Noncommutative crepant resolutions of cA_n singularities via Fukaya categories

Jonny Evans Yankı Lekili

Abstract

We compute the wrapped Fukaya category $\mathcal{W}(T^*S^1, D)$ of a cylinder relative to a divisor $D = \{p_0, \ldots, p_n\}$ of $n+1$ points, proving a mirror equivalence with the category of perfect complexes on a crepant resolution (over $k[t_0, \ldots, t_n]$) of the singularity $uv = t_0 t_1 \ldots t_n$. Upon making the base-change $t_i = f_i(x, y)$, we obtain the derived category of any crepant resolution of the cA_n singularity given by the equation $uv = f_0 \ldots f_n$. These categories inherit braid group actions via the action on $\mathcal{W}(T^*S^1, D)$ of the mapping class group of T^*S^1 fixing D. We also give geometric models for the derived contraction algebras associated to a cA_n singularity in terms of the relative Fukaya category of the disc.

1 Introduction

§1.1 Consider the Fukaya category of a point with coefficients in a ring R. Before taking the triangulated envelope, there is only one object: the point itself, with endomorphism algebra R. If R is not a field then there are non-invertible non-zero endomorphisms which allow us to construct new twisted complexes in the derived Fukaya category. Via the Yoneda embedding, we can think of the derived Fukaya category of a point with coefficients in R as $\mathrm{perf}(R)$. We can think of this as the world's lousiest A-model mirror to $\mathrm{Spec}\, R$. It is lousy in the precise sense that symplectic geometry has given us absolutely no information here: all of the interesting information is contained in the coefficient ring. The moral of the current paper is that there is a whole spectrum of ways we can get at a single triangulated A_∞-category by combining symplectic manifolds with coefficient rings. We work out in detail some examples where the symplectic manifold is a 2-dimensional cylinder.

§1.2 The starting point for these examples is the mirror symmetry result proved in [24] between (on the A-side) T^*S^1 with a collection D of $n+1$ punctures and (on the B-side) a certain reducible curve C_{n+1} with $n+1$ nodes. The two sides of the mirror, together with dual Lagrangian torus fibrations are shown in Figure 1 (the noncompact fibres on the A-side are dual to the point-like fibres on the B-side). The precise statement of mirror symmetry identifies the wrapped Fukaya category of Lagrangian branes avoiding the punctures with the derived category of perfect complexes on the nodal curve.

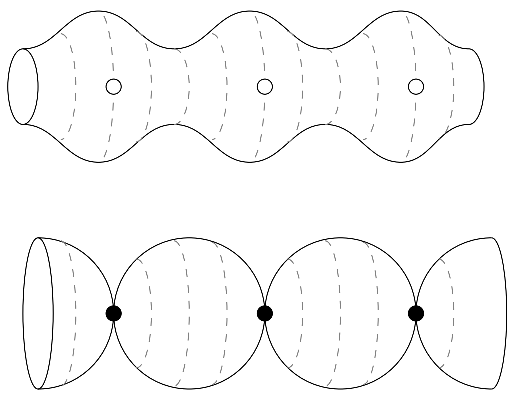


Figure 1: A punctured cylinder $T^*S^1 \setminus D$ and a nodal curve C_{n+1}. Both are equipped with dual Lagrangian torus fibrations—the fibres are the dashed curves. The fibres above are dual to those below in the sense of having reciprocal radii; the noncompact fibres ("infinite radius") through the punctures are dual to the nodes ("zero radius").

§1.3 Consider the versal deformation $\{uv = t_0 \cdots t_n\}$ of an A_n-curve singularity; this admits a crepant resolution $\mathcal{Y}$ with a morphism to $\operatorname{Spec} k[t_0, \ldots, t_n]$ whose central fibre is C_{n+1}. The B-model in our main example will be $\mathcal{Y}$. To build an A-model mirror to this, we need to find a Fukaya category which is linear over $R = k[t_0, \ldots, t_n]$ and which specialises to the Fukaya category of the $(n+1)$-punctured cylinder when the t-variables are set equal to zero. We therefore use R as the coefficient ring[1] for Floer theory on T^*S^1 and work relative to D, using intersections with D to weight polygons contributing to the Floer A_∞-operations.[2] We will further base-change coefficient rings to find mirrors to non-versal deformations.

§1.4 Here is the general setting. Let Σ be a surface (possibly non-compact) and let $D = \{z_0, \ldots, z_n\} \subset \Sigma$ be a finite set of marked points. Fix a field k, let $n = |D| - 1$, and let $R := k[t_0, \ldots, t_n]$. We consider the following wrapped Fukaya category of Σ relative to D:

- The objects are properly-immersed, exact, graded Lagrangian branes in Σ avoiding the marked points D and asymptotic to conical Lagrangians near the ends of Σ. The brane-data comprises a choice of orientation, relative spin-structure, grading, and local system.
- The hom-spaces are given by wrapped intersections (see [1] or [11, Appendix B]).

[1]to get R-linearity.
[2]to get the deformation.

- The A_∞-operations are given by counting holomorphic polygons with boundaries on (wrapped) Lagrangians, but each polygon P contributes to the corresponding operation with a weight of $\prod_{i=0}^n t_i^{\mathrm{mult}(P,z_i)} \in R$.
- Finally, we take the split-closed triangulated envelope to get an R-linear triangulated A_∞-category which we will write as $\mathcal{W}(\Sigma, D)$.

§1.5 We will frequently change our coefficient ring R. If S is an R-algebra (i.e. a ring with a morphism $R \to S$) then we will write $\mathcal{W}(\Sigma, D) \otimes_R S$ for the corresponding S-linear A_∞-category where all hom-spaces are tensored with S.

§1.6 Relative Fukaya categories have played an important role in Floer theory starting with Seidel's paper on mirror symmetry for the quartic surface [33], and the idea of deforming Floer cohomology by weighting operations according to how many times a polygon passes through a point goes back to Ozsváth and Szabo [28] in their work on Heegaard Floer homology. For a detailed exposition of Fukaya categories in the exact setting, see [31]; for wrapped categories in general, see [1] or [11, Appendix B], but for a very explicit model of the wrapped Fukaya category of a surface, see [4] and [15, Section 3.3]. For relative Fukaya categories see [30, 35] and for a very similar example of a relative Fukaya category of a surface, see [23], and for a version with an arithmetic flavour see [27].

§1.7 Main Theorem. We will focus on the specific case where Σ is the cotangent bundle T^*S^1. We will pick a collection of Lagrangian arcs $L_0, \ldots, L_n$ as shown in Figure 2. Let S be an R-algebra. We will prove the following results:

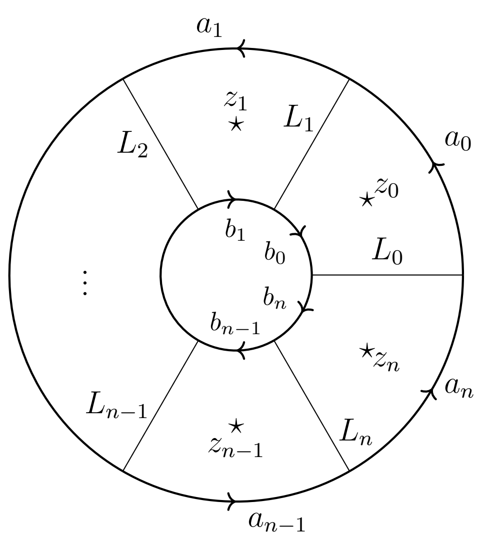


Figure 2: The surface T^*S^1 together with its Lagrangian arcs $L_0, \ldots, L_n$, marked points $z_0, \ldots, z_n$ and some of the Reeb chords a_i and b_i.

A. *The endomorphism A_∞-algebra of $\bigoplus_{i=0}^n L_i$ in $\mathcal{W}(T^*S^1, D) \otimes_R S$ is quasi-isomorphic to the algebra $\mathcal{A}(T^*S^1, D) \otimes_R S$ where $\mathcal{A}(T^*S^1, D)$ is defined in §2.1 below. This algebra is supported in degree zero, and hence has no nontrivial higher products.* (See Section 2.)

B. *Let $\mathcal{L} \subset \mathcal{W}(T^*S^1, D)$ denote the subcategory split-generated by the Lagrangian arcs $L_0, \ldots, L_n$. Then $\mathcal{L} \otimes_R S$ is preserved by the action of the mapping class group $\Gamma(T^*S^1, D)$ of compactly-supported graded symplectomorphisms of T^*S^1 fixing D pointwise.* (See Section 3.)

§1.8 Remarks. (i) In Appendix A, we will show that the arcs split-generate the category $\mathcal{W}(T^*S^1, D) \otimes_R \bar{R}$ where $\bar{R}$ is the completion $k[\![t_0, \ldots, t_n]\!]$. We expect that the arcs split-generate $\mathcal{W}(T^*S^1, D)$ itself, which would render §1.7(B) redundant, but we cannot currently see how to prove this without passing to the completion.

(ii) We will prove something slightly more general than §1.7(B) which gives quasi-equivalences for symplectomorphisms which permute the points of D. For some choices of R-algebra S, these will be autoequivalences of $\mathcal{L}$. See §3.1 for details.

(iii) By construction the algebra $\mathcal{A}(T^*S^1, D)$ is linear over R but, in fact, it turns out that it has a bigger center given by $R[u, v]/(uv - t_0 t_1 \ldots t_n)$. We expect that the autoequivalences given in §1.7(B) are linear over this bigger ring (not just linear over R). The main reason to expect this is that the additional variables u and v come from Hochschild cohomology classes of $\mathcal{A}(T^*S^1, D)$ associated with the infinite ends of T^*S^1, whereas our autoequivalences are induced by compactly supported symplectomorphisms.

§1.9 Mirror symmetry interpretation. Theorem §1.7(A) implies that

$$\mathcal{L} \simeq \mathrm{perf}(\mathcal{A}(T^*S^1, D)).$$

This category has an interpretation on the B-side. Consider the singular variety given by

$$\mathcal{Y}_0 = \operatorname{Spec} R[u, v]/(uv - t_0 \cdots t_n) \subset \mathbb{A}^{n+3}$$

This is a toric singularity. Indeed, consider the vector space $V = \mathbb{A}^{2(n+1)}$ generated by the entries of the 2-by-$(n+1)$ matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \end{pmatrix}$$

and consider the action of the torus $T = \mathbb{G}_m^n$ whose i^{th} component acts as follows:

$$\lambda : \begin{pmatrix} x_0 & \ldots & x_{i-1} & x_i & \ldots & x_n \\ y_0 & \ldots & y_{i-1} & y_i & \ldots & y_n \end{pmatrix} \to \begin{pmatrix} x_0 & \ldots & \lambda x_{i-1} & \lambda^{-1} x_i & \ldots & x_n \\ y_0 & \ldots & \lambda^{-1} y_{i-1} & \lambda y_i & \ldots & y_n \end{pmatrix}$$

Then $\mathcal{Y}_0$ can be identified with the affine GIT quotient $V /\!\!/ T$, where we can see that $t_i = x_i y_i$, $u = x_0 x_1 \dots x_n$ and $v = y_0 y_1 \dots, y_n$. The generic GIT quotients $V /\!\!/_\theta T$ provide toric crepant resolutions of $\mathcal{Y}_0$. These correspond to triangulations of $[0,1] \times \Delta_n$ where Δ_n denotes the n-simplex. All of these are (non-canonically) isomorphic to a toric Calabi-Yau variety, which we denote by $\mathcal{Y}$. These toric Calabi-Yau varieties are well-known ([9], [25]). We have a map $\mathcal{Y} \to \operatorname{Spec} R$ given by projection to $(t_0, \dots t_n)$. The fiber of this map over 0 is a nodal curve given by a chain of $\mathbb{P}^1$'s together with two $\mathbb{A}^1$'s attached at the two ends, and the total space $\mathcal{Y}$ is the versal deformation of this nodal curve.

There is a tilting bundle $\mathcal{V}$ on $\mathcal{Y}$ constructed by Van den Bergh [38]; we review this construction in Section 4. In §4.6, we will see that $\mathrm{End}_{\mathcal{Y}}(\mathcal{V})$ is precisely our algebra $\mathcal{A}(T^*S^1, D)$ and since $\mathcal{Y}$ is smooth, this means that

$$\mathcal{L} \simeq D^b(\mathrm{coh}(\mathcal{Y}))$$

which can be regarded as a relative version of homological mirror symmetry for $\mathcal{Y}$ (see also Remark §1.13).

The braid group action on $D^b(\mathrm{coh}(\mathcal{Y}))$ is constructed by Donovan-Segal [9] by the variation of GIT method, and previously by Bezrukavnikov-Riche [7] via Springer theory. Under the mirror symmetry equivalence discussed above their action on the B-side almost certainly corresponds to our braid group action on the A-side given by Theorem §1.7(B) but we do not check the details here.

§1.10 Base change. We get further results by working over an R-algebra S. Let $\mathcal{Y}_{S,0} = \mathrm{Spec}(\mathcal{O}_{Y_0} \otimes_R S)$. Let $\mathcal{Y}_S$ be the fibre product:

$$\begin{array}{ccc} \mathcal{Y}_S & \xrightarrow{j} & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{Y}_{S,0} & \longrightarrow & \mathcal{Y}_0 \end{array}$$

In §4.7, we will show that the pullback $j^*\mathcal{V}$ is still a tilting object with

$$\mathrm{End}(j^*\mathcal{V}) \cong \mathcal{A}(T^*S^1, D) \otimes_R S.$$

The variety $\mathcal{Y}_S$ is a partial resolution of $\mathcal{Y}_{S,0}$, and Theorem §1.7(B) now yields an action of $\Gamma(T^*S^1, D)$ by autoequivalences on $\mathrm{perf}(\mathcal{Y}_S)$. If $\mathcal{Y}_S$ is itself smooth, this category is quasi-equivalent to $D^b(\mathrm{coh}(\mathcal{Y}_S))$.

§1.11 Example. If we take $S = k[t]$ considered as an R-module via the homomorphism $t_i \mapsto t$ then $\mathcal{Y}_{S,0} = \mathrm{Spec}\,(k[u,v,t]/(uv - t^{n+1}))$ is the A_n surface singularity and $\mathcal{Y}_S$ is its minimal resolution, so we get a $\Gamma(T^*S^1, D)$ action on $D^b(\mathrm{coh}(\mathcal{Y}_S))$. This is one of the examples where we get a bigger group action: any compactly-supported graded symplectomorphism of T^*S^1 fixing D *setwise* acts as an autoequivalence of $\mathcal{L}$. This yields an action of the annular (extended) braid group by autoequivalences. In this example, an action of the (usual) braid group was known to Seidel and Thomas [34] and an extended braid group action was constructed by Gadbled, Thiel and Wagner in [13].

§1.12 Example. Let $f(x,y)$ be a polynomial whose lowest order term has degree $n+1$ and consider the compound A_n singularity $\{uv = f(x,y)\} \subset \mathbb{C}^4$. If f factors as $f_0 \cdots f_n$ with each curve $\{f_i(x,y) = 0\}$ smooth then the singularity admits a small resolution. This resolution has the form $\mathcal{Y}_S$ where $S = k[x,y]$ is considered as an R-algebra via the homomorphism $t_i \mapsto f_i(x,y)$. The algebra $\mathcal{A}(T^*S^1, D) \otimes_R S$ is called a *noncommutative crepant resolution* (NCCR) of this singularity: it is a noncommutative algebra whose derived category is equivalent to the derived category of the resolution.

Theorem §1.7(B) yields an action of $\Gamma(T^*S^1, D)$ on $D^b(\mathrm{coh}(\mathcal{Y}_S))$. This can be enhanced to the bigger group of symplectomorphisms: let ψ be a symplectomorphism of T^*S^1 fixing D setwise and let σ be the permutation $\psi(z_i) = z_{\sigma(i)}$; we get an autoequivalence from ψ if $f_{\sigma(i)} = f_i$ for all i. Autoequivalences of $D^b(\mathrm{coh}(\mathcal{Y}_S))$ called "mutation functors" were constructed by Iyama and Wemyss [20] using flops along the exceptional curves.

§1.13 These examples show that, although this Fukaya category leaves much of the heavy-lifting to the module category of the coefficient ring, it does readily give geometric insights which are nontrivial on the B-side. The relative Fukaya category $\mathcal{W}(T^*S^1, D)$ is appealing because working with Fukaya categories of surfaces reduces to combinatorial algebra. However, in view of [26, Conjecture E], it is possible to relate the relative Fukaya category $\mathcal{W}(T^*S^1, D)$ to an appropriate subcategory of an absolute Fukaya category of a higher dimensional symplectic manifold X. See [26, Example 2.5] for a detailed exposition of the case $D = \{1\}$.

§1.14 Derived contraction algebra. The derived contraction algebra is a DG-algebra associated to a small resolution $\mathcal{Y} \to \mathcal{Y}_0$ that prorepresents derived deformations of the irreducible components of the reduced exceptional fiber of the contraction. Concretely, it is a non-positively graded DG-algebra whose zeroth cohomology recovers the contraction algebra of Donovan and Wemyss [10]. See the papers by Hua–Toda [18], Hua [16], Hua–Keller [17], and Booth [5] for more background. The derived contraction algebra is obtained by localising a noncommutative resolution away from an idempotent. From the Fukaya-categorical description of the noncommutative resolution in the cA_n case from §1.12, we can give a geometric interpretation of this localisation: the derived contraction algebra can be described using the relative Fukaya category of the punctured disc $(T^*S^1 \setminus L_0, D)$. We discuss this in Section 6.

§1.15 Acknowledgements. JE is supported by EPSRC grant EP/W015749/1. YL is partially supported by the Royal Society URF\R\180024 and EPSRC grant EP/W015889/1. We would like to thank Michael Wemyss for enlightening discussions which led to a much

cleaner approach, and Matt Booth, Gustavo Jasso, Daniil Mamaev and Richard Thomas for helpful conversations.

2 The Floer cohomology algebra

§2.1 Definition of $\mathcal{A}(T^*S^1, D)$. Let Q_{n+1} be the quiver in Figure 3 with vertices $L_0, \ldots, L_n$ and arrows[3] $a_i \colon L_{i-1} \to L_i$, $b_i \colon L_i \to L_{i-1}$.

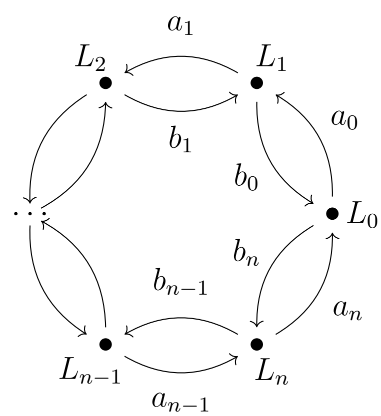


Figure 3: The quiver Q_{n+1}.

Recall that $R = k[t_0, \ldots, t_n]$. Consider the path algebra RQ_{n+1} of Q_{n+1} with coefficients in the ring R; that is elements of RQ_{n+1} are R-linear combinations of paths in Q_{n+1} and multiplication is given by concatenate-or-die. We write e_i for the idempotent corresponding to the constant (lazy) path at the vertex L_i. Let $I_R \subset RQ_n$ be the ideal of RQ_n generated by

$$a_i b_i - t_i e_{i+1}, \quad b_i a_i - t_i e_i, \quad i = 0, \ldots, n.$$

Write $\mathcal{A}(T^*S^1, D)$ for the algebra RQ_n/I_R, considered as an A_∞-algebra concentrated in degree zero with no differential or higher operations.

Theorem §1.7(A) follows immediately from the next proposition.

§2.2 Proposition. *The A_∞-algebra $\bigoplus_{i,j=0}^n CF(L_i, L_j)$ is quasi-equivalent to $\mathcal{A}(T^*S^1, D)$. Note that, in this proof, we write CF to mean* $\hom_{\mathcal{W}(T^*S^1,D)}$.

Proof. We will use the model of the Fukaya category from [15]. The arrows labelled a and b in Figure 3 represent the Reeb chords with the same names in Figure 2, considered as wrapped intersection points $a_i \in CF^0(L_i, L_{i+1})$, $b_i \in CF^0(L_{i+1}, L_i)$. All Reeb chords (called "boundary paths" in [15]) can be obtained by concatenating these, and therefore

[3] Indices are taken to belong to the cyclic group $\mathbb{Z}/(n+1)$.

the R-module $CF(L_i, L_j)$ has as a basis the set of all paths from L_i to L_j in Q_{n+1}. Here, we include the constant path e_i at L_i, thought of as the identity element of $CF(L_i, L_i)$.

Since all of these chords are concatenations of chords of degree zero, everything is in degree zero, which implies that the only nontrivial μ_k-operation on $\bigoplus_{i,j} CF(L_i, L_j)$ is μ_2: the differential and higher products all vanish. To compute μ_2, aside from concatenation of chords, we need to count polygons. The arcs L_i cut Σ into $m+1$ quadrilaterals $D_0, \ldots, D_n$, where we write D_i for the quadrilateral containing the point z_i. Using the formula[4] [15, Eq. 3.18] and keeping track of our additional weighting from the marked points, we see that:

$$\mu_2(a_i, b_i) = t_i e_{i+1} \qquad \mu_2(b_i, a_i) = t_i e_i$$

for all i, where these contributions come from D_i. Any other contributions to μ_2 would need to come from quadrilaterals, and any quadrilateral can be decomposed as a union of D_is, so any other μ_2 product can be deduced from these. □

3 Autoequivalences

§3.1 Group action. Let $R = k[t_0, \ldots, t_n]$. Given a permutation σ of $\{0, 1, \ldots, n\}$, let R_σ denote the R-module whose underlying vector space is R but t_i acts as multiplication by $t_{\sigma(i)}$. Consider the triangulated A_∞-category

$$\mathcal{W}(T^*S^1, D) \rtimes S_{n+1} := \coprod_{\sigma \in S_{n+1}} \mathcal{W}(T^*S^1, D) \otimes_R R_\sigma$$

where the morphism spaces between different components are zero. Given a graded symplectomorphism $\psi \colon T^*S^1 \to T^*S^1$ satisfying $\psi(D) = D$, we get a permutation $\sigma \in S_{n+1}$ defined by $\psi(z_i) = z_{\sigma(i)}$. This induces an autoequivalence

$$\mathcal{W}(T^*S^1, D) \rtimes S_{n+1} \to \mathcal{W}(T^*S^1, D) \rtimes S_{n+1}$$

sending $\mathcal{W}(T^*S^1, D) \otimes_R R_\tau$ to $\mathcal{W}(T^*S^1, D) \otimes_R R_{\sigma\tau}$. In particular, this gives an action of the pure annular braid group by autoequivalences on $\mathcal{W}(T^*S^1, D)$.

§3.2 Theorem. *Let $\mathcal{L}_\sigma$ denote the subcategory of $\mathcal{W}(T^*S^1, D) \otimes_R R_\sigma$ generated by the arcs $L_0, \ldots, L_n$. Then the autoequivalences from §3.1 preserve $\coprod_{\sigma \in S_{n+1}} \mathcal{L}_\sigma$.*

We now begin the proof of this theorem, which will conclude in §3.10. We will focus on the case $n \geq 2$ because it can be handled uniformly: for small n the arguments are similar but the pictures are slightly different because $L_1 = L_n$ or $L_0 = L_1 = L_n$. Throughout the argument we will ignore signs and orientations of moduli spaces. The reason we can get away with this is explained in Remark §3.11.

[4] The authors of [15] state this formula for μ_k with $k \geq 3$ only because they do not have any quadrilaterals like D_i in [15].

§3.3 Let $\psi_i\colon T^*S^1 \to T^*S^1$ denote the half-twist around the arc connecting z_i to z_{i+1} (indices taken modulo $n+1$). Let $\rho\colon T^*S^1 \to T^*S^1$ denote the symplectomorphism which preserves concentric circles in T^*S^1, fixing the two boundary components pointwise and rotating the points of D by $2\pi/(n+1)$. Let δ be a boundary-parallel Dehn twist parallel to the inner boundary of T^*S^1. The mapping classes $\psi_0, \ldots, \psi_n, \rho, \delta$ generate the graded symplectic mapping class group: see[5] [13, Section 1]. The symplectomorphism δ acts trivially on our Lagrangians: they are objects of the wrapped category and δ is part of the wrapping that we would do anyway to compute hom-spaces. The symplectomorphism ρ cyclically permutes the L_i. So to prove that $\Gamma(T^*S^1, D)$ preserves $\mathcal{L}$, it suffices to check that $\psi_i(L_j)$ is generated by the arcs $L_0, \ldots, L_n$ for all i, j. In fact, $\psi_i(L_j) = L_j$ unless $i = j$, so we just need to study $\psi_i(L_i)$. Moreover, by cyclic symmetry of (T^*S^1, D) we can assume that $i = 0$.

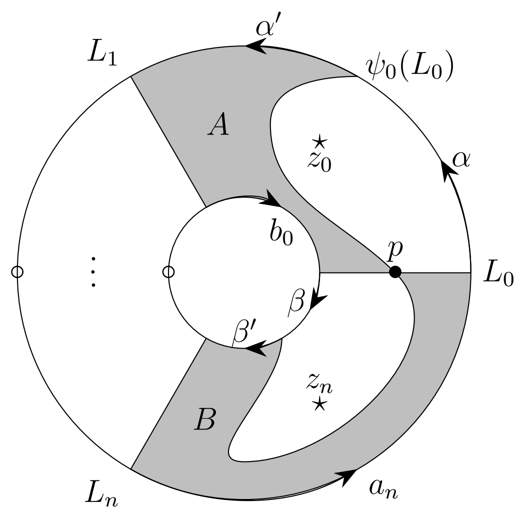


Figure 4: The half-twisted arc $\psi_0(L_0)$, perturbed slightly along the Reeb flow to separate it from L_0. We have added two stops on the boundary for convenience; these are labelled $\circ$. We have also labelled the Reeb orbits connecting the Lagrangian arcs. Note that $a_0 = \alpha'\alpha$ and $b_n = \beta'\beta'$. The point p (marked with a $\bullet$) is an intersection point of L_0 with $\psi_0(L_0)$. Two important polygonal regions A and B are shaded.

§3.4 The half-twisted arc $\psi_0(L_0)$ is shown in Figure 4. To localise the calculation near the diagram, we will insert a stop (in the sense of Sylvan [37]) on each of the two boundary components and work first in the partially wrapped Fukaya category. We will write down a twisted complex $\mathbb{L}'$ built out of L_n, L_0 and L_1 and a quasi-isomorphism $q \in CF(\mathbb{L}', \psi_0(L_0))$. If we then apply Sylvan's stop removal functor to this twisted complex, we obtain a twisted complex $\mathbb{L}$ in $\mathcal{W}(T^*S^1, D)$ which is quasi-isomorphic to $\psi_0(L_0)$.

[5]Gadbled, Thiel and Wagner treat the inner boundary as a puncture, so do not need δ.

§3.5 The advantage of inserting stops is that the partially wrapped Floer cohomology is easy to read off from Figure 4:

$$
\begin{aligned}
CF(\psi_0(L_0), L_0) &= R \cdot p, & CF(L_0, \psi_0(L_0)) &= R \cdot p \oplus R \cdot \alpha \oplus R \cdot \beta, \\
CF(\psi_0(L_0), L_1) &= R \cdot \alpha', & CF(L_1, \psi_0(L_0)) &= R \cdot (\beta b_0), \\
CF(\psi_0(L_0), L_n) &= R \cdot \beta', & CF(L_n, \psi_0(L_0)) &= R \cdot (\alpha a_n).
\end{aligned}
$$

All of these morphisms are in degree zero except for p which is in degree 1.

§3.6 Consider the twisted complex

$$\mathbb{L}' := \left(L_1 \oplus L_n \xrightarrow{(b_0, a_n)} L_0 \right)$$

and the morphisms $q_1 \colon \mathbb{L}' \to \psi_0(L_0)$ and $q_2 \colon \psi_0(L_0) \to \mathbb{L}'$ defined by[6]

$$
q_1 : \begin{tikzcd} L_1 \oplus L_n \arrow[r, "(b_0, a_n)"] & L_0 \arrow[dl, "p"] \\ \psi_0(L_0) & \end{tikzcd}
\qquad\qquad
q_2 : \begin{tikzcd} \psi_0(L_0) \arrow[d, "(\alpha', \beta')"] & \\ L_1 \oplus L_n \arrow[r, "(b_0, a_n)"'] & L_0 \end{tikzcd}
$$

We need to show that $\mu_2^{Tw}(q_1, q_2)$ and $\mu_2^{Tw}(q_2, q_1)$ are equal to the identity elements of $CF(\psi_0(L_0), \psi_0(L_0))$ and $CF(\mathbb{L}', \mathbb{L}')$ respectively (we are using Seidel's convention for composition, right-to-left). We compute μ_2^{Tw} by stacking the morphisms and then taking all possible paths through the resulting diagram, composing wherever possible.

§3.7 To calculate $\mu_2^{Tw}(q_2, q_1)$, we have the following diagram:

$$
\begin{tikzcd}
L_1 \oplus L_n \arrow[r, "(b_0, a_n)"] & L_0 \arrow[dl, "p"] \\
\psi_0(L_0) \arrow[d, "(\alpha', \beta')"'] & \\
L_1 \oplus L_n \arrow[r, "(b_0, a_n)"'] & L_0
\end{tikzcd}
$$

There are several routes through the diagram connecting the top row to the bottom. There are two paths that involve three morphisms:

[6] We will write twisted complexes horizontally and morphisms between them vertically.

$$\begin{tikzcd}[column sep=large]
L_1 \oplus L_n \arrow[r, "{(b_0, a_n)}"] \arrow[d, "{\begin{pmatrix} \mu_3(\alpha', p, b_0) & \mu_3(\alpha', p, a_n) \\ \mu_3(\beta', p, b_0) & \mu_3(\beta', p, a_n) \end{pmatrix}}"'] & L_0 \arrow[d, "{\mu_3(b_0, \alpha', p) + \mu_3(a_n, \beta', p)}"] \\
L_1 \oplus L_n \arrow[r, "{(b_0, a_n)}"'] & L_0
\end{tikzcd}$$

There is also a path of length 2 connecting L_0 to $L_1 \oplus L_n$ and one of length 4 connecting $L_1 \oplus L_n$ to L_0. Both of these concatenations vanish for degree reasons.

§3.8 The μ_3 products contributing to $\mu_2^{Tw}(q_2, q_1)$ are:

$$\begin{tikzcd}[column sep=large]
L_1 \oplus L_n \arrow[r, "{(b_0, a_n)}"] \arrow[d, "{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}"'] & L_0 \arrow[d, "1"] \\
L_1 \oplus L_n \arrow[r, "{(b_0, a_n)}"'] & L_0
\end{tikzcd}$$

For example, the products $\mu_3(\alpha', p, b_0) = 1$ and $\mu_3(\beta', p, b_0) = 1$ come from the shaded polygons in Figure 4. To understand $\mu_3(\alpha', p, b_0) = 1$, we make partially wrapped perturbations (Figure 5). We are trying to compute

$$\mu_3 \colon CF(\psi_0(L_0)', L_1) \otimes CF(L_0'', \psi_0(L_0)') \otimes CF(L_1''', L_0'') \to CF(L_1''', L_1).$$

Here, each prime indicates that we have partially wrapped, and that we have wrapped *more* when there are more primes. We see that the polygonal region A from Figure 4 becomes a quadrilateral with vertices at α', p, b_0, and at the unique intersection point $L_1''' \cap L_1$ which represents $1 \in CF(L_1''', L_1)$. The calculation of $\mu_3(\beta', p, a_n)$ is similar.

§3.9 The calculation of $\mu_3(b_0, \alpha', p) + \mu_3(a_n, \beta', p)$ comes down to the same two polygons, but it is more subtle. Either polygon A contributes to $\mu_3(a_n, \beta', p) = 1$ and the other term is zero, or else polygon B contributes to $\mu_3(b_0, \alpha', p) = 1$. Which of these eventualities occurs depends on the choice of partial wrappings; see Figure 6.

§3.10 To calculate $\mu_2^{Tw}(q_1, q_2)$, we have the following diagram:

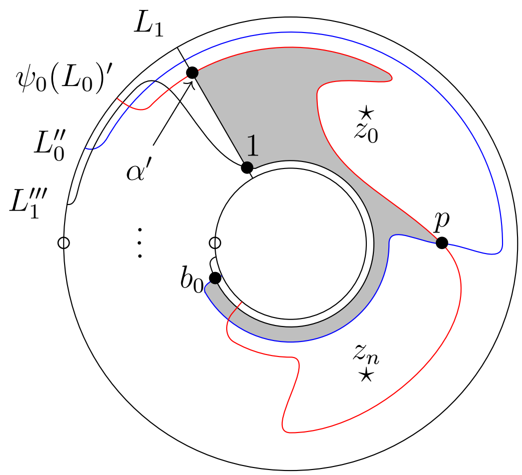


Figure 5: The computation $\mu_3(\alpha', p, b_0) = 1$ via partial wrapping.

$$\begin{tikzcd}
\psi_0(L_0) \arrow[d, "(\alpha',\beta')"'] & \\
L_1 \oplus L_n \arrow[r, "(b_0,a_n)"] & L_0 \arrow[dl, "p"] \\
\psi_0(L_0) &
\end{tikzcd}$$

There is only one route from the top row to the bottom, which means that

$$\begin{aligned}
\mu_2^{Tw}(q_1, q_2) &= \mu_3(p, (b_0, a_n), (\alpha', \beta')) \\
&= \mu_3(p, b_0, \alpha') + \mu_3(p, a_n, \beta')
\end{aligned}$$

As with the calculation in §3.9, this yields $1 \in CF(\psi_0(L_0), \psi_0(L_0))$. This shows that q_1 and q_2 are mutually inverse quasi-isomorphisms, which completes the proof. $\square$

§3.11 Remark about signs. In this proof, we completely ignored signs. If we insert all the undetermined signs, the arguments yield

$$\mu_2^{Tw}(q_1, q_2) = \pm \operatorname{id}_{\psi_0(L_0)}, \qquad \mu_{q_1}^{Tw}(q_2, q_2) = (\pm \operatorname{id}_{L_1}) \oplus (\pm \operatorname{id}_{L_n}) \oplus (\pm \operatorname{id}_{L_0}).$$

At this point, we pass to cohomology and consider the morphisms $[q_1] \in HF(\mathbb{L}, \psi_0(L_0))$ and $[q_2] \in HF(\psi_0(L_0), L_0)$. The morphisms

$$[q_1] \in HF(\mathbb{L}, \psi_0(L_0)), \qquad [q_2] \circ [q_1] \circ [q_2] \in HF(\psi_0(L_0), \mathbb{L})$$

are now mutually inverse because all signs are squared in the composites $[q_1] \circ [q_2] \circ [q_1] \circ [q_2]$ and $[q_2] \circ [q_1] \circ [q_2] \circ [q_1]$.

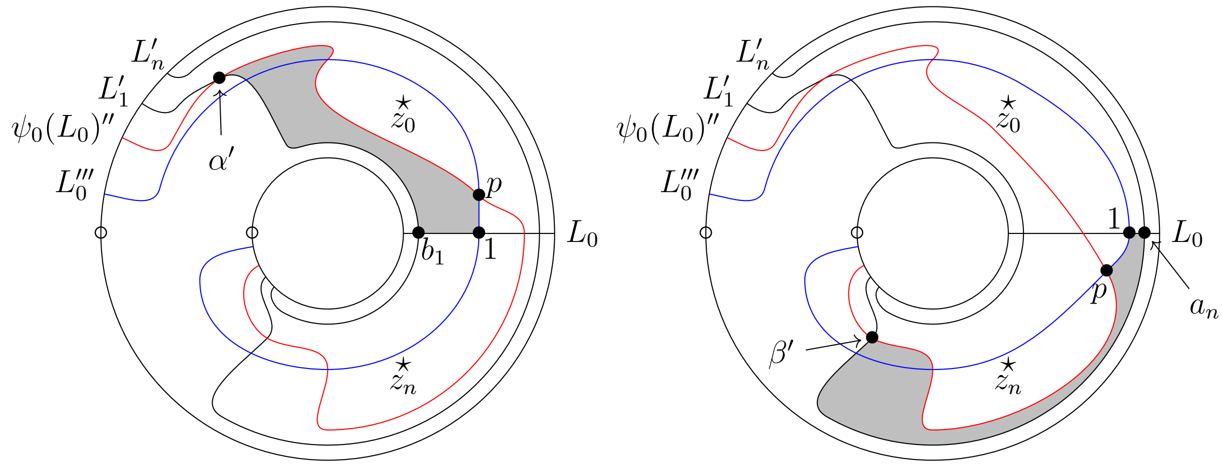


Figure 6: The computation $\mu_3((b_0, a_n), (\alpha', \beta'), p) = 1$ via partial wrapping. Depending on the choice of wrapping, either the polygon A or the polygon B yields a contribution. Left: With this choice of wrapping, we have $\mu_3(b_0, \alpha', p) = 1$ and $\mu_3(a_n, \beta', p) = 0$. Right: With this choice of wrapping, we have $\mu_3(b_0, \alpha', p) = 0$ and $\mu_3(a_n, \beta', p) = 1$.

4 B-side

§4.1 Setup. As in the introduction, let $R = k[t_0, \dots, t_n]$, let $\mathcal{Y}_0 = \operatorname{Spec} R[u, v]/(uv - t_0 \cdots t_n))$, and let $f\colon \mathcal{Y}_0 \to \mathbb{A}^{n+1}$ be the morphism given by $(t_0, \dots, t_n)$. This morphism f is the versal deformation of the A_n curve singularity. We have a toric crepant resolution $\pi\colon \mathcal{Y} \to \mathcal{Y}_0$ given by a triangulation of $[0,1] \times \Delta_n$.

§4.2 The Van den Bergh tilting bundle. We now describe a tilting bundle on $\mathcal{Y}$, making explicit the construction of Van den Bergh [38, Propositions 3.2.5, 3.2.10] in this example. Recall from §1.9 that $\mathcal{Y}$ is the GIT quotient $V /\!\!/_\theta T$, where V is the space of 2-by-$(n+1)$ matrices

$$\begin{pmatrix} x_0 & \dots & x_i & x_{i+1} & \dots & x_n \\ y_0 & \dots & y_i & y_{i+1} & \dots & y_n \end{pmatrix}$$

and the torus $T = \mathbb{G}_m^n$ acts as

$$\begin{pmatrix} \lambda_1 x_0 & \dots & \lambda_i^{-1}\lambda_{i+1} x_i & \lambda_{i+1}^{-1}\lambda_{i+2} x_{i+1} & \dots & \lambda_n^{-1} x_n \\ \lambda_1^{-1} y_0 & \dots & \lambda_i \lambda_{i+1}^{-1} y_i & \lambda_{i+1}\lambda_{i+2}^{-1} y_{i+1} & \dots & \lambda_n y_n \end{pmatrix}$$

and θ is the character $\theta(\lambda_1, \dots, \lambda_n) = \lambda_1 \cdots \lambda_n$ of T.

Given another character $\chi\colon T \to \mathbb{C}^*$, we get a line bundle $(V \times \mathbb{C}) /\!\!/_\theta T$ over $\mathcal{Y}$, where T acts with weight χ on $\mathbb{C}$. Let $\mathcal{M}_i$ be the line bundle corresponding to the character $\chi_i(\lambda_1, \dots, \lambda_n) = \lambda_i$. The sections of $\mathcal{M}_i$ are in bijection with the polynomials in the variables x_i, y_i which have weight χ under the action of T. For example, x_0 is a section of $\mathcal{M}_1$ and y_n is a section of $\mathcal{M}_n$.

§4.3 Lemma. *The sections of $\mathcal{O}_\mathcal{Y}$ form a ring isomorphic to $R[u,v]/(uv - t_0 \cdots t_n)$. The sections of $\mathcal{M}_i$ form a module over this ring which is generated by $\sigma_i \coloneqq x_0 \cdots x_{i-1}$ and $\tau_i \coloneqq y_i \cdots y_n$.*

Note that since $\pi_* \mathcal{O}_\mathcal{Y} = \mathcal{O}_{\mathcal{Y}_0}$ we can think of $H^0(\mathcal{M}_i)$ as an $\mathcal{O}_\mathcal{Y}$-module or an $\mathcal{O}_{\mathcal{Y}_0}$-module. It is isomorphic to the $R[u,v]/(uv - t_0 \cdots t_n)$-module $(u, t_0 \cdots t_{i-1})$ by identifying σ_i with u and τ_i with $t_0 \cdots t_{i-1}$.

Proof. Consider the monomial $x_0^{c_0} \cdots x_n^{c_n} y_0^{d_0} \cdots y_n^{d_n}$. The condition that this defines a section of $\mathcal{O}_\mathcal{Y}$ is that $c_i + d_{i+1} - c_{i+1} - d_i = 0$ for all $i = 0, \dots, n-1$. This implies that $c_0 - d_0 = \cdots = c_n - d_n$. If this common value is positive then the monomial can be written as

$$t_0^{d_0} \cdots t_n^{d_n} u^{c_0 - d_0}$$

otherwise it can be written as

$$t_0^{c_0} \cdots t_n^{c_n} v^{d_0 - c_0}$$

where we are defining

$$u = x_0 \cdots x_n, \qquad v = y_0 \cdots y_n, \qquad t_i = x_i y_i$$

as in §1.9. The argument for the sections of $\mathcal{M}_i$ is similar except one is left with an additional factor of $x_0 \cdots x_{i-1}$ or $y_{i+1} \cdots y_n$ depending on whether $c_i > d_i$ or $d_{i+1} > c_i$. $\square$

§4.4 Lemma. *Let $\mathcal{M} = \bigoplus_{i=1}^n \mathcal{M}_i$. Consider the $n-1$ sections*

$$\begin{aligned} s_1 &= (\sigma_1, \tau_2, 0, \dots, 0) \\ s_2 &= (0, \sigma_2, \tau_3, 0, \dots, 0) \\ &\vdots \\ s_{n-1} &= (0, \cdots, 0, \sigma_{n-1}, \tau_n) \end{aligned}$$

These sections are everywhere linearly independent, and hence span a copy of the trivial bundle of rank $n-1$ inside $\mathcal{M}$.

Proof. At each point of $\mathcal{Y}$, the wedge product $s_1 \wedge s_2 \wedge \cdots \wedge s_{n-1}$ has components

$$\begin{gathered} \tau_2 \cdots \tau_n, \\ \sigma_1 \tau_3 \cdots \tau_n, \\ \sigma_1 \sigma_2 \tau_4 \cdots \tau_n, \\ \vdots \\ \sigma_1 \cdots \sigma_{n-1}. \end{gathered}$$

If the sections are linearly dependent somewhere then all of these components vanish at that point. Let j be minimal such that $\sigma_j = 0$; note that this implies $x_j = 0$. Since

$\sigma_1 \cdots \sigma_{j-1}\tau_{j+1} \cdots \tau_n = 0$ we deduce that some $\tau_k = 0$ for $k > j$, and for the maximal such k we have that $y_k = 0$. But the unstable locus for GIT is the union of the subvarieties $\{x_j = y_k = 0\}$ for $0 \leq j < k \leq n$, so on the GIT quotient $\mathcal{Y}$ there are no points where these sections vanish simultaneously. □

§4.5 Corollary. *Let $\mathcal{L}$ be the quotient of $\mathcal{M}$ by the trivial subbundle spanned by these sections. Then $\mathcal{L}$ is an ample line bundle on $\mathcal{Y}$ and $\mathcal{V} := \mathcal{O}_\mathcal{Y} \oplus \mathcal{M}$ is a tilting bundle.*

Proof. The quotient is a line bundle and is therefore determined by its first Chern class, which is in turn determined by its restriction to the curve $\{t_0 = \cdots = t_n = 0\} \subset \mathcal{Y}$. This curve is a chain comprising n copies of $\mathbb{P}^1$ which generate $H_2(\mathcal{Y};\mathbb{Z})$ as well as two copies of $\mathbb{A}^1$ at either end of the chain. The bundle $\mathcal{M}_i$ restricts to the bundle $\mathcal{O}(1)$ on the ith $\mathbb{P}^1$ and to the trivial bundle on the other $\mathbb{P}^1$s, which means that $\mathcal{L}$ restricts to $\mathcal{O}(1)$ on all the $\mathbb{P}^1$s. Since the compact irreducible components of fibres of $\pi \colon \mathcal{Y} \to \mathcal{Y}_0$ are chains of $\mathbb{P}^1$s homologous to the positive linear combinations of $\mathbb{P}^1$s in this chain, this implies that $\mathcal{L}$ is relatively ample.

Since the bundles $\mathcal{M}_i$ are toric line bundles generated by global sections, we have [12, Corollary on p.74]

$$\mathrm{Ext}^j(\mathcal{O}_\mathcal{Y}, \mathcal{M}_i) = 0 \text{ for all } j > 0.$$

If we can show that $\mathrm{Ext}^1(\mathcal{M}_i, \mathcal{O}_\mathcal{Y}) = 0$ then we can use [38, Lemma 3.2.3] to deduce that $\mathrm{Ext}^*(\mathcal{O}_\mathcal{Y} \oplus \mathcal{M}, \mathcal{O}_\mathcal{Y} \oplus \mathcal{M})$ is supported in degree zero and argue as in [38, Proposition 3.2.5] to deduce that $\mathcal{O}_\mathcal{Y} \oplus \mathcal{M}$ generates.

Tensoring with $\mathcal{M}_i^{-1}$ we see that $\mathrm{Ext}^1(\mathcal{O}_\mathcal{Y}, \mathcal{M}_i) \cong H^1(\mathcal{M}_i^{-1})$. By projecting to $(t_0, \ldots, t_n)$, we can view $\mathcal{Y}$ as a family over $\mathbb{A}^{n+1}$ which is the versal family of deformations of the nodal curve of the form $\mathbb{A}^1 \cup_{pt} \mathbb{P}^1 \cup_{pt} \mathbb{P}^1 \cup_{pt} \ldots \mathbb{P}^1 \cup_{pt} \mathbb{A}^1$ with $n+1$ nodes. Any other fiber C_t of this family is given by a nodal curve obtained from C_0 by smoothing the nodes corresponding the non-zero component of $t = (t_0, \ldots, t_n)$. The restriction of $\mathcal{M}_i^{-1}$ to these curves gives a line bundle on C_t whose restriction to the rational components of C_t are either all trivial or in at most one component it restricts to $\mathcal{O}(-1)$. In any case, $H^1(\mathcal{M}_i^{-1}|_{C_t}) = 0$ for any t, which then implies $H^1(\mathcal{M}_i^{-1}) = 0$ as claimed. □

§4.6 Corollary. *The derived category of $\mathcal{Y}$ is quasi-equivalent to the derived category of modules over $\mathcal{A}(T^*S^1, D)$.*

Proof. Since $\mathcal{O}_\mathcal{Y} \oplus \mathcal{M}$ is a tilting object, the derived category of $\mathcal{Y}$ is quasi-equivalent to the derived category of modules of $\mathrm{End}_\mathcal{Y}(\mathcal{O}_\mathcal{Y} \oplus \mathcal{M})$. This can be computed directly via toric geometry. Indeed, we have $\mathrm{End}_\mathcal{Y}(\mathcal{M}_i, \mathcal{M}_j) \cong H^0(\mathcal{M}_j \otimes \mathcal{M}_i^{-1})$ which, as in §4.3, can be identified with the set of polynomials $p \in k[x_i, y_j]$ in the Cox ring such that $p(\lambda \cdot x) = \chi_{-i,j}(\lambda)p(x)$ for all $\lambda \in T$, where $\chi_{-i,j}(\lambda_1, \ldots, \lambda_n) = \lambda_i^{-1}\lambda_j$. Assuming $i \geq j$ without loss of generality, such polynomials are generated freely over R by

$$x_i x_{i+1} \ldots x_n x_0 \ldots x_{j-1} u^r, \qquad y_j y_{j+1} \ldots y_{i-1} v^s \text{ for } r, s \in \mathbb{Z}_{\geq 0}.$$

Note that $\mathrm{End}_{\mathcal{Y}}(\mathcal{M}_i, \mathcal{M}_i) \cong \mathcal{O}_{\mathcal{Y}}$ itself is freely generated over R by $\{1, u^r, v^s : r, s \in \mathbb{Z}_{\geq 0}\}$.

The identification with the algebra $\mathcal{A}(T^*S^1, D)$ follows immediately by sending the above basis over $R = k[t_0, t_1, \dots, t_n]$ to

$$(a_i \dots a_n a_0 \dots a_{j-1}) \cdot (\sum_{i=0}^{n} a_i \dots a_n a_0 \dots a_{i-1})^r, \qquad b_j b_{j+1} \dots b_{i-1} \cdot (\sum_{i=0}^{n} b_i \dots b_n b_0 \dots b_{i-1})^s$$

Finally, multiplication is induced by the product in the ring of n-by-n matrices with entries in the Cox ring $k[x_i, y_j]$ which then gives the desired isomorphism with $\mathcal{A}(T^*S^1, D)$. $\square$

One can also perform this calculation entirely within the category of Cohen-Macaulay modules over $\mathcal{O}_{\mathcal{Y}_0}$; for details, see the forthcoming work of Zhang [40].

§4.7 Corollary (Base-change). *Let S be a finitely generated R-algebra. Let $\mathcal{Y}_{S,0} = \mathrm{Spec}(\mathcal{O}_{\mathcal{Y}_0} \otimes_R S)$ and consider the diagram*

$$\begin{tikzcd}
\mathcal{Y}_S \arrow[r, "j"] \arrow[d] \arrow[dd, bend right=60, "g"'] & \mathcal{Y} \arrow[d] \arrow[dd, bend left=60, "f"] \\
\mathcal{Y}_{S,0} \arrow[r] \arrow[d] & \mathcal{Y}_0 \arrow[d] \\
\mathrm{Spec}(S) \arrow[r, "i"] & \mathrm{Spec}(R)
\end{tikzcd}$$

where $\mathcal{Y}_S$ is the fibre product. The pullback $j^*\mathcal{V}$ is a tilting bundle on $\mathcal{Y}_S$ with $\mathrm{End}_{\mathcal{Y}_S}(j^*\mathcal{V}) \cong \mathcal{A}(T^*S^1, D) \otimes_R S$. In particular, by §1.7(B), the derived category of perfect modules on $\mathcal{Y}_S$ inherits an action of $\Gamma(T^*S^1, D)$.

Proof. The map $\mathcal{Y} \to \mathrm{Spec}(R)$ is a conic fibration over $\mathbb{A}^{n+1}$ with equidimensional fibres and smooth (in particular, Cohen-Macaulay) total space, hence flat. The endomorphism bundle $\mathrm{End}_{\mathcal{Y}}(\mathcal{V})$ is a locally free $\mathcal{O}_{\mathcal{Y}}$-module, so $\mathcal{V}$ is flat over $\mathrm{Spec}(R)$ by [6, Lemma 2.2]. By [6, Lemma 2.9], this implies that $j^*\mathcal{V}$ is a tilting bundle with $\mathrm{End}_{\mathcal{Y}_S}(j^*\mathcal{V}) \cong g_* \mathrm{End}_{\mathcal{Y}_S}(j^*\mathcal{V}) \cong i^* f_* \mathrm{End}_{\mathcal{Y}}(\mathcal{V}) \cong i^*\mathcal{A}(T^*S^1, D) \cong \mathcal{A}(T^*S^1, D) \otimes_R S$. This base-change formula is used in the proof of [6, Lemma 2.9] but can also be found in [22, Lemma 2.10] where the pullbacks are left-derived; in our case all the modules are either free or locally free, so derived pullback equals pullback. $\square$

5 A 1-d picture of a 3-d sphere

We conclude by discussing an example which displays how one can draw 1-dimensional pictures corresponding to sheaves on the higher dimensional mirrors. Let $n = 1$; in this case $\mathcal{Y}$ is the usual small-resolved conifold which is the total space of the vector bundle

$\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$. The pushforward of the structure sheaf of $\mathbb{P}^1$ is well-known to be a 3-spherical object S in $D^b \operatorname{coh}(\mathcal{Y})$. It can be resolved by line bundles as follows:

$$\mathcal{O}(2) \xrightarrow{(y_0,-x_1)} \mathcal{O}(1)^{\oplus 2} \xrightarrow{(x_1,y_0)} \mathcal{O}$$

and $\mathcal{O}(2)$ in turn is equivalent to $\mathcal{O} \xrightarrow{(x_0,y_1)} \mathcal{O}(1)^{\oplus 2}$, where $\mathcal{O}(i)$ denote the line bundles on $\mathcal{Y}$ with degree i on $\mathbb{P}^1$. We can, therefore, express the mirror to the 3-spherical object S, in terms of the generators of $\mathcal{W}(T^*S^1, D)$ and then work out, using the surgery exact triangle on the A-side, which immersed Lagrangian it corresponds to. In Figure 7, the thick curve is this immersed Lagrangian. Note that this immersed Lagrangian is unobstructed: it does bound four "teardrops" (monogons) which would contribute to the curved A_∞-operation μ_0, but these appear in cancelling pairs passing through the same marked point (and hence weighted by the same variable).

The gray curve is a small pushoff. The Floer complex between these two curves has eight generators, living in the following degrees:

degree	-2	-1	0	1	2	3
generators	y	x, z	e	m	$\overline{x}, \overline{z}$	$\overline{y}$

The Floer differential can be computed as follows:

$$\begin{aligned} &\partial y = t_1 z - t_0 x, && \partial x = t_1 e, && \partial z = t_0 e \\ &\partial e = 0, && \partial m = t_1 \overline{x} - t_0 \overline{z} && \\ &\partial \overline{x} = t_1 \overline{y}, && \partial \overline{z} = t_0 \overline{y}, && \partial \overline{y} = 0. \end{aligned}$$

which yields cohomology of $k[t_0, t_1]/(t_0, t_1) = k$ in degrees 0 and 3.

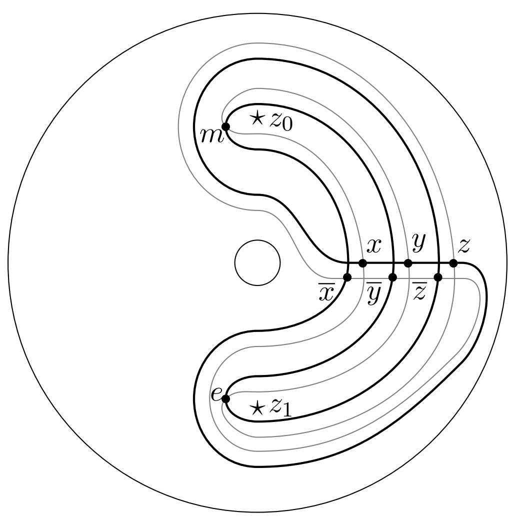


Figure 7: A 3-spherical object in $\mathcal{W}(T^*S^1, D)$ where $|D| = 2$. The gray curve is a small pushoff, used to compute the Floer complex.

It is also possible to verify directly that this immersed Lagrangian corresponds to a simple module of $\mathcal{A}(T^*S^1, D)$ dual to L_0.

6 Derived contraction algebra

§6.1 Let $\mathcal{Y}_0$ be a 3-fold compound Du Val singularity admitting a small resolution $\mathcal{Y}$. The *derived contraction algebra* of $\mathcal{Y}$ is an enhancement of the contraction algebra Λ of Donovan and Wemyss [10] in the sense that $\Lambda = H^0(\Gamma)$. The derived contraction algebra can be understood as the Drinfeld localisation of the endomorphism algebra $\mathrm{End}(\mathcal{V})$ of the tilting bundle on $\mathcal{Y}$ with respect to the idempotent $e = \mathrm{id}_{\mathcal{O}_\mathcal{Y}}$ corresponding to the structure sheaf $\mathcal{O}_\mathcal{Y}$. Recall that the Drinfeld localisation is given by

$$\mathrm{End}(\mathcal{V})_e = \mathrm{End}(\mathcal{V})\langle \epsilon \rangle / (\epsilon e = e\epsilon = \epsilon,\, d\epsilon = e),$$

that is we freely introduce an element ϵ to $\mathrm{End}(\mathcal{O}_\mathcal{Y})$ of degree -1 with $d\epsilon = e$. This kills the corresponding object in $D^b(\mathrm{End}(\mathcal{V})) \simeq D^b(\mathcal{Y})$ after localisation:

$$\mathrm{perf}(\mathrm{End}(\mathcal{V})_e) \simeq D^b(\mathcal{Y})/\langle \mathcal{O}_\mathcal{Y} \rangle.$$

§6.2 Let us consider the case of a compound A_N singularity. Recall that in this case we have a 3-fold singularity given by $uv = f_0(x,y)f_1(x,y)\dots f_n(x,y)$. The relative Fukaya category is derived equivalent to the algebra $\mathcal{A}(T^*S^1, D) \otimes_R S$ where $S := k[x,y]$ is viewed as an R-algebra by the homomorphism $t_i \to f_i(x,y)$. By Corollary §4.7, $\mathcal{A}(T^*S^1, D) \otimes_R S$ is isomorphic to the algebra $\mathrm{End}_{Y_S}(j^*\mathcal{V})$ of endomorphisms of the tilting bundle $j^*V = \mathcal{O}_{Y_S} \oplus j^*\mathcal{M}$. Hence the derived contraction algebra is given by

$$\left(\mathcal{A}(T^*S^1, D) \otimes_R S\right)_{e_0}, \quad e_0 = \mathrm{id}_{L_0}.$$

That is, the localisation of $D^b(\mathcal{Y}_S)$ away from $\mathcal{O}_{Y_S}$ corresponds to localisation away from the Lagrangian L_0 in the relative Fukaya category $\mathcal{W}(T^*S^1, D) \otimes_R S$. In the remainder of this section, we will give an alternative, more geometric, description of the derived contraction algebra in terms of the relative Fukaya category of a disc.

§6.3 Theorem. *Let Δ be the disc obtained by excising L_0 from T^*S^1 (Figure 8). The derived contraction algebra of a 3-fold compound A_n singularity is quasi-equivalent to the endomorphism algebra of $\bigoplus_{i=1}^n L_i$ in the relative Fukaya category $\mathcal{W}(\Delta, D) \otimes_R S$.*

Proof. We can think of the annulus T^*S^1 as the result of attaching a Weinstein 1-handle to the disc, with L_0 as the cocore of the handle. By Ganatra, Pardon and Shende [14, Proposition 11.2], this means that the localisation

$$\left(\mathcal{W}(T^*S^1, D) \otimes_R S\right) / \langle L_0 \rangle$$

is quasi-equivalent to the relative Fukaya category of the disc Δ we get by excising L_0 from T^*S^1. This proves the theorem. ∎

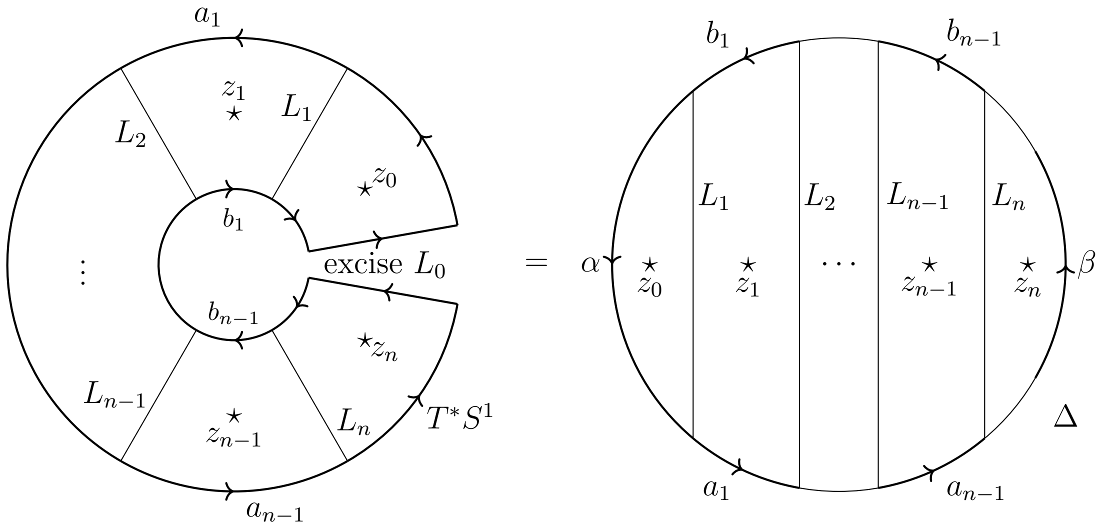
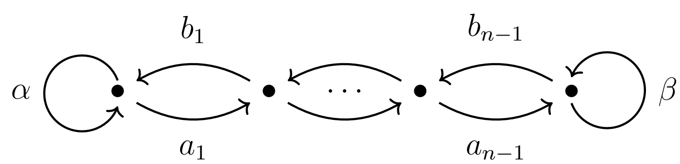


Figure 8: Relative Fukaya category of the disc as a localisation.

§6.4 A model for the derived contraction algebra. We now give a model for the A_∞-algebra $\mathrm{End}_{\mathcal{W}(\Delta,D)}(\bigoplus_{i=1}^n L_i)$. This can be calculated directly. It is given by taking the R-linear path algebra of the following quiver



imposing the relations (coming from the quadrilaterals with boundary $b_i \cup L_i \cup a_i \cup L_{i+1}$ in Δ):

$$b_i a_i = t_i e_i,\ a_i b_i = t_i e_{i+1},\ \text{for } i = 1, \ldots, n-1,$$
$$\alpha^2 = 0, \qquad \beta^2 = 0,$$

and defining the differential (coming from the bigons with boundary $\alpha \cup L_1$ and $L_n \cup \beta$) by

$$da_i = db_i = 0 \text{ for } i = 1, \ldots, n-1, \qquad d\alpha = t_0 e_1, \quad d\beta = t_n e_n,$$

extending to longer paths by the graded Leibniz rule. Note that a_i, b_i, $i = 1, \ldots, n-1$, are in degree zero whilst α and β are in degree -1.

To see that there are no higher products, we appeal to a Maslov index calculation of Ozsváth and Szabo [29, Proposition 6.2] who studied these relative categories in the context of Heegaard–Floer theory (where it is called the *pong algebra*). A rigid $(k+1)$-gon contributing to a μ_k ¡operation has Maslov index $2-k$; Ozsváth and Szabo show that the Maslov index of a holomorphic disk u with boundaries on $L_1, \ldots, L_n$ is given by

$\mathrm{mult}(u, z_1) + \mathrm{mult}(u, z_n)$, which is non-negative since u is holomorphic. It follows that $k \leq 2$. A similar argument appears in [3, Proposition 3.6].

§6.5 Remark. The relative wrapped Fukaya category $\mathcal{W}(\Delta, D)$ is acted on by its center given by its Hochschild cohomology which can be identified with the symplectic cohomology $SH(\Delta, D)$. There is a closed orbit η that corresponds to the boundary of Δ which has degree -2. Thus, $\mathcal{W}(\Delta, D)$ can be seen as a category over $k[\eta]$. This recovers the familiar structure of the derived contraction algebra studied in detail in [17, Section 6].

§6.6 Example. We can compute the case where $n = 1$ and $f_0 = x, f_1 = y$. This corresponds to the conifold singularity. We get that $\Gamma = k[x,y]\langle\alpha,\beta\rangle$ with $\alpha^2 = \beta^2 = 0$, $d\alpha = x$ and $d\beta = y$. It is easy to determine that $H^*(\Gamma) = k[\eta]$ with $\eta = \alpha\beta + \beta\alpha$ of degree -2. This coincides with Booth's calculation [5, Section 4.2].

§6.7 Example. Consider the Pagoda flop $f_0 = y + x^n$, $f_1 = y - x^n$. Our model for the derived contraction algebra gives

$$k[x,y]\langle\alpha,\beta\rangle/(\alpha^2,\beta^2), \qquad d\alpha = y + x^n,\, d\beta = y - x^n.$$

Assuming we are not in characteristic 2, we can define

$$\zeta_1 = (\alpha+\beta)/2, \qquad \zeta_2 = (\alpha-\beta)/2$$

so that $d\zeta_1 = y$ and $d\zeta_2 = x^n$. This DG-algebra is isomorphic to the graded commutative algebra

$$k[x,y,\zeta_1,\zeta_2]/(\zeta_1^2+\zeta_2^2), \qquad d\zeta_1 = y,\, d\zeta_2 = x^n$$

Now, it is easy to see that the map from

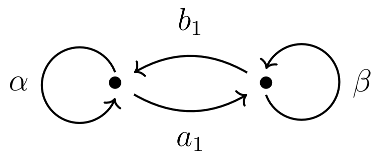
$$k[x,\zeta], \qquad d\zeta = x^n$$

sending $\zeta \to \zeta_2$ and $x \to x$ is a quasi-isomorphism. This latter model for the derived contraction algebra of the Pagoda flop is given by Booth in [5, Lemma 4.3.8]. Note that in characteristic 2, the class $x^n \in H^0(\Gamma)$ is non-trivial, so the assumption on characteristic is important here.

§6.8 Example. Consider the 3-fold $uv = xy(x^2+y^3)$. This has six different partial resolutions corresponding to different permutations of

$$f_1 = x, f_2 = x^2+y^3, f_3 = y.$$

We just focus on this particular choice and compare the answer our model gives for $\Lambda = H^0(\Gamma)$ with that computed by August [2, Example 4.5, Figure 2]. Our model gives an algebra over $k[x,y]$ described by the following quiver:



with differential

$$d\alpha = xe_1, \quad d\beta = ye_2,$$

and relations

$$a_1 b_1 = (x^2 + y^3)e_1, \quad b_1 a_1 = (x^2 + y^3)e_2, \quad \alpha^2 = 0, \quad \beta^2 = 0.$$

At the chain level, in degree zero, we have the free $k[x, y]$-module spanned by e_1, e_2, a_1, b_1. We need to quotient by

$$xe_1, \quad ye_2, \quad a_1 b_1 = (x^2 + y^3)e_1, \quad b_1 a_1 = (x^2 + y^3)e_2.$$

The quotient algebra is therefore generated by $m \coloneqq ye_1$, $\ell \coloneqq xe_2$, $a \coloneqq b_1$, $c \coloneqq a_1$ and these satisfy precisely the relations

$$\ell^2 = ac, \quad m^3 = ca, \quad \ell a = am = c\ell = mc = 0$$

given for B_{con} in [2, Figure 2]. For example:

$$m^3 = y^3 e_1 = (y^3 + x^2)e_1 = a_1 b_1 = ca.$$

A Generation of the relative Fukaya category

§A.1 Proposition *Let $\mathfrak{m} = (t_0, \ldots, t_n) \subset R$ be the maximal ideal and write k for the module $R/\mathfrak{m}$. The category $\mathcal{W}(T^*S^1, D) \otimes_R k$ is split-generated by the Lagrangian arcs $L_0, \ldots, L_n$.*

Proof. There is a tautological identification of $\mathcal{W}(T^*S^1, D) \otimes_R R/\mathfrak{m}$ with the full subcategory

$$\mathcal{B}(D) \subset \mathcal{W}(T^*S^1 \setminus D)$$

corresponding to Lagrangian branes which do not go near the punctures along D. The manifold $T^*S^1 \setminus D$ is a $(n+3)$-punctured sphere with the grading structure restricted from the standard one on T^*S^1. In [24], a mirror equivalence was established giving

$$\mathcal{W}(T^*S^1 \setminus D) \simeq D^b \operatorname{coh}(C)$$

where $C = \mathbb{A}^1 \cup_{pt} \mathbb{P}^1 \cup_{pt} \mathbb{P}^1 \ldots \cup_{pt} \mathbb{P}^1 \cup_{pt} \mathbb{A}^1$ is a nodal curve with $n + 2$ irreducible toric components glued together at the toric fixed points. Under this equivalence, the full subcategory $\mathcal{B}(D)$ gets identified with the full subcategory $\operatorname{perf}(C) \subset D^b \operatorname{coh}(C)$, and the

Lagrangians L_i go to line bundles $\mathcal{L}_i$ on C. In particular, one can arrange that $\mathcal{L}_0$ is the trivial bundle (i.e. the structure sheaf $\mathcal{O}_C$).

In the case $n = 0$, the mirror curve C is simply the *affine* curve $\mathbb{A}^1 \cup_{pt} \mathbb{A}^1 = \operatorname{Spec} k[x, y]/(xy)$, and the category $D^b \operatorname{coh}(C)$ is quasi-equivalent to the derived category of modules over $\operatorname{End}(\mathcal{O}_C)$. The subcategory of perfect objects is then generated by $\operatorname{End}(\mathcal{O}_C)$ itself [36, Lemma 15.78.1].

For higher n, there is an $n+1$-fold covering map $\pi\colon T^*S^1 \setminus D \to T^*S^1 \setminus \{p\}$ which respects the grading. The graph of π is a Lagrangian submanifold of $(T^*S^1 \setminus D)^- \times (T^*S^1 \setminus \{p\})$ (where $^-$ indicates that we reverse the sign of the symplectic form on this factor). This induces triangulated A_∞ quilt functors

$$\pi_*\colon \mathcal{W}(T^*S^1 \setminus D) \to \mathcal{W}(T^*S^1 \setminus \{p\}) \quad \text{respectively} \quad \pi^*\colon \mathcal{W}(T^*S^1 \setminus \{p\}) \to \mathcal{W}(T^*S^1 \setminus D).$$

Geometrically, a Lagrangian brane is sent under π_*, respectively π^*, to its (possibly immersed) image, respectively preimage, under π. These functors restrict to give functors

$$\pi_*\colon \mathcal{B}(D) \to \mathcal{B}(p) \quad \text{respectively} \quad \pi^*\colon \mathcal{B}(p) \to \mathcal{B}(D).$$

Given an object of $\mathcal{B}(D)$, it follows as in [32, Section 9] that the object $\pi^*\pi_*(L)$ is the sum $\bigoplus_{g\in G} g(L)$ where G is the deck group of the covering map π.

Write $L_0, \dots, L_n$ for the arcs in $T^*S^1 \setminus D$ and $\bar{L}_0$ for the arc in $T^*S^1 \setminus \{p\}$. By the $n = 0$ case of the proposition, if $L \in \mathcal{B}(D)$ then $\pi_*(L)$ is generated by $\bar{L}_0 \subset T^*S^1 \setminus \{p\}$. Therefore $\bigoplus_G g(L)$ is generated by $\pi^*\bar{L}_0 = \bigoplus_{i=0}^n L_i$, and since L is a summand of $\bigoplus_{g\in G} g(L)$, we see that L is split-generated by $\bigoplus_{i=0}^n L_i$, as required. $\square$

§A.2 Remark. Obviously, the Lagrangians $L_0, \dots, L_n$ do not generate $\mathcal{W}(T^*S^1 \setminus D)$, since the Lagrangian branes that are allowed in $\mathcal{W}(T^*S^1 \setminus D)$ can have ends near the punctures along D.

§A.3 Proposition (Generation with coefficients). *Let L be an object of $\mathcal{W}(T^*S^1, D)$. If L generates $\mathcal{W}_0(T^*S^1, D) := \mathcal{W}(T^*S^1, D) \otimes_R R/\mathfrak{m}$ then it also generates the relative wrapped category with coefficients in $\bar{R}$, that is $\mathcal{W}(T^*S^1, D) \otimes_R \bar{R}$.*

As a corollary, the category $\mathcal{W}(T^*S^1, D) \otimes_R \bar{R}$ is split-generated by the Lagrangian arcs $L_0, \dots, L_n$. The proof of this proposition will take up the rest of the appendix.

§A.4 Proof. Let

$$\bar{\mathcal{A}} = \operatorname{End}_{\mathcal{W}(T^*S^1,D)}(L) \otimes_R \bar{R}, \qquad \mathcal{A}_0 = \operatorname{End}_{\mathcal{W}_0(T^*S^1,D)}(L) = \operatorname{End}_{\mathcal{W}(T^*S^1,D)}(L) \otimes_R R/\mathfrak{m}.$$

We have Yoneda-type functors

$$\bar{Y}\colon \mathcal{W}(T^*S^1, D) \otimes_R \bar{R} \to \operatorname{mod}(\bar{\mathcal{A}})$$

and

$$Y_0\colon \mathcal{W}_0(T^*S^1, D) \to \mathrm{mod}(\mathcal{A}_0).$$

The module $Y_0(L) = \mathcal{A}_0$ (respectively $\bar{Y}(L) = \bar{\mathcal{A}}$) generates the subcategory $\mathrm{perf}(\mathcal{A}_0)$ (respectively $\mathrm{perf}(\bar{\mathcal{A}})$) of perfect objects. Since L generates $\mathcal{W}_0(T^*S^1, D)$, the functor Y_0 lands in $\mathrm{perf}(\mathcal{A}_0)$ and corestricts to give a quasi-equivalence

$$Y_0\colon \mathcal{W}_0(T^*S^1, D) \to \mathrm{perf}(\mathcal{A}_0)$$

(i.e. the induced functor on homotopy categories is fully faithful and essentially surjective). We want to show that

(a) $\bar{Y}$ lands in $\mathrm{perf}(\bar{\mathcal{A}})$;

(b) the induced functor $H(\bar{Y})$ on homotopy categories is (i) essentially surjective and (ii) fully faithful.

§A.5 Proof of (a): The subcategory $\mathrm{perf}(\bar{\mathcal{A}}) \subset \mathrm{mod}(\bar{\mathcal{A}})$ is precisely the triangulated subcategory of compact objects (see for example [36, Proposition 15.78.3]). An object C in a pre-triangulated A_∞ category is compact if and only if the functor it corepresents $\hom(C, \ )$ preserves coproducts, that is,

$$\oplus_i \hom(C, E_i) = \hom(C, \oplus_i E_i)$$

for arbitrary direct sums $\oplus_i E_i$. So it suffices to show that if $K \in \mathcal{W}(T^*S^1, D) \otimes_R \bar{R}$ is an object then

$$\oplus_i \hom_{\mathrm{mod}(\bar{\mathcal{A}})}\left(\bar{Y}(K), E_i\right) = \hom_{\mathrm{mod}(\bar{\mathcal{A}})}\left(\bar{Y}(K), \oplus_i E_i\right)$$

for arbitrary direct sums $\oplus_i E_i$ in $\mathrm{mod}(\bar{\mathcal{A}})$.

The complexes $\oplus_i \hom_{\mathrm{mod}(\bar{\mathcal{A}})}\left(\bar{Y}(K), E_i\right)$ and $\hom_{\mathrm{mod}(\bar{\mathcal{A}})}\left(\bar{Y}(K), \oplus_i E_i\right)$ are complete filtered $\bar{R}$-modules with the filtration coming from the action of powers of the maximal ideal; the canonical map

$$\oplus_i \hom_{\mathrm{mod}(\bar{\mathcal{A}})}\left(\bar{Y}(K), E_i\right) \to \hom_{\mathrm{mod}(\bar{\mathcal{A}})}\left(\bar{Y}(K), \oplus_i E_i\right) \tag{A.1}$$

is a morphism of filtered complexes. There are therefore spectral sequences computing both sides, and a morphism of spectral sequences induced by (A.1). By the Eilenberg-Moore comparison theorem, it suffices to check that this morphism is an isomorphism on the E_0 pages. Note that Eilenberg-Moore requires completeness of the filtration, which is why we are working over $\bar{R}$ instead of R.

The E_0 pages are respectively

$$E_0^{pq} = \oplus_i \hom^{p+q}_{\mathrm{mod}(\mathcal{A}_0)}\left(Y_0(K), \mathrm{gr}^p(E_i)\right) \text{ and } E_0^{pq} = \hom^{p+q}_{\mathrm{mod}(\mathcal{A}_0)}\left(Y_0(K), \oplus_i \mathrm{gr}^p(E_i)\right)$$

where gr^p denotes the pth graded piece of the associated graded module. The morphism on E_0-pages is induced by the canonical map

$$\oplus_i \hom_{\mathrm{mod}(\mathcal{A}_0)}\left(Y_0(K), \mathrm{gr}(E_i)\right) \to \hom_{\mathrm{mod}(\mathcal{A}_0)}\left(Y_0(K), \oplus_i \mathrm{gr}(E_i)\right).$$

Since $Y_0(K)$ is perfect, this is an isomorphism, which proves (a).

§A.6 Proof of (b.i): We have $\bar{\mathcal{A}} = \bar{Y}(L)$, and since $\bar{\mathcal{A}}$ generates $\mathrm{perf}(\bar{\mathcal{A}})$, the essential image of $\bar{Y}$ in $\mathrm{mod}(\bar{\mathcal{A}})$ contains $\mathrm{perf}(\bar{\mathcal{A}})$.

§A.7 Proof of (b.ii): Given objects $K, K' \in \mathcal{W}(T^*S^1, D) \otimes_R \bar{R}$, the complexes

$$CF(K, K') \otimes_R \bar{R} \quad \text{and} \quad \hom_{\mathrm{mod}(\bar{\mathcal{A}})}\left(\bar{Y}(K), \bar{Y}(K')\right)$$

are filtered by powers of the maximal ideal. These filtrations give us spectral sequences and the functor $\bar{Y}$ gives a map of filtered complexes $CF(K, K') \otimes_R \bar{R} \to \hom_{\mathrm{mod}(\bar{\mathcal{A}})}\left(\bar{Y}(K), \bar{Y}(K')\right)$ and hence a morphism of spectral sequences. On the E_1 page this is just the map

$$H\left(\hom_{\mathcal{W}_0(T^*S^1,D)}(K, K')\right) \otimes_R \mathrm{gr}(\bar{R}) \to H\left(\hom_{\mathrm{mod}(\mathcal{A}_0)}(Y_0(K), Y_0(K'))\right) \otimes_R \mathrm{gr}(\bar{R})$$

induced from $H(Y_0)\colon H\left(\hom_{\mathcal{W}_0(T^*S^1,D)}(K, K')\right) \to H\left(\hom_{\mathrm{mod}(\mathcal{A}_0)}(Y_0(K), Y_0(K'))\right)$ (because any polygons which pass through the marked points have their contributions weighted by an element of $\mathfrak{m}$). This is an isomorphism because Y_0 is cohomologically full and faithful. The Eilenberg-Moore comparison theorem then implies that the map $H(\bar{Y})$ is an isomorphism, which proves that $\bar{Y}$ is cohomologically full and faithful. □

References

[1] M. Abouzaid and P. Seidel. An open string analogue of Viterbo functoriality. *Geom. Topol.*, 14(2):627–718, 2010.

[2] J. August. On the finiteness of the derived equivalence classes of some stable endomorphism rings. *Math. Z.* 296(3-4)1157–1183, 2020.

[3] D. Auroux. Fukaya categories of symmetric products and bordered Heegaard-Floer homology. J. Gökova Geom. Topol. GGT 4 (2010), 1–54.

[4] R. Bocklandt. *A gentle introduction to homological mirror symmetry*, volume 99 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2021.

[5] M. Booth. The derived contraction algebra. arXiv:1903.12156, 2019.

[6] R.-O. Buchweitz, and L. Hille. *Hochschild (co-)homology of schemes with tilting object Trans. Amer. Math. Soc.*, 365(6):2823–2844, 2013.

[7] R. Bezrukavnikov, S. Riche. *Affine braid group actions on derived categories of Springer resolutions*, *Ann. Sci. Éc. Norm. Supér.*, (4)45(2012), no.4, 535–599.

[8] V. Drinfeld. *DG quotients of DG categories*, *J. Algebra* 272(2004), no.2, 643—691.

[9] W. Donovan and E. Segal. Mixed braid group actions from deformations of surface singularities. *Comm. Math. Phys.*, 335(1):497–543, 2015.

[10] W. Donovan and M. Wemyss. Noncommutative deformations and flops. *Duke Math. J.*, 165(8):1397–1474, 2016.

[11] T. Ekholm and Y. Lekili. Duality between Lagrangian and Legendrian invariants. *arXiv:1701.01284, to appear in Geometry & Topology*, 2017.

[12] W. Fulton. *Introduction to toric varieties.* Annals of Mathematics Studies, Number 131. Princeton University Press, Princeton, NJ, 1993.

[13] A. Gadbled, A.-L. Thiel, and E. Wagner. Categorical action of the extended braid group of affine type A. *Commun. Contemp. Math.*, 19(3):1650024, 39, 2017.

[14] S. Ganatra, J. Pardon, and V. Shende Sectorial descent for wrapped Fukaya categories arXiv:1809.03427, 2018.

[15] F. Haiden, L. Katzarkov, and M. Kontsevich. Flat surfaces and stability structures. *Publ. Math. Inst. Hautes Études Sci.*, 126:247–318, 2017.

[16] Z. Hua. Contraction algebra and singularity of three-dimensional flopping contraction. Math. Z. 290(1-2):431–443, 2018.

[17] Z. Hua and B. Keller Cluster categories and rational curves. arXiv:1810.00749 to appear in Geom. Topol.

[18] Z. Hua and Y. Toda. Contraction algebra and invariants of singularities. Int. Math. Res. Not. IMRN Volume 2018 No. 10 pp. 3173–3198, 2018.

[19] A. Ishii and K. Ueda Dimer models and exceptional collections. arXiv:0911.4529, 2009.

[20] O. Iyama and M. Wemyss. Reduction of triangulated categories and maximal modification algebras for cA_n singularities. *J. Reine Angew. Math.*, 738:149–202, 2018.

[21] M. Kalck and D. Yang. Relative singularity categories II: DG models. arXiv:1803.08192, 2018.

[22] J. Karmazyn Deformations of algebras defined by tilting bundles. J. Algebra 513:388–434, 2018.

[23] Y. Lekili and A. Polishchuk. Arithmetic mirror symmetry for genus 1 curves with n marked points. *Selecta Math. (N.S.)*, 23(3):1851–1907, 2017.

[24] Y. Lekili and A. Polishchuk. Auslander orders over nodal stacky curves and partially wrapped Fukaya categories. *J. Topol.*, 11(3):615–644, 2018.

[25] Y. Lekili and A. Polishchuk. Homological mirror symmetry for the symmetric squares of punctured spheres. *Adv. Math.*, 418:Paper No. 108942, 63, 2023.

[26] Y. Lekili and E. Segal. Equivariant Fukaya categories at singular values. *arXiv:2304.10969*, 2023.

[27] Y. Lekili and D. Treumann. A symplectic look at the Fargues-Fontaine curve. *Forum Math. Sigma* 10 (2022), 40 pp.

[28] P. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)*, 159(3):1027–1158, 2004.

[29] P. Ozsváth and Z. Szabó. *The Pong algebra and wrapped Fukaya category https://arxiv.org/abs/2212.14420*, 2022.

[30] T. Perutz and N. Sheridan. Constructing the relative Fukaya category. *arXiv:2203.15482*, 2022.

[31] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[32] P. Seidel. Homological mirror symmetry for the genus two curve. *J. Algebraic Geom.*, 20(4):727–769, 2011.

[33] P. Seidel. Homological mirror symmetry for the quartic surface. *Mem. Amer. Math. Soc.*, 236(1116):vi+129, 2015.

[34] P. Seidel and R. Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.

[35] N. Sheridan. Versality of the relative Fukaya category. *Geom. Topol.*, 24(2):747–884, 2020.

[36] The Stacks Project authors. The Stacks Project. `https://stacks.math.columbia.edu`, 2023.

[37] Z. Sylvan. On partially wrapped Fukaya categories. *J. Topol.*, 12(2):372–441, 2019.

[38] M. Van den Bergh. Three-dimensional flops and noncommutative rings. *Duke Math. J.*, 122(3):423–455, 2004.

[39] M. Wemyss. Chapter IV. Noncommutative resolutions. In *Noncommutative Algebraic Geometry*, M.S.R.I. Publications, Vol. 64. Eds. G. Bellamy, D. Rogalski, T. Schedler, J. T. Stafford, M. Wemyss. Cambridge University Press, Cambridge, 2016.

[40] H. Zhang. In preparation, 2023.

J. D. Evans, University of Lancaster

`j.d.evans@lancaster.ac.uk`

Y. Lekili, Imperial College London

`y.lekili@imperial.ac.uk`