s')}, \phi^s \xi_j^{(s')}, z_i^{(s)}\right)$ we have the triangle

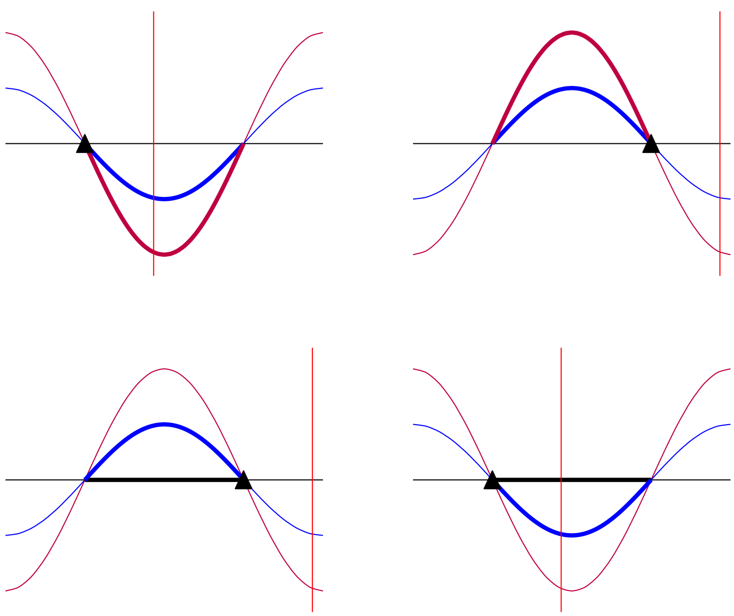
$$\begin{matrix} (4.9.2) \cup \{\min(i - s\cos(2\pi x), i+j-(s+s')\cos(2\pi x))) \geq y \geq 0\} & \text{if } \frac{j}{s'} < \frac{i}{s} \\ (4.9.3) \cup \{\max(i - s\cos(2\pi x), i+j-(s+s')\cos(2\pi x)) \leq y \leq 0\} & \text{if } \frac{j}{s'} > \frac{i}{s} \end{matrix} \tag{4.9.5}$$



Now we discuss the triangles with $i/s = j/s'$. For generic s and s', it is only possible that $i/s = j/s'$ when $i = j = 0$. In that case $\mathcal{M}\left(z_0^{(s+s')}, \phi^s z_0^{(s')}, z_0^{(s)}\right)$ again contains a single point (the constant map with value $z_0 = (.25, 0)$), and is again transversely cut, but these two assertions are not true for $\mathcal{M}\left(\xi_0^{(s+s')}, \phi^s z_0^{(s')}, \xi_0^{(s)}\right)$ or for $\mathcal{M}\left(\xi_0^{(s+s')}, \phi^s \xi_0^{(s')}, z_0^{(s)}\right)$. These spaces each contain two points, which are degenerate triangles (they are at the boundary of the Deligne-Mumford-Stasheff compactification) which are not maps out of a triangle but out of a wedge sum of triangle and a bigon:



The degenerate maps in $\mathcal{M}\left(\xi_0^{(s+s')}, \phi^s z_0^{(s')}, \xi_0^{(s)}\right)$ and $\mathcal{M}\left(\xi_0^{(s+s')}, \phi^s \xi_0^{(s')}, z_0^{(s)}\right)$ collapse the triangle part to a point (to $\xi_0 = (.75, 0)$) but are nontrivial along the bigon:



(4.9.6)

The top two figures indicate the two points of $\mathcal{M}\left(\xi_0^{(s+s')}, \phi^s z_0^{(s')}, \xi_0^{(s)}\right)$, the bottom two are the two points of $\mathcal{M}\left(\xi_0^{(s+s')}, \phi^s \xi_0^{(s')}, z_0^{(s)}\right)$. Though they are not transversely cut they have analytic index zero — more precisely they have index $+1$ along the constant triangle and index -1 along the bigon.

4.10. **Triangle products on** $\mathrm{CF}_{\mathrm{loud}}$**.** For short, let us put $A^{(s)} := \mathrm{CF}(\phi^s L_{(0)}, L_{(0)}; \underline{C})$. The triangles in the previous section, together with the identification $\mathrm{CF}(\phi^{s+s'}, \phi^s L_{(0)}; \underline{C})$ of and $\mathrm{CF}(\phi^{s'} L_{(0)}, L_{(0)}; \underline{C})$, give a multiplication $A^{(s)} \times A^{(s')} \to A^{(s+s')}$ specifically

- (Coming from (4.9.2) and (4.9.3))

$$(z_i^{(s)} \cdot a, z_j^{(s')} \cdot b) \mapsto z_{i+j}^{(s+s')} \cdot ab$$

- (Coming from (4.9.4))

$$(\xi_i^{(s)} \cdot a, z_j^{(s')} \cdot b) \mapsto \xi_{i+j}^{(s+s')} \cdot \begin{cases} a\sigma(b) & \text{if } j/s' < i/s \\ ab & \text{if } j/s' > i/s \end{cases} \tag{4.10.1}$$

- (Coming from (4.9.5))

$$(z_i^{(s)} \cdot a, \xi_j^{(s')} \cdot b) \mapsto \xi_{i+j}^{(s+s')} \cdot \begin{cases} ab & \text{if } j/s' < i/s \\ \sigma(a)b & \text{if } j/s' > i/s \end{cases} \tag{4.10.2}$$

Since $\mathcal{M}\left(\xi_0^{(s+s')}, \phi^s z_0^{(s')}, \xi_0^{(s)}\right)$ and $\mathcal{M}\left(\xi_0^{(s+s')}, \phi^s \xi_0^{(s')}, z_0^{(s)}\right)$ are not transversely cut, they carry a virtual fundamental class rather than an orientation. We will simply put

$$(\xi_0^{(s)} \cdot a, z_0(s') \cdot b) \mapsto \xi_0^{(s+s')} \cdot ab \qquad (z_0^{(s)}a, \xi_0^{(s')}b) \mapsto \xi_0^{(s+s')} \cdot \sigma(a)b \tag{4.10.3}$$

as though the left two degenerate triangles displayed in (4.9.6) contributed nothing. The same issue can also be addressed by introducing a Hamiltonian perturbation ψ of $L_{(0)}$ (but not $\phi^s L_{(0)}$ or $\phi^{s+s'} L_{(0)}$) supported in a very small neighborhood of z_0 and ξ_0. In that case all moduli spaces are transversely cut and the triangle products $\mu_2(\psi z_0^{(s)} \cdot a, \phi^s \xi_0^{(s')} \cdot b)$ and $\mu_2(\psi \xi_0^{(s)} \cdot a, \phi^s z_0^{(s')} \cdot b)$ are well-defined, though the specific formula will depend on ψ — (4.10.3) is consistent with some of these ψ.

Now we use the products $A^{(s)} \times A^{(s')} \to A^{(s+s')}$ to define a multiplication on $\lim_s A^{(s)}$, i.e. on $\mathrm{CF}_{\mathrm{loud}}(L_{(0)}, L_{(0)}; \underline{C})$. It is not quite straightforward, because the products (4.10.1) and (4.10.2) are not eventually constant as s and s' grow — it depends on which of i/s and j/s' are larger. The square

$$\begin{array}{ccc} A^{(s)} \times A^{(s')} & \longrightarrow & A^{(s+s')} \\ \downarrow & & \downarrow \\ A^{(S)} \times A^{(S')} & \longrightarrow & A^{(S+S')} \end{array} \tag{4.10.4}$$

does not commute for all s, s', S, S' with $s < S$ and $s' < S'$. We address this in the following crude way: we choose an irrational number $e > 0$, and note that since $i/s - j/(es)$ has constant sign for $s > 0$, (4.10.4) does commute when $s' = es$ and $S' = eS$. The induced multiplication on the colimit is explicitly $(z_i \cdot a, z_j \cdot b) \mapsto z_{i+j} \cdot ab$ and

$$(\xi_i \cdot a, z_j \cdot b) \mapsto \xi_{i+j} \cdot \begin{cases} a\sigma(b) & \text{if } j/e < i \\ ab & \text{if } j/e \geq i \end{cases} \qquad (z_i \cdot a, \xi_j \cdot b) \mapsto \xi_{i+j} \cdot \begin{cases} ab & \text{if } j/e < i \\ \sigma(a)b & \text{if } j/e \geq i \end{cases}$$

Thus we get one binary operation on $\mathrm{CF}_{\mathrm{loud}}(L_{(0)}, L_{(0)}; \underline{C})$ for every irrational $e > 0$. These multiplications are genuinely different for different e. Moreover, they are not associative; they do however, obey the Leibniz rule $\mu_1(ww') = \mu_1(w)w' + w\mu_1(w')$, with μ_1 as in

(4.8.3). It is likely that they can be extended to an A_∞-structure on $\mathrm{CF}_{\mathrm{loud}}(L_{(0)}, L_{(0)}; \underline{C})$ (and even more likely that there is such an A_∞-structure on a complex quasi-isomorphic to it, defined along the lines of [AS]), but we will not construct it. Instead we simply note that the induced multiplication on $\mathrm{HF}^0_{\mathrm{loud}} := \ker(\mu_1)$ (and even on $\mathrm{HF}^0_{\mathrm{loud}} \oplus \mathrm{HF}^1_{\mathrm{loud}}$, though the degree 1 part vanishes if C is algebraically closed and σ is the pth root map), is associative, and independent of e. Indeed it is simply the Laurent polynomial ring $C^\sigma[z^{\pm 1}]$ under the assignment $\sum c_i z^i \mapsto \sum z_i \cdot c_i$.

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