RABINOWITZ FUKAYA CATEGORIES AS CLUSTER CATEGORIES

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Abstract. We discuss homological mirror symmetry for Rabinowitz Fukaya categories of Milnor fibers of double suspensions of invertible polynomials, and prove it for Brieskorn–Pham polynomials which are not of Calabi–Yau type. This allows a calculation of the Rabinowitz Floer homology of the Milnor fiber as the Hochschild homology of the dg category of equivariant matrix factorizations.

1. Introduction

1.1. For a pair (k,m) of integers satisfying $k \geq 2$ and $m-k \geq 2$, let

$$\check{U} := \left\{ (x,y,z,w) \in \mathbb{C}^4 \;\middle|\; x^k + y^{m-k} + z^2 + w^2 = 1 \right\} \tag{1.1}$$

be the Milnor fiber of the double suspension of the Brieskorn–Pham polynomial $x^k + y^{m-k}$. The cases $(k,m) = (2, r+3)$, $(3,6)$, $(3,7)$, and $(3,8)$ give simple singularities of types A_r, D_4, E_6, and E_8 respectively, for whom homological mirror symmetry conjectured in [LU22] is proved in [LU21].

1.2. Fix an algebraically closed field $\mathbf{k}$ of characteristic zero as coefficients for Fukaya categories. The *stable Fukaya category* of the Liouville manifold $\check{U}$ is defined as the quotient

$$\mathcal{S}\left(\check{U}\right) := \mathcal{W}\left(\check{U}\right) / \mathcal{F}\left(\check{U}\right) \tag{1.2}$$

of the wrapped Fukaya category $\mathcal{W}\left(\check{U}\right)$ by its full subcategory $\mathcal{F}\left(\check{U}\right)$ consisting of compact Lagrangian submanifolds.

1.3. Set

$$R := \mathbf{k}[x,y]/(x^k + y^{m-k}), \tag{1.3}$$

and let G be the diagonal subgroup of SL_2 isomorphic to $\boldsymbol{\mu}_m = \operatorname{Spec} \mathbf{k}[\xi]/(\xi^m - 1)$. The stable category $\underline{\mathrm{CM}}_G(R)$ of the category $\mathrm{CM}_G(R)$ of G-equivariant maximal Cohen–Macaulay R-modules is the homotopy category of the dg category $\operatorname{mf}\left([\mathbb{A}^2/G], x^k + y^{m-k}\right)$ of G-equivariant matrix factorizations of $x^k + y^{m-k}$ on $\mathbb{A}^2 = \operatorname{Spec} \mathbf{k}[x,y]$ [Eis80], which is quasi-equivalent to the *stable derived category* $\operatorname{scoh} X := \operatorname{coh} X / \operatorname{perf} X$ defined as the dg quotient of the bounded derived category of coherent sheaves on the quotient stack $X := [\operatorname{Spec} R/G]$ by the full subcategory consisting of perfect complexes [Buc87, Orl04].

1.4. As shown in [JKS16], the category $\mathrm{CM}_G(R)$ gives an additive categorification of the cluster algebra $\mathbf{k}[\mathrm{Gr}_{k,m}]$. Moreover, the endomorphism ring of a cluster-tilting module is described by a dimer model on a disk [BKM16], which is originally introduced in [Pos] to describe parametrizations of cells in totally nonnegative Grassmannians, and used in [Sco06] to give a structure of a cluster algebra on the homogeneous coordinate ring of $\mathrm{Gr}(k,m)$.

1.5. In this paper, we give two proofs of an equivalence

$$\operatorname{mf}\left(\left[\mathbb{A}^2/G\right], x^k + y^{m-k}\right) \simeq \mathcal{S}\left(\check{U}\right) \tag{1.4}$$

and its generalizations. From (1.4) and on, dg categories of matrix factorizations and Fukaya categories are completed with respect to cones and direct summands, so that they are idempotent-complete stable ∞-categories over $\mathbf{k}$.

1.5.1. One proof is based on homological mirror symmetry for (exact symplectic Lefschetz fibrations associated with) invertible polynomials, which is known for Brieskorn–Pham singularities [FU11].

1.5.2. The other proof is based on Conjecture 3.2, which is homological mirror symmetry for Milnor fibers of invertible polynomials [LU22]. We prove Conjecture 3.2 for Brieskorn–Pham singularities in Theorem 4.1.

1.6. Let $\operatorname{RFH}_*(\check{U})$ and $\mathcal{R}\left(\check{U}\right)$ be the Rabinowitz Floer homology and the Rabinowitz Fukaya category of $\check{U}$ respectively. Koszul duality holds for $\check{U}$ by [LU22, Theorem 6.11], so that one has

$$\mathcal{S}\left(\check{U}\right) \simeq \mathcal{R}\left(\check{U}\right) \tag{1.5}$$

by [GGV, Corollary 1.4] (cf. also [KS]), and

$$\operatorname{RFH}_*(\check{U}) \simeq \operatorname{HH}_*(\mathcal{R}\left(\check{U}\right)) \tag{1.6}$$

by [GGV, Corollary 1.7]. Hence one can compute the Rabinowitz Floer homology as the Hochschild homology of the dg category of equivariant matrix factorizations.

1.7. This paper is organized as follows:

1.7.1. In Section 2, we prove Theorem 2.2, which shows that a generalization of (1.4) to double suspensions of invertible polynomials follows from Conjecture 2.1. Conjecture 2.1 is homological mirror symmetry for invertible polynomials. Theorem 2.2 implies (1.4) since Conjecture 2.1 is known for Brieskorn–Pham polynomials.

1.7.2. In Section 3, we show that Conjecture 3.2 (which is homological mirror symmetry for Milnor fibers of invertible polynomials) implies (3.19) (which is a 'stable' version of homological mirror symmetry for Milnor fibers of invertible polynomials). Then we prove Theorem 3.3 (which shows that a generalization of (1.4) to double suspensions of invertible polynomials follows from Conjecture 3.2) using (3.19) and the Knörrer periodicity.

1.7.3. *Remark.* Theorem 2.2 and Theorem 3.3 have slightly different (but closely related) hypotheses (homological mirror symmetry for $\mathbf{w}$ for the former and that for the Milnor fiber of $\mathbf{W} = \mathbf{w} + z^2 + w^2$ for the latter) and the same conclusion.

1.7.4. In Section 4, we prove Conjecture 3.2 for Brieskorn–Pham polynomials. The proof is based on

- a deformation-theoretic argument going back to Seidel and Sheridan, and
- the Koszul duality between the Fukaya category and the wrapped Fukaya category.

1.7.5. The 'stable' homological mirror symmetry (3.19) gives an algorithm to compute the Rabinowitz Floer homology explicitly. We give sample calculations in Section 5.

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2. Double suspensions of invertible polynomials

A weighted homogeneous polynomial $\mathbf{w} \in \mathbb{C}[x_1, \dots, x_n]$ with an isolated critical point at the origin is *invertible* if there is an integer matrix $A = (a_{ij})_{i,j=1}^n$ with non-zero determinant such that

$$\mathbf{w} = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}. \tag{2.1}$$

The *transpose* of $\mathbf{w}$ is defined in [BH93] as

$$\check{\mathbf{w}} = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ji}}, \tag{2.2}$$

whose exponent matrix $\check{A}$ is the transpose matrix of A. The group

$$\Gamma \coloneqq \{(t_1, \dots, t_n) \in (\mathbb{G}_\mathrm{m})^n \mid t_1^{a_{11}} \cdots t_n^{a_{1n}} = \cdots = t_1^{a_{n1}} \cdots t_n^{a_{nn}}\} \tag{2.3}$$

acts naturally on $\mathbb{A}^n$. The group $\widehat{\Gamma} := \operatorname{Hom}(\Gamma, \mathbb{G}_m)$ of characters of Γ is generated by $\chi_i \colon (t_j)_{j=1}^n \mapsto t_i$ for $i = 1, \dots, n$ with relations $\chi := \sum_{i=1}^n a_{1i}\chi_i = \cdots = \sum_{i=1}^n a_{ni}\chi_i$. Here, the group structure on $\widehat{\Gamma}$ is written additively.

Let $\operatorname{mf}([\mathbb{A}^n/\Gamma], \mathbf{w})$ be the idempotent completion of the dg category of $\Gamma_{\mathbf{w}}$-equivariant matrix factorizations of $\mathbf{w}$, and $\mathcal{F}(\check{\mathbf{w}})$ be the Fukaya–Seidel category of (a Morsification of) $\check{\mathbf{w}}$.

Conjecture 2.1. *For any invertible polynomial* $\mathbf{w}$*, one has an equivalence*

$$\operatorname{mf}([\mathbb{A}^n/\Gamma], \mathbf{w}) \simeq \mathcal{F}(\check{\mathbf{w}}) \tag{2.4}$$

of ∞*-categories.*

Conjecture 2.1 is stated for Brieskorn–Pham singularities in three variables in [Ued06], for polynomials in three variables associated with a regular system of weights of dual type in the sense of Saito in [Tak10], and for invertible polynomials in three variables in [ET11]. It is proved for $n = 2$ in [HS20], and for Sebastiani–Thom sums of polynomials of type A and D in [FU11, FU13]. The conjecture that $\operatorname{mf}([\mathbb{A}^n/\Gamma], \mathbf{w})$ has a full exceptional collection, which is implied by Conjecture 2.1, is stated in [HO23, Conjecture 1.4], and proved in [FKK23].

Let $(\check{d}_1, \dots, \check{d}_n, \check{h})$ be the sequence of positive integers such that $\gcd(\check{d}_1, \dots, \check{d}_n, h) = 1$ and

$$\check{\mathbf{w}}(t^{\check{d}_1}x_1, \dots, t^{\check{d}_n}x_n) = t^{\check{h}}\check{\mathbf{w}}(x_1, \dots, x_n), \tag{2.5}$$

which is unique since $\check{\mathbf{w}}$ is an invertible polynomial. We say that $\check{\mathbf{w}}$ is of Calabi–Yau type if

$$\check{h} = \check{d}_1 + \cdots + \check{d}_n. \tag{2.6}$$

Let

$$\check{U} := \left\{(x_1, \dots, x_n, z, w) \in \mathbb{C}^{n+2} \;\middle|\; \check{\mathbf{w}} + z^2 + w^2 = 1\right\} \tag{2.7}$$

be the Milnor fiber of the double suspension of $\check{\mathbf{w}}$ and set

$$G := \{(t_1, \dots, t_n) \in \Gamma \mid t_1 \cdots t_n = 1\}. \tag{2.8}$$

We say that $\mathcal{F}(\check{U})$ and $\mathcal{W}(\check{U})$ are *Koszul dual* to each other if there exist collections $(S_i)_{i=1}^\mu$ and $(L_i)_{i=1}^\mu$ of objects generating $\mathcal{F}(\check{U})$ and $\mathcal{W}(\check{U})$ such that

$$\dim_{\mathbf{k}} \hom^*(L_i, S_j) = \delta_{ij}, \qquad 1 \le i, j \le \mu, \tag{2.9}$$

where δ_{ij} is the Kronecker delta, so that the augmented endomorphism A_∞-algebras of $\bigoplus_{i=1}^\mu S_i$ and $\bigoplus_{i=1}^\mu L_i$ are Koszul dual to each other. By [LU22, Theorem 6.11], this assumption is satisfied if $\check{\mathbf{w}}$ is a Brieskorn–Pham polynomial not of Calabi–Yau type.

Theorem 2.2. *Conjecture 2.1 and Koszul duality between* $\mathcal{F}(\check{U})$ *and* $\mathcal{W}(\check{U})$ *implies an equivalence*

$$\operatorname{mf}([\mathbb{A}^n/G], \mathbf{w}) \simeq \mathcal{S}(\check{U}) \tag{2.10}$$

of ∞*-categories.*

Theorem 2.2 can be regarded as a 'stable' homological mirror symmetry for $\check{U}$. Theorem 2.2 implies (1.4) since Conjecture 2.1 and Koszul duality are known in this case.

The *n-cluster category* of a pretriangulated A_∞-category $\mathscr{A}$ with a Serre functor $\mathbb{S}$ is defined as the orbit category with respect to the shift $\mathbb{S}[-n]$ of the Serre functor (see e.g. [Iya18] and references therein). (2.10) is obtained as the composite of equivalences

$$\operatorname{mf}([\mathbb{A}^n/G], \mathbf{w}) \simeq C_n(\operatorname{mf}([\mathbb{A}^n/\Gamma], \mathbf{w})) \tag{2.11}$$

and

$$C_n(\mathcal{F}(\check{\mathbf{w}})) \simeq \mathcal{S}(\check{U}). \tag{2.12}$$

The relation between stable Fukaya categories and cluster categories was first pointed out in [LU22] and studied further in [BJK].

Proof of (2.11). Note that one has an isomorphism

$$(\chi) \simeq [2] \tag{2.13}$$

of endofunctors of $\operatorname{mf}([\mathbb{A}^n/\Gamma], \mathbf{w})$. Graded Auslander–Reiten duality [AR87] shows that

$$\mathbb{S} := (\chi - \chi_1 - \cdots - \chi_n)[n-2] \tag{2.14}$$

is a Serre functor of $\operatorname{mf}([\mathbb{A}^n/\Gamma], \mathbf{w})$ (see [IT13, Theorem 2.5]). Hence one has

$$\mathbb{S}[-n] \simeq (-\chi_1 - \cdots - \chi_n), \tag{2.15}$$

so that the orbit category of $\operatorname{mf}([\mathbb{A}^n/\Gamma], \mathbf{w})$ with respect to $\mathbb{S}[-n]$ is equivalent to the category of matrix factorizations of $\mathbf{w}$ graded by $\widehat{\Gamma}/(-\chi_1 - \cdots - \chi_n) \simeq \widehat{G}$, which is nothing but the category $\operatorname{mf}([\mathbb{A}^n/G], \mathbf{w})$ of G-equivariant matrix factorizations of $\mathbf{w}$. This concludes the proof of (2.11). □

Proof of (2.12). Let $\mathscr{G}$ be the $(n+1)$-Calabi–Yau completion of $\mathscr{A} := \mathcal{F}(\mathbf{w})$ in the sense of [Kel11]. Let further

$$\mathscr{B} := \mathscr{A} \oplus \mathscr{A}^\vee[-n-1] \tag{2.16}$$

be the trivial extension algebra of degree $n+1$ of $\mathscr{A}$, where the $\mathscr{A}$-bimodule

$$\mathscr{A}^\vee := \hom_{\mathbf{k}}(\mathscr{A}, \mathbf{k}) \tag{2.17}$$

is the graph of the Serre functor. Then $\mathscr{G}$ is Koszul dual to $\mathscr{B}$ by [HLW23, Theorem 6]. It follows from [Kel05, Theorem 2] that

$$\operatorname{pseu}\mathscr{B}/\operatorname{perf}\mathscr{B} \simeq C_n(\mathscr{A}), \tag{2.18}$$

where $\operatorname{pseu}\mathscr{B}$ is the category of *pseudo-perfect $\mathscr{B}$-modules* (i.e., dg modules over $\mathscr{B}$ which are perfect as $\mathbf{k}$-modules).

One has

$$\mathcal{F}\left(\check{U}\right) \simeq \operatorname{perf}\mathscr{B} \tag{2.19}$$

by [Sei10, Corollary 6.5]. Since $\mathcal{F}\left(\check{U}\right)$ and $\mathcal{W}\left(\check{U}\right)$ are Koszul dual to each other, one has

$$\mathcal{W}\left(\check{U}\right) \simeq \operatorname{perf}\mathscr{G}. \tag{2.20}$$

The Koszul duality between $\mathscr{G}$ and $\mathscr{B}$ implies

$$\operatorname{pseu}\mathscr{B} \simeq \operatorname{perf}\mathscr{G}, \tag{2.21}$$

and (2.12) is proved. □

3. More general invertible polynomials

Let

$$\check{U} := \left\{(x_1, \ldots, x_{n+2}) \in \mathbb{C}^{n+2} \;\middle|\; \check{\mathbf{W}} = 1\right\} \tag{3.1}$$

be the Milnor fiber of an invertible polynomial

$$\check{\mathbf{W}} = \sum_{i=1}^{n+2} \prod_{j=1}^{n+2} x_j^{a_{ji}} \tag{3.2}$$

in $n+2$ variables. The group

$$K := \left\{(t_0, \ldots, t_{n+2}) \in (\mathbb{G}_{\mathrm{m}})^{n+3} \;\middle|\; t_1^{a_{11}} \cdots t_n^{a_{1,n+2}} = \cdots = t_1^{a_{n+2,1}} \cdots t_n^{a_{n+2,n+2}} = t_0 \cdots t_{n+2}\right\} \tag{3.3}$$

acts diagonally on $\mathbb{A}^{n+3}$ making $\mathbf{W} - x_0 \cdots x_{n+2} \colon \mathbb{A}^{n+3} \to \mathbb{A}^1$ equivariant, where

$$\mathbf{W} = \sum_{i=1}^{n+2} \prod_{j=1}^{n+2} x_j^{a_{ij}} \tag{3.4}$$

is the transpose of $\check{\mathbf{W}}$. Set $\widetilde{U} := \operatorname{Spec} \mathbf{k}[x_0, \ldots, x_{n+2}]/(\mathbf{W} - x_0 \cdots x_{n+2})$ and $U := \left[\widetilde{U}\middle/K\right]$, so that

$$\operatorname{mf}\left(\left[\mathbb{A}^{n+3}/K\right], \mathbf{W} - x_0 \cdots x_{n+2}\right) \simeq \operatorname{scoh} U. \tag{3.5}$$

Lemma 3.1. *The singular locus of $\widetilde{U}$ is the x_0-axis.*

Proof. The singular locus of $\widetilde{U}$ is defined inside the ambient space $\mathbb{A}^{n+3} = \operatorname{Spec} \mathbf{k}[x_0, \ldots, x_{n+2}]$ by

$$\mathbf{W} - x_0 \cdots x_{n+2} = x_1 \cdots x_{n+2} = \frac{\partial \mathbf{W}}{\partial x_1} - x_0 x_2 \cdots x_{n+2} = \cdots = \frac{\partial \mathbf{W}}{\partial x_{n+2}} - x_0 \cdots x_{n+1} = 0. \tag{3.6}$$

It follows from $x_1 \cdots x_{n+2} = 0$ that $x_i = 0$ for some $i \in \{1, \ldots, n+2\}$, and one may assume $i = 1$ without loss of generality. Then one has

$$\frac{\partial \mathbf{W}}{\partial x_2} = \cdots = \frac{\partial \mathbf{W}}{\partial x_{n+2}} = 0, \tag{3.7}$$

and one can show using the classification of invertible polynomials [KS92] that (3.7) implies $x_j = 0$ for some $j \in \{2, \ldots, n+2\}$. Then one has

$$\frac{\partial \mathbf{W}}{\partial x_1} = x_0 x_2 \cdots x_{n+2} = 0, \tag{3.8}$$

which together with (3.7) implies $x_1 = \cdots = x_{n+2} = 0$ since $\mathbf{W}$ is an invertible polynomial. ☐

If $\mathbf{W}$ is not of Calabi–Yau type, then any point of $\widetilde{U}$ satisfying $x_0 \neq 0$ can be brought to $x_0 = 1$ by the action of K, so that the complement $D := U \setminus E$ of the closed substack E of U defined by $x_0 = 0$ can be identified with $\left[\widetilde{D} \middle/ H\right]$ where

$$\widetilde{D} := \operatorname{Spec} \mathbf{k}[x_1, \ldots, x_{n+2}]/(\mathbf{W} - x_1 \cdots x_{n+2}) \tag{3.9}$$

and

$$H := \left\{(t_1, \ldots, t_{n+2}) \in (\mathbb{G}_\mathrm{m})^{n+2} \;\middle|\; t_1^{a_{11}} \cdots t_n^{a_{1,n+2}} = \cdots = t_1^{a_{n+2,1}} \cdots t_n^{a_{n+2,n+2}} = t_1 \cdots t_{n+2}\right\} \tag{3.10}$$

$$\cong K \cap \{t_0 = 1\}\,. \tag{3.11}$$

Since the intersection of the singular locus of U with E is the origin, the full subcategory $\operatorname{scoh}_E U$ of $\operatorname{scoh} U := \operatorname{coh} U / \operatorname{perf} U$ consisting of objects supported on E is equivalent to the full subcategory $\operatorname{scoh}_0 U$ consisting of objects supported at the origin;

$$\operatorname{scoh}_0 U \simeq \operatorname{scoh}_E U. \tag{3.12}$$

Conjecture 3.2. *For any invertible polynomial $\mathbf{W}$ not of Calabi–Yau type, one has equivalences*

$$\operatorname{scoh} U \simeq \mathcal{W}\left(\check{U}\right) \tag{3.13}$$

and

$$\operatorname{scoh}_0 U \simeq \mathcal{F}\left(\check{U}\right) \tag{3.14}$$

of ∞-categories.

(3.13) is [LU22, Conjecture 1.4], from which (3.14) should follow as the restriction to the full subcategories consisting of objects X such that $\hom(X, Y)$ is perfect as a $\mathbf{k}$-module for any Y. [Hab22, Theorem 1.1] gives (3.14) for $n = 0$, and a $\mathbb{Z}/2\mathbb{Z}$-graded variant of (3.13) is discussed in [Gam24].

Theorem 3.3. *If Conjecture 3.2 holds for the double suspension of an invertible polynomial $\mathbf{w}$, then one has an equivalence*

$$\operatorname{mf}\left([\mathbb{A}^n/G], \mathbf{w}\right) \simeq \mathcal{S}\left(\check{U}\right) \tag{3.15}$$

of ∞-categories.

Proof. The equivalence

$$\operatorname{coh} D \simeq \operatorname{coh} U / \operatorname{coh}_E U \tag{3.16}$$

induces an equivalence

$$\operatorname{scoh} D \simeq \operatorname{scoh} U / \operatorname{scoh}_E U, \tag{3.17}$$

which together with (3.12) gives an equivalence

$$\operatorname{scoh} D \simeq \operatorname{scoh} U / \operatorname{scoh}_0 U. \tag{3.18}$$

It follows that if $\mathbf{W}$ is an invertible polynomial not of Calabi–Yau type such that Conjecture 3.2 holds, then one has an equivalence

$$\operatorname{scoh} D \simeq \mathcal{S}\left(\check{U}\right) \tag{3.19}$$

of ∞-categories.

If $\mathbf{W}$ is not of Calabi–Yau type, then the singular locus of $\widetilde{D}$ is the origin. Indeed, the singular locus of $\widetilde{D}$ is defined inside the ambient space $\mathbb{A}^{n+2} = \operatorname{Spec} \mathbf{k}[x_1, \ldots, x_{n+2}]$ by

$$\mathbf{W} - x_1 \cdots x_{n+2} = \frac{\partial \mathbf{W}}{\partial x_1} - x_2 \cdots x_{n+2} = \cdots = \frac{\partial \mathbf{W}}{\partial x_{n+2}} - x_1 \cdots x_{n+1} = 0. \tag{3.20}$$

By multiplying $d_i x_i$ and summing over i, one obtains

$$h\mathbf{W} - \sum_{i=1}^{n+2} d_i x_1 \cdots x_{n+2} = 0, \tag{3.21}$$

which together with $\mathbf{W} - x_1 \cdots x_{n+2} = 0$ implies $\mathbf{W} = x_1 \cdots x_{n+2} = 0$. Now we can argue as in the proof of Lemma 3.1 that $x_1 = \cdots = x_{n+2} = 0$.

The double suspension $\mathbf{w}(x_1, \ldots, x_n) + x_{n+1}^2 + x_{n+2}^2$ is not of Calabi–Yau type since $2d_{n+1} = 2d_{n+2} = h$ and hence $\sum_{i=1}^{n+2} d_i > h$.

Set $D' := \left[\widetilde{D}' \middle/ H\right]$ where $\widetilde{D}' := \operatorname{Spec} \mathbf{k}[x_1, \ldots, x_{n+2}]/(\mathbf{W})$. The singular loci of both $\widetilde{D}$ and $\widetilde{D}'$ are the origin, and the formal completions of $\widetilde{D}$ and $\widetilde{D}'$ at the origin are isomorphic since

$$\begin{aligned} &\mathbf{w}(x_1, \ldots, x_n) + x_{n+1}^2 + x_{n+2}^2 - x_1 \cdots x_{n+2} \\ &\qquad = \mathbf{w}(x_1, \ldots, x_n) + \left(\sqrt{1 - \frac{1}{4}(x_1 \cdots x_n)^2} x_{n+1}\right)^2 + \left(x_{n+2} - \frac{1}{2} x_1 \cdots x_{n+1}\right)^2 \end{aligned} \tag{3.22}$$

in $\mathbf{k}[\![x_1, \ldots, x_{n+2}]\!]$. It follows that

$$\operatorname{scoh} D \simeq \operatorname{scoh} D' \tag{3.23}$$

by [Orl11, Theorem 2.10]. The isomorphism

$$H \cong G \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \tag{3.24}$$

and the Knörrer periodicity [Knö87, Proposition 2.1] shows

$$\operatorname{scoh} D' \simeq \operatorname{mf}\left(\left[\mathbb{A}^n / G\right], \mathbf{w}\right), \tag{3.25}$$

and Theorem 2.2 is proved. □

4. Homological mirror symmetry for Milnor fibers of Brieskorn–Pham singularities

We use the same notations as in Section 3. We prove the following theorem in this section:

Theorem 4.1. *Conjecture 3.2 holds if* $\mathbf{W}$ *is a Brieskorn–Pham polynomial.*

Let M be the free abelian group generated by $\{\mathbf{e}_i\}_{i=1}^{n+2}$, and $\widetilde{M}$ be the subgroup generated by $\left\{\mathbf{f}_j := \sum_{i=1}^{n+2} a_{ji} \mathbf{e}_i\right\}_{i=1}^{n+2}$. We write the inclusion $\widetilde{M} \hookrightarrow M$ as φ.

Let

$$\mathcal{P} := \left\{(y_1, \ldots, y_{n+2}) \in M_{\mathbb{C}^\times} := M \otimes \mathbb{C}^\times \;\middle|\; y_1 + \cdots + y_{n+2} + 1 = 0\right\} \tag{4.1}$$

be an n-dimensional pair of pants, and set $\widetilde{\mathcal{P}} := \varphi_{\mathbb{C}^\times}^{-1}(\mathcal{P})$ where

$$\varphi_{\mathbb{C}^\times} := \varphi \otimes \mathbb{C}^\times \colon \widetilde{M}_{\mathbb{C}^\times} \to M_{\mathbb{C}^\times}, \quad (x_i)_{i=1}^{n+2} \mapsto \left(y_i = \prod_{j=1}^{n+2} x_j^{a_{ji}}\right)_{i=1}^{n+2}. \tag{4.2}$$

The closure of $\widetilde{\mathcal{P}}$ in $\widetilde{M}_{\mathbb{C}}$ is identified with the Milnor fiber

$$\check{U} = \left\{ (x_1, \ldots, x_{n+2}) \in \mathbb{C}^{n+2} \;\middle|\; \check{\mathbf{W}}(x_1, \ldots, x_{n+2}) + 1 = 0 \right\} \tag{4.3}$$

of $\check{\mathbf{W}}$.

We equip $\check{U}$ with the grading defined by the tensor square $\Omega_{\check{U}}^{\otimes 2}$ of the holomorphic volume form

$$\Omega_{\check{U}} := \operatorname{Res} \frac{dx_1 \wedge \cdots \wedge dx_{n+2}}{\check{\mathbf{W}}(x_1, \ldots, x_{n+2}) + 1}. \tag{4.4}$$

If $n \geq 1$, then the choice of a grading of $\check{U}$ is unique because of the simple connectivity of $\check{U}$.

The divisors

$$E_i = \left\{ (x_1, \ldots, x_{n+2}) \in \check{U} \;\middle|\; x_i = 0 \right\}, \quad i = 1, \ldots, n+2 \tag{4.5}$$

are smooth, and one has $\check{U} \setminus \widetilde{\mathcal{P}} = \bigcup_{i=1}^{n+2} E_i$. If $n \geq 1$, then E_i is non-empty and connected for any $i = 1, \ldots, n+2$.

Let $\mathcal{P}^{\mathrm{ua}}$ be the universal abelian cover of $\mathcal{P}$, which agrees with the universal cover if $n \geq 1$. The inclusion $\mathcal{P} \hookrightarrow M_{\mathbb{C}^\times}$ induces an isomorphism $H_1(\mathcal{P}) \xrightarrow{\sim} H_1(M_{\mathbb{C}^\times}) \cong M$, so that M is identified with the group $\operatorname{Deck}(\mathcal{P}^{\mathrm{ua}} \to \mathcal{P})$ of deck transformations. The group $\operatorname{Deck}\left(\mathcal{P}^{\mathrm{ua}} \to \widetilde{\mathcal{P}}\right)$ is naturally identified with $\widetilde{M}$, so that $\overline{M} := \operatorname{Deck}\left(\widetilde{\mathcal{P}} \to \mathcal{P}\right)$ is identified with $M/\widetilde{M}$.

The Lagrangian immersion L^{SS} from an n-sphere to $\mathcal{P}$ introduced by Seidel and Sheridan [Sei11, She11] lifts to a Lagrangian immersion $\tilde{L}^{\mathrm{SS}}$ from the disjoint union of $|\overline{M}|$ copies of spheres to $\widetilde{\mathcal{P}}$. Let $\mathcal{F}_0(U)$ be the full subcategory of $\mathcal{F}(U)$ split-generated by $\tilde{L}^{\mathrm{SS}}$.

Let

$$\mathbf{V}(z_0, \ldots, z_{n+2}) := \mathbf{W}(z_1, \ldots, z_{n+2}) - \prod_{i=0}^{n+2} z_i \tag{4.6}$$

be a semi-invariant element of $\mathbf{k}[z_0, \ldots, z_{n+2}]$ with respect the natural action of

$$K := \left\{ (t_0, \ldots, t_{n+2}) \in (\mathbb{G}_{\mathrm{m}})^{n+2} \;\middle|\; \prod_{j=1}^{n+2} t_j^{a_{ij}} = \prod_{j=0}^{n+2} t_j \text{ for any } i \in \{1, \ldots, n+2\} \right\}, \tag{4.7}$$

and $\operatorname{mf}_0([\mathbb{A}^{n+2}/K], \mathbf{V})$ be the full subcategory of the dg category $\operatorname{mf}([\mathbb{A}^{n+2}/K], \mathbf{V})$ of K-equivariant matrix factorizations of $\mathbf{V}$ split-generated by the structure sheaf of the origin.

Proposition 4.2. *One has an equivalence $\mathcal{F}_0(U) \simeq \operatorname{mf}_0([\mathbb{A}^{n+2}/K], \mathbf{V})$.*

Proof. Let $\operatorname{LGr}(\mathcal{P})^{\mathrm{ua}} \to \operatorname{LGr}(\mathcal{P})$ be the universal abelian cover of $\operatorname{LGr}(\mathcal{P})$, whose group of deck transformations can be identified with $\mathbb{G} := H_1(\operatorname{LGr}(\mathcal{P}))$. A $\mathbb{G}$-*grading* of a Lagrangian L in $\mathcal{P}$ is a lift $\tilde{s}$ of the tautological section $s_L \colon L \to \operatorname{LGr}(\mathcal{P})$ to $\operatorname{LGr}(\mathcal{P})^{\mathrm{ua}}$. The Floer cohomology of $\mathbb{G}$-graded Lagrangians is $\mathbb{G}$-graded.

The tensor square $\Omega_{\mathcal{P}}^{\otimes 2}$ of the holomorphic volume form

$$\Omega_{\mathcal{P}} := \operatorname{Res} \frac{1}{1 + y_1 + \cdots + y_{n+2}} \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{n+2}}{y_{n+2}} \tag{4.8}$$

induces a splitting of the exact sequence

$$0 \to H_1(\operatorname{LGr}(T_p\mathcal{P})) \to H_1(\operatorname{LGr}(\mathcal{P})) \to H_1(\mathcal{P}) \to 0, \tag{4.9}$$

where $\operatorname{LGr}(T\mathcal{P})$ is the Lagrangian Grassmannian bundle of the tangent bundle $T\mathcal{P}$, and $p \in \mathcal{P}$ is an arbitrarily chosen base point. We identify $\mathbb{G} := H_1(\operatorname{LGr}(\mathcal{P}))$ with $H_1(\mathcal{P}) \oplus H_1(\operatorname{LGr}(T_p\mathcal{P})) \cong M \oplus \mathbb{Z}$ by this splitting. Similarly, we identify $\operatorname{Deck}\left(\operatorname{LGr}(\mathcal{P})^{\mathrm{ua}} \to \operatorname{LGr}\left(\widetilde{\mathcal{P}}\right)\right)$ with $\widetilde{M} \oplus \mathbb{Z}$ using the tensor square (of the restriction to $\widetilde{\mathcal{P}}$) of Ω_U.

The cohomology algebra A of the endomorphism A_∞-algebra $\mathcal{A}$ of the immersed Lagrangian sphere L^{SS} in the $\mathbb{G}$-graded Fukaya category of $\mathcal{P}$ is computed in [She11] as the exterior algebra generated by elements $\theta_0, \ldots, \theta_{n+2}$ of degrees

$$\deg \theta_i = \begin{cases} -(\mathbf{e}_1 + \cdots + \mathbf{e}_{n+2}) - 1 & i = 0, \\ 1 + \mathbf{e}_i & i = 1, \ldots, n+2. \end{cases} \tag{4.10}$$

The Kontsevich formality lifts the Hochschild–Kostant–Rosenberg isomorphism to an L_∞-quasi-isomorphism

$$\Phi_{\mathrm{HKR}} \colon \mathrm{CC}^\bullet(A) \to \mathbb{C}[z_0, \ldots, z_{n+2}][\theta_0, \ldots, \theta_{n+2}] \tag{4.11}$$

from the Hochschild cochain complex of A to the graded Lie algebra of polyvector fields on $\mathbb{C}[z_0, \ldots, z_{n+2}]$. The latter is a formal L_∞-algebra, whose underlying graded vector space is the the free graded commutative algebra generated by even variables z_i of degrees

$$\deg z_i = -\deg \theta_i + 1 \tag{4.12}$$

and odd variables θ_i of degrees (4.10). The non-trivial L_∞-operation $\mathfrak{l}_2$ is identified with the Schouten bracket by sending θ_i to $\frac{\partial}{\partial z_i}$. As explained in [SS21, Lemma 3.3], the Maurer-Cartan element $\mu^{\geq 3}$ describing the deformation of A to $\mathcal{A}$ is sent to

$$\mathbf{V}_0 \coloneqq -z_0 \cdots z_{n+2} \tag{4.13}$$

by Φ_{HKR}. As shown in [SS21, Lemma 3.5], this implies that the Hochschild cohomology of $\mathcal{A}$ is isomorphic to the quotient

$$\mathbb{C}[z_0, \ldots, z_{n+2}][u_0, \ldots, u_{n+2}]/\mathcal{J} \tag{4.14}$$

of the free graded commutative algebra generated by even variables z_i of degrees (4.12) and odd variables u_i of degrees 1 by the ideal

$$\mathcal{J} = \left(\prod_{i \notin I} z_i \prod_{i \in I} u_i \right)_{I \subset \{0, \ldots, n+2\}}. \tag{4.15}$$

Let $\widetilde{\mathcal{A}}$ be the endomorphism A_∞-algebra of $\tilde{L}^{\mathrm{SS}}$ in the $\widetilde{\mathbb{G}}$-graded Fukaya category of $\widetilde{\mathcal{P}}$. It is obtained from $\mathcal{A}$ as follows:

- Take a complete set $\{m_g\}_{g \in \overline{M}} \subset M \subset \mathbb{G} \cong M \oplus \mathbb{Z}$ of representatives of $\overline{M} \cong M/\widetilde{M}$.
- Take the $\mathbb{G}$-graded endomorphism A_∞-algebra of the direct sum $\bigoplus_{g \in \overline{M}} \mathcal{A}(m_g)$ of shifted free modules in $\operatorname{mod} \mathcal{A}$.
- Take the A_∞-subalgebra consisting of homogeneous elements whose degrees are in $\widetilde{\mathbb{G}}$.

Let $\widetilde{\mathcal{A}}_R$ be the $\widetilde{\mathbb{G}}$-graded deformation of $\widetilde{\mathcal{A}}$ over the polynomial ring

$$R \coloneqq \mathbb{C}\left[r_1, \ldots, r_{n+2}\right] \tag{4.16}$$

whose A_∞-operations are given by counting pseudo-holomorphic disks in U weighted by intersection numbers with E_i. The degree

$$\deg r_i = \mathbf{f}_i + 2 = \sum_{j=1}^{n+2} a_{ij} \mathbf{e}_j + 2 \tag{4.17}$$

of the variable counting intersection numbers with E_i is defined by first choosing a small disk in U intersecting simply and transversely to E_i and disjoint from all the other E_j's, lifting it to $\mathrm{LGr}(U)$, and taking the class of its boundary in $\widetilde{\mathbb{G}}$. Let $\mathfrak{m} = (r_1, \ldots, r_{n+2})$ be the maximal ideal of R at the origin. The first order deformation class of $\widetilde{\mathcal{A}}_R$ belongs to the $\overline{M}$-invariant degree 2 part $\mathrm{HH}^2\left(\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} \otimes \mathfrak{m}/\mathfrak{m}^2\right)^{\overline{M}}$ of the $\widetilde{\mathbb{G}}$-graded Hochschild cohomology of the $\widetilde{\mathcal{A}}$-bimodule $\widetilde{\mathcal{A}} \otimes \mathfrak{m}/\mathfrak{m}^2$, which is isomorphic to the degree 2 part $\mathrm{HH}^2\left(\mathcal{A}, \mathcal{A} \otimes \mathfrak{m}/\mathfrak{m}^2\right)$ of the $\mathbb{G}$-graded Hochschild cohomology of the $\mathcal{A}$-bimodule $\mathcal{A} \otimes \mathfrak{m}/\mathfrak{m}^2$ by [She15, Remark 2.66].

One has

$$\deg\left(\prod_{i=1}^{n+2} r_i^{b_i} \prod_{i=0}^{n+2} z_i^{c_i} \prod_{i\in I} u_i\right) \tag{4.18}$$

$$= \sum_{i=1}^{n+2} b_i \left(\sum_{j=1}^{n+2} a_{ij}\mathbf{e}_j + 2\right) - \sum_{i=1}^{n+2} c_i \mathbf{e}_i + c_0(\mathbf{e}_1 + \cdots + \mathbf{e}_{n+2} + 2) + |I| \tag{4.19}$$

$$= \sum_{i=1}^{n+2}\left(\sum_{j=1}^{n+2} a_{ji}b_j - c_i + c_0\right)\mathbf{e}_i + \sum_{i=1}^{n+2} 2b_i + 2c_0 + |I|. \tag{4.20}$$

For this to be 2, one needs

$$c_i = \sum_{j=1}^{n+2} a_{ji}b_j + c_0 \tag{4.21}$$

for $i = 1, \ldots, n+2$ and

$$2\sum_{i=1}^{n+2} b_i + 2c_0 + |I| = 2. \tag{4.22}$$

This is the case if and only if one of $b_1, \ldots, b_n$, c_0, or $|I|/2$ is 1 and others are 0. If $b_j = \delta_{ij}$ for $i \in \{1, \ldots, n+2\}$, then one has

$$c_j = \sum_{k=1}^{n+2} a_{kj}\delta_{ki} = a_{ij} \tag{4.23}$$

and

$$r_i \prod_{j=1}^{n+2} z_j^{c_j} = r_i \prod_{j=1}^{n+2} z_j^{a_{ij}}. \tag{4.24}$$

This argument also shows $\mathrm{HH}^2(\mathcal{A}, \mathcal{A} \otimes \mathfrak{m}^i) = 0$ for $i \geq 2$.

The first order deformation class of $\widetilde{\mathcal{A}}_R$ is the sum

$$\sum_{i=1}^{n+2} r_i \prod_{j=1}^{n+2} z_j^{a_{ij}} \tag{4.25}$$

of (4.24) for all $i \in \{1, \ldots, n+2\}$. If $n > 0$, this follows from [She20, Lemma 4.22] and [She20, Assumption 5.3].[1] If $n = 0$, then one has either

(1) Brieskorn–Pham: $\mathbf{W} = \check{\mathbf{W}} = x^p + y^q$,
(2) chain: $\mathbf{W} = x^p y + y^q$ and $\check{\mathbf{W}} = x^p + xy^q$, or
(3) loop: $\mathbf{W} = \check{\mathbf{W}} = x^p y + xy^q$.

In the Brieskorn–Pham case, [She20, Lemma 4.22] and [She20, Assumption 5.3] shows that the first order deformation class of $\widetilde{\mathcal{A}}_R$ is given by (4.25) just as in the case $n > 0$. In the chain case, since the divisor in $\check{U}$ defined by $x = 0$ is empty, [She20, Lemma 4.22] and [She20, Assumption 5.3] shows that the first order deformation class of $\widetilde{\mathcal{A}}_R$ is given by $r_2 y^q$, which is equivalent to (4.25) since

$$r_1 x^p y + r_2 y^q - xyz = r_2 y^q - xy(z - r_1 x^{p-1}), \tag{4.26}$$

so that the term $r_1 x^p y$ can be absorbed into a coordinate change of z. Similarly, in the loop case, both of the divisors in $\check{U}$ defined by $x = 0$ and $y = 0$ are empty, and one has

$$r_1 x^p y + r_2 xy^q - xyz = xy(z - r_1 x^{p-1} - r_2 y^{q-1}), \tag{4.27}$$

so that both of the terms $r_1 x^p y$ and $r_2 xy^q$ can be absorbed into a coordinate change of z.

[1] Although the proof of [She20, Assumption 5.3] is relegated to future work, there is no difficulty in the present setting, where U is exact.

Now $\widetilde{\mathcal{A}}_R$ and hence

$$\widetilde{\mathcal{A}}_1 := \widetilde{\mathcal{A}}_R \otimes_R R/(r_i - 1)_{i=1}^n \tag{4.28}$$

are determined uniquely up to quasi-isomorphism by [She16, Proposition 6.6], so that

$$\mathcal{F}_0(U) \simeq \operatorname{mod} \widetilde{\mathcal{A}}_1. \tag{4.29}$$

Similarly, $\mathrm{mf}_0(\mathbf{k}[z_0, \ldots, z_{n+2}], \mathbf{V}_0)$ admits a $\mathbb{G}$-grading by (4.12), and

$$\mathrm{mf}_0\left(R[z_0, \ldots, z_{n+2}], \mathbf{V}_R := \mathbf{V}_0 + \sum_{i=1}^{n+2} r_i \prod_{j=1}^{n+2} z_j^{a_{ij}}\right) \tag{4.30}$$

is a $\widetilde{\mathbb{G}}$-graded deformation whose first order deformation class is (4.25) by [She15, Proposition 7.1]. It follows that

$$\mathrm{mf}_0\left(\left[\mathbb{A}^{n+2}/K\right], \mathbf{V}\right) \simeq \operatorname{mod} \widetilde{\mathcal{A}}_1, \tag{4.31}$$

and Proposition 4.2 is proved. □

Now we specialize to Brieskorn–Pham singularities where $a_{ij} = p_i \delta_{ij}$ and $p_i > 2$ for $i \in \{1, \ldots, n+2\}$.

Proposition 4.3. *One has an equivalence*

$$\mathcal{F}_0(\check{U}) \simeq \mathcal{F}(\check{U}). \tag{4.32}$$

Proof. Let

$$\varpi \colon \check{U} \to \mathbb{C}, \qquad (x_k)_{k=1}^{n+2} \mapsto x_{n+2} \tag{4.33}$$

be the projection to the last coordinate. The fiber

$$\varpi^{-1}(x_{n+2}) = \left\{(x_1, \ldots, x_n) \in \mathbb{C}^n \;\middle|\; x_1^{p_1} + \cdots + x_n^{p_n} + (x_{n+2}^{p_{n+2}} + 1) = 0\right\} \tag{4.34}$$

is a Milnor fiber of a lower-dimensional Brieskorn–Pham singularity, unless x_{n+2} belongs to the set

$$\operatorname{Critv} \varpi = \left\{\zeta_k := \exp\left(2(k+1/2)\pi\sqrt{-1}/p_{n+2}\right)\right\}_{k=0}^{p_{n+2}-1} \tag{4.35}$$

of critical values of ϖ, in which case the fiber has the lower-dimensional Brieskorn–Pham singularity at the origin.

Set

$$I := \left\{\boldsymbol{i} = (i_1, \ldots, i_{n+2}) \in \mathbb{Z}^{n+2} \;\middle|\; 0 \le i_k \le p_k - 2 \text{ for any } 1 \le k \le n+2\right\}. \tag{4.36}$$

A distinguished basis $(V_{\boldsymbol{i}})_{\boldsymbol{i} \in I}$ of vanishing cycles is described inductively in [FU11] as a fibration of lower-dimensional vanishing cycles above the matching path on the x_{n+2}-plane connecting $\zeta_{i_{n+2}}$ and $\zeta_{i_{n+2}+1}$.

There is another distinguished basis $(S_{\boldsymbol{i}})_{\boldsymbol{i} \in I}$ of vanishing cycles, which can similarly be constructed inductively as a fibration of lower-dimensional vanishing cycles above the matching path on the x_{n+2}-plane connecting ζ_0 and $\zeta_{i_{n+2}+1}$. Note that $S_{\boldsymbol{i}} = V_{\boldsymbol{i}}$ for $\boldsymbol{i} = \mathbf{0} := (0, \ldots, 0)$. The collection $(V_{\boldsymbol{i}})_{\boldsymbol{i} \in I}$ in the Fukaya–Seidel category of the Brieskorn–Pham polynomial corresponds to simple modules of the tensor product of the path algebras of A_{p_k-1}-quivers, whereas the collection $(S_{\boldsymbol{i}})_{\boldsymbol{i} \in I}$ corresponds to projective modules.

The collection $(S_{\boldsymbol{i}})_{\boldsymbol{i} \in I}$ of objects in $\mathcal{F}(\check{U})$ has a Koszul dual collection $(L_{\boldsymbol{i}})_{\boldsymbol{i} \in I}$ of objects in $\mathcal{W}(\check{U})$, which are connected components of the inverse image of

$$\left\{(x_1, \ldots, x_{n+2}) \in \mathcal{P} \cap \mathbb{R}^{n+2} \;\middle|\; x_i > 0 \text{ for } 1 \le i \le n \text{ and } x_{n+2} < -1\right\} \tag{4.37}$$

by the covering map $\varphi_{\mathbb{C}^\times} : \check{U} \to \mathcal{P}$. The Lagrangian $L_{\boldsymbol{i}}$ is a Lefschetz thimble for ϖ over the half line on the x_{n+2}-planes from $\zeta_{i_{n+2}+1}$ to infinity associated with the $(n-1)$-dimensional vanishing cycle $V_{\bar{\boldsymbol{\imath}}}$ where $\bar{\boldsymbol{\imath}} = (i_k)_{k=1}^n$.

Since L^{SS} intersects (4.37) only at one point [She11, Corollary 2.9], there is an irreducible component $\tilde{L}_{\mathbf{0}}^{\mathrm{SS}}$ of $\tilde{L}^{\mathrm{SS}}$ such that $\tilde{L}_{\mathbf{0}}^{\mathrm{SS}} \cap L_{\boldsymbol{i}}$ consists of one point if $\boldsymbol{i} = \mathbf{0}$, and is empty otherwise. It follows that $\tilde{L}_{\mathbf{0}}^{\mathrm{SS}} \simeq S_{\mathbf{0}}$ in $\mathcal{F}(\check{U})$. The vanishing cycle $V_{\boldsymbol{i}}$ for other $\boldsymbol{i}$ is the image of $V_{\mathbf{0}} = S_{\mathbf{0}}$ by the map $(x_k)_{k=1}^{n+2} \mapsto \left(\exp\left(2 i_k \pi \sqrt{-1}/p_k\right) x_k\right)_{k=1}^{n+2}$. Since vanishing cycles generate $\mathcal{F}\left(\check{U}\right)$, Proposition 4.3 is proved. □

The closed immersion

$$\iota\colon \left[\mathbf{W}^{-1}(0)/K\right] \to \left[\mathbf{V}^{-1}(0)/K\right] \tag{4.38}$$

induces a push-forward

$$\iota_*\colon \operatorname{scoh}\left[\mathbf{W}^{-1}(0)/K\right] \to \operatorname{scoh}\left[\mathbf{V}^{-1}(0)/K\right] \tag{4.39}$$

since the push-forward of the structure sheaf of $\mathbf{W}^{-1}(0)$ is defined by the non-zero-divisor x_0 and hence has projective dimension one. The adjunction with the pull-back

$$\iota^*\colon \operatorname{scoh}\left[\mathbf{W}^{-1}(0)/K\right] \to \operatorname{scoh}\left[\mathbf{V}^{-1}(0)/K\right] \tag{4.40}$$

follows from that for coherent sheaves. Let

$$\mathcal{L}_{\boldsymbol{i}} := \mathcal{O}_{\mathbb{A}^1 \times \mathbf{0}}\left(-i_1\chi_1 - \cdots - i_{n+2}\chi_{n+2}\right) \tag{4.41}$$

be the structure sheaf of the closed subscheme $\mathbb{A}^1 \times \mathbf{0}$ of $\mathbf{V}^{-1}(0)$ defined by $x_1 = \cdots = x_{n+2} = 0$, twisted by an element of $\widehat{K} := \operatorname{Hom}(K, \mathbb{G}_{\mathrm{m}})$ which is generated by

$$\chi_i\colon K \to \mathbb{G}_{\mathrm{m}}, \quad (\alpha_i)_{i=0}^{n+2} \mapsto \alpha_i^{-1} \tag{4.42}$$

with relations

$$p_1\chi_1 = \cdots = p_{n+2}\chi_{n+2} = \chi_0 + \cdots + \chi_{n+2}. \tag{4.43}$$

Then the sequence $(\mathcal{E}_{\boldsymbol{i}} := \iota^*\mathcal{L}_{\boldsymbol{i}})_{\boldsymbol{i}\in I}$ is the full strong exceptional collection given in [FU11]. Let $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in I}$ be the full exceptional collection right dual to $(\mathcal{E}_{\boldsymbol{i}})_{\boldsymbol{i}\in I}$ so that

$$\dim\hom\left(\mathcal{E}_{\boldsymbol{i}}, \mathcal{E}_{\boldsymbol{j}}\right) = \delta_{\boldsymbol{i},\boldsymbol{j}}, \tag{4.44}$$

and set

$$\mathcal{S}_{\boldsymbol{i}} := \iota_*\mathcal{F}_{\boldsymbol{i}}. \tag{4.45}$$

Then $(\mathcal{L}_{\boldsymbol{i}})_{\boldsymbol{i}\in I}$ and $(\mathcal{S}_{\boldsymbol{i}})_{\boldsymbol{i}\in I}$ are Koszul dual;

$$\dim\hom(\mathcal{L}_{\boldsymbol{i}}, \mathcal{S}_{\boldsymbol{j}}) \cong \dim\hom(\mathcal{E}_{\boldsymbol{i}}, \mathcal{F}_{\boldsymbol{j}}) \simeq \delta_{\boldsymbol{i},\boldsymbol{j}}. \tag{4.46}$$

Proposition 4.4. *If $\check{\mathbf{w}}$ is not of Calabi–Yau type, then one has an equivalence*

$$\operatorname{Fun}^{\mathrm{ex}}\left(\operatorname{mf}_0\left(\left[\mathbb{A}^{n+3}/K\right], \mathbf{V}\right), \operatorname{perf}\mathbf{k}\right) \simeq \operatorname{mf}\left(\left[\mathbb{A}^{n+3}/K\right], \mathbf{V}\right) \tag{4.47}$$

of ∞-categories.

Proof. This follows from (4.46) and the fact that the right orthogonal to $(\mathcal{S}_{\boldsymbol{i}})_{\boldsymbol{i}\in I}$ is zero since $\mathbb{G}_{\mathrm{m}} \subset K$ is a dilating action on the critical locus $\mathbb{A}^1 \times \mathbf{0}$ of $\mathbf{V}$ with the origin as the unique fixed point. □

Proposition 4.4 combined with [LU22, Theorem 6.11] proves [LU22, Conjecture 1.4] for non-log Calabi–Yau Brieskorn–Pham singularity.

5. Rabinowitz Floer homology from mirror symmetry

5.1. We use the same notations as in Section 3. We regard x_0 as a section of the line bundle $\mathcal{L}$ on U associated with the character χ_0 of K, which in turn gives a natural transformation s from the autoequivalence $\mathcal{L}^\vee \otimes (-) \simeq (-\chi_0)$ on $\operatorname{mf}\left(\left[\mathbb{A}^{n+3}/K\right], \mathbf{W} - x_0 \cdots x_{n+2}\right) \simeq \operatorname{scoh} U$ to the identity functor. The localization of $\operatorname{scoh} U$ along s gives $\operatorname{scoh} D \simeq \operatorname{scoh} U \setminus E$ since E is defined by $x_0 = 0$ (cf. [Sei08, Section 1]).

Let $(d_1, \ldots, d_{n+2}, h)$ be the sequence of positive integers such that $\gcd(d_1, \ldots, d_{n+2}, h) = 1$ and

$$\mathbf{W}(t^{d_1}x_1, \ldots, t^{d_{n+2}}x_{n+2}) = t^h\mathbf{W}(x_1, \ldots, x_{n+2}). \tag{5.1}$$

Then h is the minimal positive integer such that x_0^h is invariant under the action of $\ker\chi \subset K$, and one has

$$\left(\mathcal{L}^\vee\right)^{\otimes h} \otimes (-) \simeq (-h\chi_0) \simeq (-d_0\chi) \simeq [-2d_0] \tag{5.2}$$

where

$$d_0 \coloneqq h - d_1 - \cdots - d_{n+2}. \tag{5.3}$$

One can regard s^h as an element of $\mathrm{HH}^{2d_0}(\operatorname{scoh} U)$, so that $\operatorname{scoh} U$ is linear over $\mathbf{k}[s^h]$. The localization of $\operatorname{scoh} U$ along s is equivalent to that along s^h;

$$\operatorname{scoh} D \simeq \operatorname{scoh} U \otimes_{\mathbf{k}[s^h]} \mathbf{k}[s^h, s^{-h}]. \tag{5.4}$$

Assume Conjecture 3.2, so that $\operatorname{scoh} D \simeq \mathcal{R}\left(\check{U}\right)$ by (3.19). This implies

$$\begin{aligned} \mathrm{RFH}^*\left(\check{U}\right) &\simeq \mathrm{HH}^*\left(\mathcal{R}\left(\check{U}\right)\right) && (5.5)\\ &\simeq \mathrm{HH}^*\left(\operatorname{scoh} U \otimes_{\mathbf{k}[s^h]} \mathbf{k}[s^h, s^{-h}]\right) && (5.6)\\ &\simeq \mathrm{HH}^*\left(\operatorname{scoh} U\right) \otimes_{\mathbf{k}[s^h]} \mathbf{k}[s^h, s^{-h}]. && (5.7) \end{aligned}$$

5.2. Let $\check{\mathbf{W}}$ be the transpose of $\mathbf{W}$ and $(\check{d}_1, \ldots, \check{d}_{n+2}, \check{h})$ be the sequence of positive integers such that $\gcd(\check{d}_1, \ldots, \check{d}_{n+2}, \check{h}) = 1$ and

$$\check{\mathbf{W}}(t^{\check{d}_1} x_1, \ldots, t^{\check{d}_{n+2}} x_{n+2}) = t^{\check{h}} \check{\mathbf{W}}(x_1, \ldots, x_{n+2}). \tag{5.8}$$

Then the link

$$C \coloneqq \left\{ (x_1, \ldots, x_{n+2}) \in \check{\mathbf{W}}^{-1}(0) \;\middle|\; |x_1|^2 + \cdots + |x_{n+2}|^2 = 1 \right\} \tag{5.9}$$

of the singularity of $\check{\mathbf{W}}^{-1}(0)$ at the origin has an S^1-action defined by

$$S^1 \ni t \colon (x_1, \ldots, x_{n+2}) \mapsto \left(t^{\check{d}_1} x_1, \ldots, t^{\check{d}_{n+2}} x_{n+2} \right), \tag{5.10}$$

which lifts to an S^1-action

$$S^1 \ni t \colon (x_0, \ldots, x_{n+2}) \mapsto \left(t^{\check{h}} x_0, t^{\check{d}_1} x_1, \ldots, t^{\check{d}_{n+2}} x_{n+2} \right) \tag{5.11}$$

on the total space of the family

$$\varphi \colon \check{\mathfrak{U}} \coloneqq \left\{ (x_0, \ldots, x_{n+2}) \in \mathbb{C}^{n+3} \;\middle|\; \check{\mathbf{W}}(x_1, \ldots, x_{n+2}) = x_0 \right\} \to \mathbb{C}^1, \qquad (x_0, \ldots, x_{n+2}) \mapsto x_0. \tag{5.12}$$

Let μ be the endofunctor of $\mathcal{W}\left(\check{U}\right)$ defined as the clockwise monodromy of the family (5.12) around the origin, which is isomorphic to the composite of inverse spherical twists along a distinguished basis of vanishing cycles of $\check{\mathbf{W}}$. It is shown in [Sei00, Section 4.c] that

$$\mu^{\check{h}} \simeq \left[-2\check{d}_0 \right] \tag{5.13}$$

where

$$\check{d}_0 \coloneqq \check{h} - \check{d}_1 - \cdots - \check{d}_{n+2}. \tag{5.14}$$

The wrapped Fukaya category of the singular hypersurface $\check{\mathbf{W}}^{-1}(0)$ in the sense of Auroux (cf. [Jef22, Definition 1]), defined as the localization of $\mathcal{W}\left(\check{U}\right)$ along the natural transformation $\check{s} \colon \mu \to \mathrm{id}$ first introduced in [Sei09], is equivalent to the quotient of $\mathcal{W}\left(\check{U}\right)$ by the split-closure of the essential image of the cap functor [Jef22, Corollary 1]. Since $\mathcal{F}\left(\check{U}\right)$ is split-generated by vanishing cycles, which are in the essential image of the cap functor, the wrapped Fukaya category of the singular hypersurface $\check{\mathbf{W}}^{-1}(0)$ is equivalent to the stable Fukaya category, which in turn is equivalent to the Rabinowitz Fukaya category $\mathcal{R}\left(\check{U}\right)$.

5.3. Set

$$V := \mathbf{k}x_0 \oplus \mathbf{k}x_1 \oplus \cdots \oplus \mathbf{k}x_{n+2}. \tag{5.15}$$

For $\gamma \in K$, let V_γ be the subspace of γ-invariant elements in V, S_γ be the symmetric algebra of V_γ, $\mathbf{V}_\gamma$ be the restriction of $\mathbf{V}$ to $\operatorname{Spec} S_\gamma$, and N_γ be the K-stable complement of V_γ in V so that $V \cong V_\gamma \oplus N_\gamma$ as a K-module. Then [Dyc11, CT13, Seg13, BFK14] (cf. also [LU22, Theorem 3.1]) shows that $\mathrm{HH}^t\left(\operatorname{mf}\left([\mathbb{A}^{n+2}/K], \mathbf{V}\right)\right)$ is isomorphic to

$$\bigoplus_{\substack{\gamma\in\ker\chi,\ l\geq 0\\ t-\dim N_\gamma=2u}} \left(H^{-2l}(d\mathbf{V}_\gamma)\otimes\Lambda^{\dim N_\gamma}N_\gamma^\vee\right)_{(u+l)\chi} \oplus \bigoplus_{\substack{\gamma\in\ker\chi,\ l\geq 0\\ t-\dim N_\gamma=2u+1}} \left(H^{-2l-1}(d\mathbf{V}_\gamma)\otimes\Lambda^{\dim N_\gamma}N_\gamma^\vee\right)_{(u+l+1)\chi}. \tag{5.16}$$

Here $H^i(d\mathbf{V}_\gamma)$ is the i-th cohomology of the Koszul complex

$$C^*(d\mathbf{V}_\gamma) := \left\{\cdots \to \Lambda^2 V_\gamma^\vee \otimes S_\gamma(-2\chi) \to V_\gamma^\vee \otimes S_\gamma(-\chi) \to S_\gamma\right\}, \tag{5.17}$$

where the rightmost term S_γ sits in cohomological degree 0, and the differential is the contraction with

$$d\mathbf{V}_\gamma \in \left(V_\gamma \otimes S_\gamma\right)_\chi. \tag{5.18}$$

5.4. As an example, consider the case $\mathbf{W} = x_1^2 + \cdots + x_{n+2}^2$, where $\check{U} \cong T^*S^{n+1}$, $h = 2$, $d_0 = -n$, and $\mathrm{HH}^*\left(\operatorname{mf}\left([\mathbb{A}^5/K], \mathbf{W}\right)\right) \otimes_{\mathbf{k}[s^h]} \mathbf{k}[s^h, s^{-h}]$ has a basis consisting of

- x_0^{2m} for $m \in \mathbb{Z}$ and $\gamma = (1, \ldots, 1)$ of degree $-2mn$,
- $x_0^{2m+1} \otimes x_0^\vee$ for $m \in \mathbb{Z}$ and $\gamma = (1, \ldots, 1)$ of degree $-2mn + 1$

and, if n is even, in addition to the above,

- $x_0^{2m} \otimes x_1^\vee \otimes \cdots x_{n+2}^\vee$ for $m \in \mathbb{Z}$ and $\gamma = (1, -1, -1, \ldots, -1)$ of degree $(2m+1)n$,
- $x_0^{2m+1} \otimes x_0^\vee \otimes \cdots x_{n+2}^\vee$ for $m \in \mathbb{Z}$ and $\gamma = (1, -1, -1, \ldots, -1)$ for $m \in \mathbb{Z}$ of degree $(2m+1)n+1$.

5.5. Let U_n be the Liouville domain $\mathbb{C}^\times \cong T^*S^1$ equipped with the grading determined by the quadratic differential $x^n(d\log x)^{\otimes 2}$. Then $\mathcal{W}(U_n)$ is generated by the cotangent fiber, whose endomorphism A_∞-algebra is isomorphic to the free algebra $\mathbf{k}\langle u, u^{-1}\rangle$ generated by an element u of degree n and its inverse u^{-1}, so that

$$\mathcal{W}(U_n) \simeq \operatorname{mod}\mathbf{k}\left\langle u, u^{-1}\right\rangle \tag{5.19}$$

$$\simeq \mathcal{R}\left(T^*S^{n+1}\right), \tag{5.20}$$

and hence

$$\mathrm{SH}^*(U_n) \simeq \mathrm{HH}^*(\mathbf{k}\left\langle u, u^{-1}\right\rangle) \tag{5.21}$$

$$\simeq \mathrm{RFH}(T^*S^{n+1}). \tag{5.22}$$

Reeb orbits in the contact boundary of U_n with winding number w come in a family parametrized by S^1, and a Bott–Morse model of the symplectic cohomology gives two generators p_w and q_w of degrees nw and $nw + 1$ in such a way that the Floer differential is given by

$$dp_w = (1 - (-1)^{nw})q_w \tag{5.23}$$

(see [BO09, Propostion 3.9]). It follows that

- if n is even, then p_w and q_w for all $w \in \mathbb{Z}$ survives,
- if n is odd, then p_w and q_w for odd w annihilates each other, and only p_w and q_w for even w survives,

and one can identify

$$p_{2m} = x_0^{2m}, \tag{5.24}$$

$$q_{2m} = x_0^{2m+1} \otimes x_0^\vee, \tag{5.25}$$

$$p_{2m+1} = x_0^m \otimes x_1^\vee \otimes \cdots \otimes x_{n+2}^\vee, \tag{5.26}$$

$$q_{2m+1} = x_0^{2m+1} x_0^\vee \otimes x_1^\vee \otimes \cdots \otimes x_{n+2}^\vee. \tag{5.27}$$

5.6. One can also compute $\operatorname{HH}^*(\mathbf{k}\langle u, u^{-1}\rangle)$ in a purely algebraic way. The enveloping algebra $R^{\mathrm{op}} \otimes R$ of $R := \mathbf{k}\langle u, u^{-1}\rangle$ is isomorphic to $E = \mathbf{k}\langle \lambda^{\pm 1}, \rho^{\pm 1}\rangle / (\lambda\rho - (-1)^n \rho\lambda)$. The diagonal bimodule Δ regarded as a right E-module is the $\mathbf{k}$-vector space $\bigoplus_{i=-\infty}^{\infty} \mathbf{k} u^i$ equipped with the action

$$u^i \cdot \lambda = (-1)^{in} u^{i+1}, \tag{5.28}$$

$$u^i \cdot \rho = u^{i+1}. \tag{5.29}$$

One has

$$\Delta \simeq \left\{ E[-n] \xrightarrow{(\lambda-\rho)\cdot} E \right\}, \tag{5.30}$$

where the right terms sits in degree zero, so that the Hochschild complex of R is given by

$$\hom_E(\Delta, \Delta) \simeq \hom_E\left(\left\{ E[-n] \xrightarrow{(\lambda-\rho)\cdot} E \right\}, \Delta \right) \tag{5.31}$$

$$\simeq \Delta \otimes_E \left\{ E \xrightarrow{\cdot(\lambda-\rho)} E[n] \right\} \tag{5.32}$$

$$\simeq \left\{ \Delta \xrightarrow{\cdot(\lambda-\rho)} \Delta[n] \right\}, \tag{5.33}$$

where the left term sits in degree zero. Since

$$u^i \cdot (\lambda - \rho) = \begin{cases} 0 & i \text{ is even}, \\ (1 - (-1)^{in}) u^{i+1} & i \text{ is odd}, \end{cases} \tag{5.34}$$

the complex (5.33) is identical to the Floer complex appearing in Section 5.5.

5.7. As another example, consider the case

$$\mathbf{W}(x_1, x_2, x_3, x_4) = x_1^k + x_2^{m-k} + x_3^2 + x_4^2. \tag{5.35}$$

The hypersurface $\mathbf{W}^{-1}(0)$ has an isolated cDV singularity at the origin.

5.8. Recall from [Rei83, Theorem 1.1] that a 3-fold singularity is terminal of index 1 if and only if it is an isolated cDV singularity. It follows from [McL16, Theorem 1.1] that the link of a terminal singularity is index-positive in the sense of [CO18, Section 9.5], so that the Rabinowitz Floer cohomology is an invariant of the link (i.e., does not depend on the symplectic filling) (see [CO18, Proposition 9.17]).

5.9. One can find a formal change of coordinates transforming $\mathbf{V}(x_0, x_1, x_2, x_3, x_4)$ to $\mathbf{W}(x_1, x_2, x_3, x_4)$ to show $\operatorname{mf}([\mathbb{A}^5/K], \mathbf{V}) \simeq \operatorname{mf}([\mathbb{A}^5/K], \mathbf{W})$ just as in Section 3. Explicit computations of $\dim \operatorname{HH}^t(\operatorname{mf}([\mathbb{A}^5/K], \mathbf{W}))$ are discussed for simple singularities in [LU21, Section 5], for more general cases in [EL23, Section 3.1], and for all of (5.35) in [APZ, Theorem C].

5.10. One has $\operatorname{HH}^t(\operatorname{mf}([\mathbb{A}^5/K], \mathbf{W})) = 0$ for $t > 3$, and a basis of $\operatorname{HH}^3(\operatorname{mf}([\mathbb{A}^5/K], \mathbf{W}))$ consists of

$$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee \in H^0(C^*(d\mathbf{W}_\gamma)) \otimes \Lambda^5 N_\gamma^\vee \cong \Lambda^5 N_\gamma^\vee \tag{5.36}$$

in the direct summand of (5.16) such that $V_\gamma = 0$ and

$$x_0^\vee x_1^\vee x_2^\vee x_3^\vee x_4^\vee \in H^{-1}(C^*(d\mathbf{W}_\gamma)) \otimes \Lambda^4 N_\gamma^\vee \cong \mathbf{k}[x_0] \otimes (\mathbf{k} x_0)^\vee \otimes \Lambda^4 N_\gamma^\vee \tag{5.37}$$

in the direct summand of (5.16) such that $V_\gamma = \mathbf{k}x_0$. It follows that $\dim \mathrm{HH}^3 (\mathrm{mf}([\mathbb{A}^5/K], \mathbf{W}))$ is equal to the number $(m-k-1)(k-1)$ of

$$\gamma \in \ker \chi = \{(t_0, t_1, t_2, t_3, t_4) \in (\mathbf{k}^\times)^5 \mid t_1^k = t_2^{m-k} = t_3^2 = t_4^2 = t_0 t_1 t_2 t_3 t_4 = 1\} \tag{5.38}$$

such that $t_i \neq 1$ for $i = 1, 2, 3, 4$, which in turn is equal to the Milnor number of the singularity defined by (5.35). One also has $\mathrm{HH}^2 (\mathrm{mf}([\mathbb{A}^5/K], \mathbf{W})) = 0$ and $\dim \mathrm{HH}^{2i} (\mathrm{mf}([\mathbb{A}^5/K], \mathbf{W})) = \dim \mathrm{HH}^{2i+1} (\mathrm{mf}([\mathbb{A}^5/K], \mathbf{W}))$ for all $i \leq 0$. A basis of $\mathrm{HH}^{-2i} (\mathrm{mf}([\mathbb{A}^5/K], \mathbf{W}))$ for $i \in \mathbb{N}$ can be divided into two kinds.

5.11. Let $m_1 \colon \mathbb{N} \to \mathbb{N}$ and $b_1 \colon \mathbb{N} \to \{0, \dots, k-1\}$ (resp. $m_2 \colon \mathbb{N} \to \mathbb{N}$ and $b_2 \colon \mathbb{N} \to \{0, \dots, m-k-1\}$) be the quotient and the remainder by k (resp. $m-k$), so that

$$b_0 = b_1(b_0) + k m_1(b_0) = b_2(b_0) + (m-k) m_2(b_0) \tag{5.39}$$

for any $b_0 \in \mathbb{N}$. A basis of the first kind in cohomological degree $-2i$ for $i \in \mathbb{N}$ is parametrized by the set

$$\mathrm{I}^i_{k,m-k} \coloneqq \{b_0 \in \mathbb{N} \mid b_1(b_0) \neq k-1, \ b_2(b_0) \neq m-k-1, \ m_1(b_0) + m_2(b_0) = i\} \tag{5.40}$$

as

$$\begin{cases} x_0^{b_0} x_1^{b_1(b_0)} x_2^{b_2(b_0)} & b_0 \text{ is even and } \gamma = (1,1,1,1,1), \\ x_0^{b_0} x_1^{b_1(b_0)} x_2^{b_2(b_0)} x_3^\vee x_4^\vee & b_0 \text{ is odd and } \gamma = (1,1,1,-1,-1). \end{cases} \tag{5.41}$$

5.12. A basis of the second kind in cohomological degree $-2i$ is parametrized by the product set of

$$\mathrm{II}^i_{k,m-k} \coloneqq \{b_0 \in \mathbb{N} \mid \exists n_1, n_2 \in \mathbb{N}, n_1 + n_2 - 1 = i \text{ and } b_0 = -1 + k n_1 = -1 + (m-k) n_2\} \tag{5.42}$$

and

$$(\boldsymbol{\mu}_k \times \boldsymbol{\mu}_{m-k}) \setminus \{1\} = \left\{\xi \in \mathbf{k} \mid \xi \neq 1 \text{ and } \xi^k = \xi^{m-k} = 1\right\} \tag{5.43}$$

as

$$\begin{cases} x_0^{b_0} x_1^\vee x_2^\vee & b_0 \text{ is even and } \gamma = (1, \xi, \xi^{-1}, 1, 1), \\ x_0^{b_0} x_1^\vee x_2^\vee x_3^\vee x_4^\vee) & b_0 \text{ is odd and } \gamma = (1, \xi, \xi^{-1}, -1, -1). \end{cases} \tag{5.44}$$

5.13. Table 1 shows $\dim \mathrm{HH}^t(\mathrm{mf}([\mathbb{A}^5/K], \mathbf{W}))$ for $3 \geq t \geq -15$ for small (m,k) to give an idea about the outcome of this computation.

(m,k)	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15
$(4,2)$	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(5,2)$	2	0	1	1	0	0	0	0	1	1	0	0	1	1	0	0	0	0	1
$(6,2)$	3	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(7,2)$	4	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	1	1	1
$(6,3)$	4	0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
$(7,3)$	6	0	2	2	0	0	1	1	1	1	0	0	2	2	0	0	2	2	0
$(8,3)$	8	0	2	2	1	1	0	0	2	2	0	0	1	1	2	2	0	0	2
$(8,4)$	9	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3

TABLE 1. Examples of $\dim \mathrm{HH}^t (\mathrm{mf}([\mathbb{A}^5/K], \mathbf{W}))$

5.14. Define

$$h^i_{k,m-k} := \dim \mathrm{HH}^{-2i}\left(\mathrm{mf}\left(\left[\mathbb{A}^5/K\right], \mathbf{W}\right)\right). \tag{5.45}$$

Set $\ell = \gcd(k, m-k)$ and write $(k, m-k) = (e\ell, f\ell)$, so that $\gcd(e,f) = 1$.

Proposition 5.1. *One has*

$$h^i_{e\ell,f\ell} = \ell h^i_{e,f} + \ell - 1. \tag{5.46}$$

Proof. If $i \not\equiv -1 \mod e+f$, then one has $\mathrm{II}^i_{e\ell,f\ell} = \emptyset$ so that

$$h^i_{e\ell,f\ell} = |\mathrm{I}^i_{e\ell,f\ell}|. \tag{5.47}$$

If we write

$$\mathrm{I}^i_{e,f} = \{b_0^{\min}, b_0^{\min}+1, \dots, b_0^{\max}\}, \tag{5.48}$$

then one has

$$\mathrm{I}^i_{\ell e,\ell f} = \{\ell b_0^{\min}, \ell b_0^{\min}+1, \dots, \ell b_0^{\max} + 2\ell - 2\}, \tag{5.49}$$

so that

$$|\mathrm{I}^i_{e\ell,f\ell}| = (\ell b_0^{\max} + 2\ell - 2) - \ell b_0^{\min} + 1 \tag{5.50}$$

$$= \ell(b_0^{\max} - b_0^{\min} + 1) + \ell - 1 \tag{5.51}$$

$$= \ell |I^i_{e,f}| + \ell - 1. \tag{5.52}$$

If $i \equiv -1 \mod e+f$, then one has $\mathrm{I}^i_{e\ell,f\ell} = \emptyset$, $|\mathrm{II}^i_{e\ell,f\ell}| = 1$, and

$$h^i_{e\ell,f\ell} = \left|\left(\boldsymbol{\mu}_{e\ell} \cap \boldsymbol{\mu}_{f\ell}\right) \setminus \{1\}\right| = \ell - 1, \tag{5.53}$$

which is equal to $\ell h^i_{e,f} + \ell - 1$ since $h^i_{e,f} = 0$. □

5.15. The following conjecture is proposed in [EL23] (see also [Pet]):

Conjecture 5.2 ([EL23, Conjecture 1.4])**.** *A compound Du Val singularity admits a small resolution if and only if the dimension of the symplectic cohomology of its Milnor fiber is constant in every negative cohomological degree. Furthermore, if this is the case, then this dimension is equal to the number of irreducible components of the exceptional locus of a small resolution.*

The following refinement of Conjecture 5.2 was also proposed by the authors of [EL23] based on unpublished calculations:

Conjecture 5.3. *Let $Y \to X$ be a small $\mathbb{Q}$-factorialization of a compound Du Val singularity $P \in X$. Let further r be the number of irreducible components of the exceptional locus, $\check{U}$ be the Milnor fiber of $P \in X$, and $\check{U}_1, \dots, \check{U}_s$ be the Milnor fibers of the resulting $\mathbb{Q}$-factorial singularities $Q_1, \dots, Q_s \in Y$. Then one has*

$$\dim \mathrm{SH}^i\left(\check{U}\right) = \sum_{j=1}^{s} \dim \mathrm{SH}^i\left(\check{U}_j\right) + r \tag{5.54}$$

for any $i < 0$.

5.16. The blow-up of

$$\operatorname{Spec} \mathbf{k}[x,y,z,w]/(xy - f(z,w)g(z,w)) \tag{5.55}$$

along the ideal $(x, f(z,w))$ is defined by $xv = f(z,w)u$ and $yu = g(z,w)v$ inside $\mathbb{A}^4_{x,y,z,w} \times \mathbb{P}^1_{u:v}$, which is the union of

$$\operatorname{Spec} \mathbf{k}[x,z,w,v]/(xv - f(z,w)) \tag{5.56}$$

and

$$\operatorname{Spec} \mathbf{k}[y,z,w,u]/(yu - g(z,w)). \tag{5.57}$$

5.17. Now we discuss Conjecture 5.3 for the case when X is defined by $\check{\mathbf{W}}$ given by (5.35) (and hence the transpose $\mathbf{W}$ of $\check{\mathbf{W}}$ is also given by (5.35)). By starting from the 3-fold

$$X = \operatorname{Spec} \mathbf{k}[x,y,z,w] / \left(x^2+y^2+z^{e\ell}+w^{f\ell}\right) \cong \operatorname{Spec} \mathbf{k}[x,y,z,w] \Bigg/ \left(xy - \prod_{i=0}^{\ell-1} \left(z^e - \zeta_\ell^i w^f\right)\right) \tag{5.58}$$

where $\zeta_\ell := \exp\left(2\pi\sqrt{-1}/\ell\right)$ and performing the blown-up in Section 5.16 $\ell-1$ times, one obtains a chain of $\ell-1$ exceptional $\mathbb{P}^1$'s and ℓ copies of $\mathbb{Q}$-factorial compound Du Val singularities isomorphic to $x^2+y^2+z^e+w^f$. It follows that one has $r=\ell-1$, $s=\ell$, and $\check{U}_1,\ldots,\check{U}_s$ are Milnor fibers of $x^2+y^2+z^e+w^f$ in this case. Since Conjecture 3.2 holds for (5.35) by Theorem 4.1, one has $\dim \mathrm{SH}^i(\check{U}) = h^i_{e\ell,f\ell}$ and $\dim \mathrm{SH}^i(\check{U}_j) = h^i_{e,f}$ for $j=1,\ldots,s$. Now Proposition 5.1 shows that Conjecture 5.3 holds for the singularity defined by (5.35).

5.18. For $\mathbf{W}$ given by (5.35), one has

$$(h, d_0) = \begin{cases} (ef\ell/2, -e-f) & \ell \text{ is even}, \\ (ef\ell, -2e-2f) & \ell \text{ is odd}. \end{cases} \tag{5.59}$$

The isomorphism $\mathrm{HH}^*(\operatorname{scoh} D) \simeq \mathrm{HH}^*(\operatorname{scoh} U) \otimes_{\mathbf{k}[s^h]} \mathbf{k}[s^h, s^{-h}]$ and the explicit description of $\mathrm{HH}^*(\operatorname{scoh} U) \cong \mathrm{HH}^*(\operatorname{mf}([\mathbb{A}^5/K], \mathbf{W}))$ above imply $\mathrm{HH}^i(\operatorname{scoh} D) \cong \mathrm{HH}^i(\operatorname{scoh} U)$ for all $i \le 0$ and that the isomorphisms $s^h\colon \mathrm{HH}^i(\operatorname{scoh} U) \xrightarrow{\sim} \mathrm{HH}^{i+2d_0}(\operatorname{scoh} U)$ for $i \le 0$ extend to isomorphisms $s^h\colon \mathrm{HH}^i(\operatorname{scoh} D) \xrightarrow{\sim} \mathrm{HH}^{i+2d_0}(\operatorname{scoh} D)$ for all $i \in \mathbb{Z}$. It follows that the symplectic cohomology and the Rabinowitz Floer cohomology of the mirror $\check{U}$ satisfies $\mathrm{RFH}^i\left(\check{U}\right) \cong \mathrm{SH}^i\left(\check{U}\right)$ for all $i \le 0$ and $\mathrm{RFH}^i\left(\check{U}\right) \cong \mathrm{RFH}^{i+2d_0}\left(\check{U}\right)$ for all $i \in \mathbb{Z}$.

5.19. As an application of the non-vanishing of the Rabinowitz Floer homology, one can obtain the non-displaceability of the contact boundary of $\check{U}$ by [CF09, Theorem 1.2].

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